Lecture 1– Definite Integrals, Theorem of Calculus

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1 Recap MATH 1012/1013

1.1 Recall definitions and main theorems

1.1.1 Limits

Let's start with limit rules and theorems blow, and suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.
Sum and difference rule: $\lim_{x\to a}(bf(x)+cg(x)) = b\lim_{x\to a} f(x) + c\lim_{x\to a} g(x)$;
Product rule: $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$;
Quotient rule: If $\lim_{x\to a} g(x)\neq 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$;
L'Hospital's rule Suppose that 1) f, g differentiable, 2) $g'(x)\neq 0$ on $x\in I$, 3) $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$, 4) $\lim_{x\to a}\frac{f'}{g'}$ exist or is ∞. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Rk: $\frac{0}{0}$ form, $\frac{\infty}{\infty}$ form.
Ex. Find the limit $\lim_{x\to 1}\frac{\ln x}{x-1}$.
Squeeze theorem: If $f(x)\leq g(x)\leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$;
Limit and one-sided limit: $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f = \lim_{x\to a^+} f = L$;
Two important limits:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \qquad \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e.$$

1.1.2 Differentiation

Derivative: $\frac{df}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, which denotes the rate of change of f with respect to (w.r.t.) x, can be understood as the slope of the tangent line.

Examples: (derivatives of some elementary functions)

- $(c)' = 0,$ for any constant;
- $(x^p)' = px^{p-1},$ for any constant p;
- $(e^x)' = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x$: the slope of $y = f(x) = e^x$ at (x, e^x);
- $(\ln|x|)' = \lim_{h\to 0}\frac{\ln|x+h|-\ln|x|}{h} = \frac{1}{x}$;

Rk: e^x and $\ln x$: inverse function of each other, since $\frac{d\ln x}{dx} = m = \frac{1}{M} = \frac{1}{e^{\ln x}}$, where m is the slope of $\ln x$ at $(x, \ln x)$, and M is the slope of e^x at $(\ln x, x)$;

- $(\sin x)' = \cos x \longleftrightarrow (\cos x)' = -\sin x$;
- $(\tan x)' = \sec^2 x \longleftrightarrow (\cot x)' = -\csc^2 x$;

note: Denote $\cot x = \tan(\frac{\pi}{2} - x) =: \frac{a}{b}$, we have

$$\begin{aligned}\frac{d\cot x}{dx} &= \frac{d}{dx}\tan\left(\frac{\pi}{2} - x\right) \\ &\overset{chain}{\underset{rule}{=}} \sec^2\left(\frac{\pi}{2} - x\right)\cdot\frac{d}{dx}\left(\frac{\pi}{2} - x\right) \\ &= \csc^2 x\cdot(-1) = -\csc^2 x,\end{aligned}$$

where $\sec\left(\frac{\pi}{2} - x\right) = \csc x$ has been used;

- $(\sec x)' = \sec x\tan x \longleftrightarrow (\csc x)' = -\csc x\cot x$;

. Chain rule: Suppose f' and g' exist. Then

$$[f(g(x))]' = f'(g(x))\cdot g'(x),$$

or in Leibniz notation,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

Example:

1. $(\ln|\sec x|)'$ 2. $(\ln|\sec x + \tan x|)'$

solution Note that

$$(\ln|g(x)|)' = \frac{1}{g(x)}\cdot g'.$$

Thus, we have

$$(\ln|\sec x|)' = \frac{1}{\sec x}\cdot(\sec x)' = \frac{1}{\sec x}\cdot\sec x\tan x = \tan x.$$

and

$$\begin{aligned}(\ln|\sec x + \tan x|)' &= \frac{1}{\sec x + \tan x}(\sec x + \tan x)' = \frac{1}{\sec x\tan x}\cdot\left(\sec x\tan x + \sec^2 x\right) \\ &= \frac{\sec x\cdot(\tan x + \sec x)}{\tan x + \sec x} = \sec x.\end{aligned}$$

Rk: $\ln|\sec x + \tan x|$ is an anti-derivative of $\sec x$.

Sum and difference rule:

$$[af(x) + bg(x)]' = af' + bg'.$$

Product rule:

$$(fg)' = f\cdot g' + g\cdot f'.$$

Quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}.$$

Example:

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cdot(\sin x)' - \sin x\cdot(\cos x)'}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

Formulas of trigonometric functions:

- $\cos^2 x + \sin^2 x = 1$;
- $1 + \tan^2 x = \sec^2 x$, divide both sides by $\cos^2 x$ above;
- $\cot^2 x + 1 = \csc^2 x$, divide both sides by $\sin^2 x$ above;

Angle sum/Difference formulas

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha,\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha,\\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.
\end{aligned}$$

Double angle formulas:

$$\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha,\\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.
\end{aligned}$$

2 Definite integrals

Definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Rk: 1) Interpretations of Definite Integral: the definite integral is to give the **net area** between the graph of $f(x)$ and the x-axis on the interval $[a, b]$; 2) the integral over the positive y axis is the area, however, the integral over the negative y axis, is negative area.

Example 2.1. *Using the definition to calculate*

$$\int_0^2 x^2 + 1\,dx,$$

solution.
For brevity, we use the right endpoints of each interval. We know that for a general n the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

The subintervals are then,

$$\left[0, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{4}{n}\right], \left[\frac{4}{n}, \frac{6}{n}\right], \ldots\ \ldots, \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right], \ldots, \left[\frac{2(n-1)}{n}, 2\right].$$

As we can see the **right endpoint** *of the i^{th} subinterval is*

$$x_i^* = \frac{2i}{n}.$$

The summation in the definition of the definite integral is then,

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i^*)\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right),\\
&= \sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2 + 1\right]\left(\frac{2}{n}\right)\\
&= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{2}{n}\right).
\end{aligned}$$

In particular, note that the summation notation

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We can evaluate the above summation as

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i^*)\Delta x &= \sum_{i=1}^{n} \frac{8i^2}{n^3} + \sum_{i=1}^{n} \frac{2}{n} \\
&= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 2 \\
&= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{n}\cdot 2n \\
&= \frac{4(n+1)(2n+1)}{3n^2} + 2 \\
&= \frac{14n^2+12n+4}{3n^2}
\end{aligned}$$

We can now compute the definite integral by usage of taking a limit of this,

$$\begin{aligned}
\int_0^2 x^2 + 1 \, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \frac{14n^2+12n+4}{3n^2} \\
&= \frac{14}{3}.
\end{aligned}$$

Properties:

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \quad \int_a^a f(x)\,dx = 0, \quad \int_a^b f(x)\,dx = \int_a^b f(t)\,dt,$
- $\int_a^b cf(x) \pm g(x)\,dx = c\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx,$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (where c is any number).

More properties that we often use:

- $\int_a^b c\,dx = c(b-a)$, c is any number;
- If $f(x) \geq 0$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$;
- If $f(x) \geq g(x)$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$;
- If $m \leq f(x) \leq M$ for $x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$;
- $|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx$.

3 Theorem of calculus

Theorem 3.1 (Fundamental Theorem of Calculus)**.** *Let f be a continuous function on $[a,b]$.*

Part I. Then the function

$$A(x) = \int_a^x f(t)\,dt,$$

is continuous on $[a, b]$, *differentiable in* (a, b), *and* $\frac{d}{dx}A(x) = f(x)$.

Part II. If F *is a differentiable function* $[a, b]$, *such that* $\frac{d}{dx}F(x) = f(x)$ *(say,* $F(x)$ *is any* **anti-derivative** *of* $f(x)$*), then*

$$\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a).$$

Rk: 1) note Part I, for some constant C, $A(x) = F(x) + C$, however, $0 = A(a) = F(a) + C$, thus $C = -F(a)$, $A(x) = F(x) - F(a)$; 2) If the interval of integration is given, then the integral is definite.

Example [other way out the definite integral: FTC] Evaluate

$$\int_0^2 (x^2 + 1)\, dx = \left(\frac{x^3}{3} + x\right)|_0^2 = \left(\frac{2^3}{3} + 2\right) - 0 = \frac{8}{3} + 2 = \frac{14}{3}.$$

Example: Evaluate

$$\int_0^1 x^2\, dx, \qquad \int_0^\pi \sin x\, dx, \qquad \int_0^1 e^x\, dx$$

solution.

- Since $\left(\frac{1}{3}x^3\right)' = x^2$, by the Fundamental Theorem of Calculus (FTC), we have

  $$\int_0^1 x^2\, dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}.$$

  Similarly, note that $(-\cos x)' = \sin x$, $(e^x)' = e^x$, (the anti-derivative once found, one can use the FTC), we have

  $$\int_0^\pi \sin x\, dx = -\cos x|_0^\pi\, dx = -\cos\pi - (-\cos 0) = 1 + 1 = 2$$
  $$\int_0^1 e^x\, dx = e^x|_0^1 = e^1 - e^0 = e - 1.$$

Example. If $F(x) = \int_0^x (t^2 + 1)\, dt$, evaluate $F'(x)$.
solution By the FTC, we have

$$f(x) = \frac{dF(x)}{dx} = x^2 + 1.$$

Example. If $F(x) = \int_0^x (t^2 + 1)e^x\, dt$, evaluate $F'(x)$.
solution note that $F(x) = e^x \int_0^x (t^2 + 1)\, dt$, by the FTC and chain rule, we have

$$f(x) = \frac{dF(x)}{dx} = e^x \int_0^x (t^2 + 1)\, dt + e^x(x^2 + 1).$$

Anti-differentiation

- Given f, to find f', called **differentiation**;
- Given f, to find F, such that $F' = f$, called **anti-differentiation**;

Ideas to find anti-differentiation:

Note that indefinite integral $\int f(x)\,dx$ means that find all the anti-differentiation of f. The idea to find one anti-differentiations of f, denoted by F, such $F' = f$, here F is called **one** anti-differentiation of f.

Note that if $F' = f(x) = G'$, we have

$$(F - G)' = f - f = 0.$$

So, the rate of change is zero, means $F - G = C$, where C is arbitrary constant, say $F(x) = G(x) + C$. Thus, we have all the anti-differentiation of f given by

$$\int f(x)\,dx = F(x) + C.$$

Example

$$\int x^2\,dx = \frac{x^3}{3} + C, \qquad \int x^p\,dx = \frac{1}{p+1}x^{p+1} + C, \qquad \text{for all const. } p \neq 1,$$
$$\int e^x\,dx = e^x + C, \qquad \int \frac{1}{x}\,dx = \ln|x| + C, \qquad \int a^x\,dx = \frac{x^x}{\ln a} + C$$
$$\int \cos x\,dx = \sin x + C, \qquad \int \sin x\,dx = -\cos x + C,$$
$$\int \tan x\,dx = \ln|\sec x| + C, \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C,$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C, \qquad \int \frac{1}{\sqrt{1-x^2}} = \arcsin x + C,$$
$$\int \sec^2 x\,dx = \tan x + C, \qquad \int \csc^2 x\,dx = -\cot x + C,$$
$$\int \sec x \tan x\,dx = \sec x + C, \qquad \int \csc x \cot x\,dx = -\csc x + C.$$

solution

since $\left(\frac{x^3}{3}\right)' = x^2$, $\frac{x^3}{3}$ is one anti-derivative of x^2. Thus, all the anti-derivatives construct $\int x^2\,dx$. Similarly, other example as remained exercise!

Example:

Evaluate

$$\int_0^1 \left(x^5 + 2\cos x - \frac{1}{x^2+1}\right)\,dx.$$

solution.

We can first evaluate the indefinite integral:

$$\int \left(x^5 + 2\cos x - \frac{1}{x^2+1}\right)\,dx = \int x^5\,dx + 2\int \cos x\,dx - \int \frac{1}{x^2+1}\,dx$$
$$= \frac{1}{6}x^6 + 2\sin x - \arctan x + C.$$

Then, by applying the FTC, we obtain

$$\int_0^1 \left(x^5 + 2\cos x - \frac{1}{x^2+1}\right)\,dx = \left(\frac{1}{6}x^6 + 2\sin x - \arctan x\right)\Big|_0^1$$
$$= \frac{1}{6} + 2\sin 1 - \arctan 1$$
$$= \frac{1}{6} + 2\sin 1 - \frac{\pi}{4}.$$

Rk: the homeworks can be found here https://www.classviva.org.

Example (from classviva.org).

Evaluate

$$\int_6^8 x^3\sqrt{4x^4-3}\,dx.$$

solution

Note that

$$\left[(4x^4-3)^{\frac{3}{2}}\right]' = \frac{3}{2}\left(4x^4-3\right)^{\frac{1}{2}}\cdot 16x^3 = 24x^3\sqrt{4x^4-3}.$$

Say by FTC,

$$\begin{aligned}\int_6^8 x^3\sqrt{4x^4-3}\,dx &= \frac{1}{24}\int_6^8 \left[(4x^4-3)^{\frac{3}{2}}\right]'\,dx = \frac{1}{24}\left(4x^4-3\right)^{\frac{3}{2}}|_6^8\\ &= \frac{1}{24}\left[\sqrt{(4\cdot 8^4-3)^3}-\sqrt{(4\cdot 6^4-3)^3}\right]\approx 71818.83.\end{aligned}$$