Lecture 10 (Applications of integration)–Volume by cylindrical shells

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1 Recap last time

volume by slicing

- if the cross-sectional area $A(x)$ is perpendicular to the x-axis over $[a, b]$, the volume of solid is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx;$$

- if the cross-sectional area $A(y)$ is perpendicular to the y-axis over $[c, d]$, the volume of solid is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(y_i^*)\Delta y = \int_c^d A(y)\, dy;$$

Specification of $A(x)$ and $A(y)$: The solid is generated by rotating some enclosed area of resolution about some line. For example,

- If $y = f(x) \geq 0$ or $y = f(x) \leq 0$ for all $x \in [a, b]$, rotating it about x-axis, we have $A(x) = \pi(f(x))^2 \Longrightarrow V = \int_a^b A(x)\, dx = \pi \int_a^b f^2\, dx$;
- If $x = g(y) \geq 0$ or $x = g(y) \leq 0$ for all $y \in [c, d]$, rotating it about y-axis, we have $A(y) = \pi(g(y))^2 \Longrightarrow V = \int_c^d A(y)\, dy = \int_c^d g^2\, dy$;
- If $0 \leq f(x) \leq g(x)$ for all $x \in [a, b]$, rotating the enclosed region of $y = f(x)$, $y = g(x)$, $x = a$, $x = b$ about x-axis, we have $A(x) = \pi[g^2 - f^2] \Longrightarrow V = \pi \int_a^b [g^2(x) - f^2(x)]\, dx$;
- If $0 \leq f(y) \leq g(y)$ for all $y \in [c, d]$, rotating the enclosed region of $x = f(y)$, $x = g(y)$, $y = c$, $y = d$ about y-axis, we have $A(y) = \pi[g^2 - f^2] \Longrightarrow V = \pi \int_c^d [g^2(y) - f^2(y)]\, dy$;

Exercise. Calculate volume V of the solid by rotating the area bounded by $y = x$ and $y = x^2$ about x-axis and y-axis, respectively.
solution. Intersections: 1) $x = 0$, $y = 0$; 2) $x = 1$, $y = 1$.

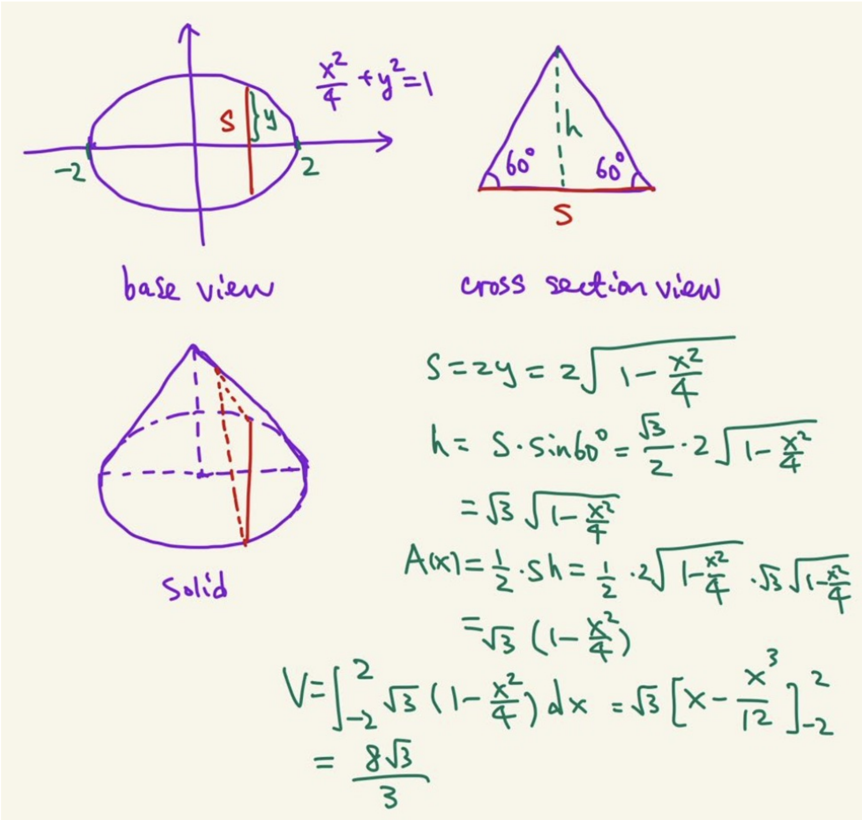
- rotation about x-axis,

$$V = \int_0^1 \pi[x^2 - (x^2)^2]\, dx = \frac{2}{15}\pi,$$

- rotation about y-axis,

$$V = \int_0^1 \pi[(\sqrt{y})^2 - y^2]\, dy = \frac{1}{6}\pi.$$

Example. A solid with base $\frac{x^2}{4} + y^2 = 1$ has cross sections along the x-axis as equilateral triangles. Find the volume of the solid.
solution. We use $V = \int_a^b A(x)\, dx$ as below.



2 Volume by cylindrical shells

volume by using shells Another way to decompose the volume of the solid of resolution,

- obtained by rotating about y-axis the region under the curve $y = f(x)$ over $[a, b]$ $(0 \le a < b)$, is

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} V_i = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x = \int_a^b 2\pi x f(x)\, dx;$$

- obtained by rotating about x-axis the region under the curve $x = f(y)$ over $[c, d]$ $(0 \le c < d)$, is

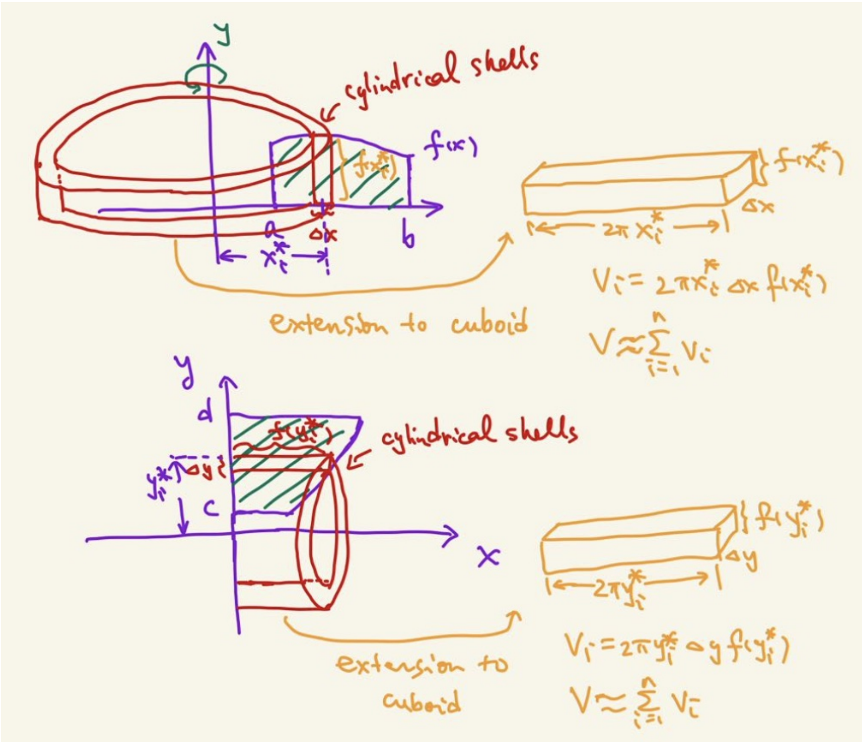
$$V = \lim_{n\to\infty}\sum_{i=1}^{n} V_i = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi y_i^* f(y_i^*)\Delta y = \int_c^d 2\pi y f(y)\, dy.$$

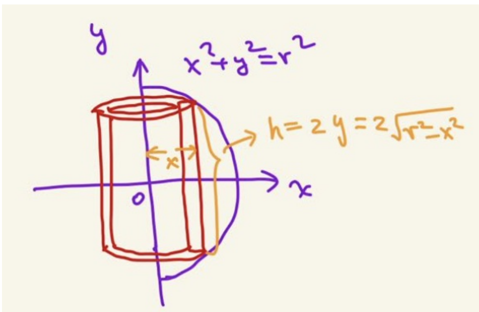
Here, in the formula,

$$V_i = \text{(circumference)} \times \text{(height)} \times \text{(thickness)}$$

and the terms can be interpreted as

- x or y: radius of a typical cylinder shell;
- $2\pi x$ or $2\pi y$: circumference;
- $f(x)$ or $f(y)$: height;
- dx or dy: thickness of the shell.


y
cylindrical shells
f(x)
a
Δx
b
extension to cuboid
$V_i = 2\pi x_i^* \Delta x f(x_i^*)$
$V \approx \sum_{i=1}^{n} V_i$
d
c
x
extension to cuboid
$V_i = 2\pi y_i^* \Delta y f(y_i^*)$
$V \approx \sum_{i=1}^{n} V_i$


y
$x^2+y^2=r^2$
$h=2y=2\sqrt{r^2-x^2}$
x
0

Example (using shells). Find the volume of the ball of radius r.

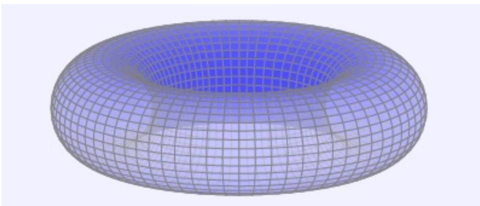
solution. The ball can be thought of the rotation about y-axis of the half curve $x = \sqrt{r^2 - y^2}$. The shell with radius x and the height $h = 2y = 2\sqrt{r^2 - x^2}$. Thus, we have

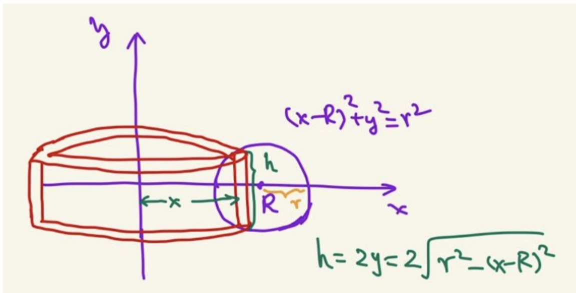
$$V = \int_0^r 2\pi x \cdot 2\sqrt{r^2 - x^2}\, dx = (-2\pi)\int_0^r \sqrt{r^2 - x^2}\, d(r^2 - x^2)$$
$$= (-2\pi) \cdot \frac{2}{3}(r^2 - x^2)^{\frac{3}{2}} \mid_0^r = \frac{4}{3}\pi r^3.$$

Example (volume of a solid torus using shells) Consider the region bounded by the curves

$$(x - R)^2 + y^2 = r^2, \quad 0 < r < R.$$

The solid obtained by rotating the region about the y-axis, is the solid torus, as shown below, compute the volume of the solid.





solution. Let $u = x - R$, we have $du = dx$, $x = u + R$, $x : R - r \to R + r$, $u : -r \to r$. Note that the function $u\sqrt{r^2 - R^2}$ is odd. In turn,

$$\int_{-r}^{r} u\sqrt{r^2 - u^2}\, du = 0.$$

The volume is given by

$$V = \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2 - (x - R)^2}\, dx = 4\pi \int_{-r}^{r} (u + R)\sqrt{r^2 - u^2}\, du$$
$$= 4\pi R \int_{-r}^{r} \sqrt{r^2 - u^2}\, du.$$

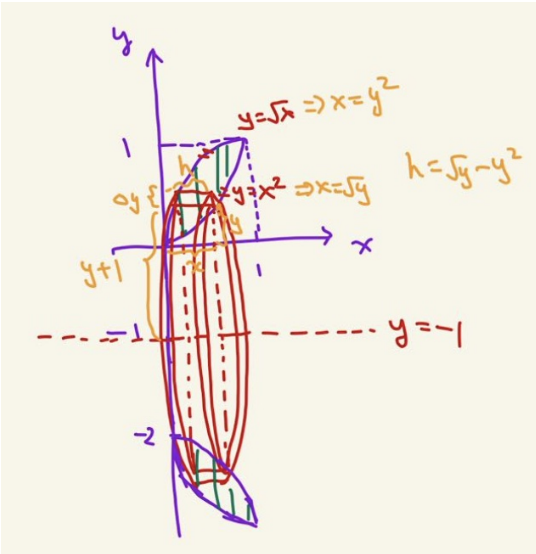
Since the last integral represents the area of the semicircular region of radius r, we have

$$V = 4\pi R \cdot \frac{1}{2}\pi r^2 = 2\pi^2 R r^2.$$

Rk. If we compute the volume by slicing, we have

$$V = \int_{-r}^{r} \pi \left[\left(R + \sqrt{r^2 - y^2}\right)^2 - \left(R - \sqrt{r^2 - y^2}\right)^2 \right] dy$$
$$= 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2}\, dy = 4\pi R \cdot \frac{\pi r^2}{2} = 2\pi R \cdot \pi r^2.$$

Example (rotating about a horizontal line using shells). Find the volume of the solid obtained by rotating the region bounded by the curves $y = f(x) = \sqrt{x}$ and $y = g(x) = x^2$ about the horizontal line $y = -1$.



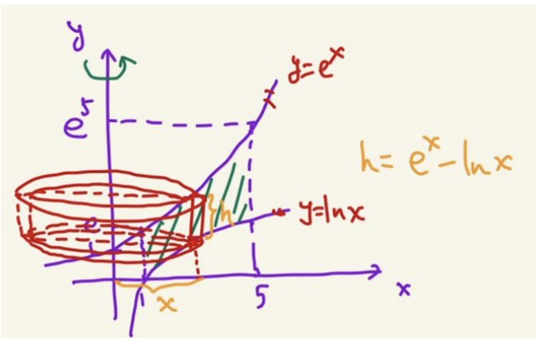
solution. To find the intersections, we have to take

$$\sqrt{x} = x^2, \quad \Longrightarrow x = 0, x = 1.$$

The typical shell is shown below. Thus, the volume is

$$V = \int_0^1 2\pi(y+1)\cdot(\sqrt{y} - y^2)\, dy = 2\pi \int_0^1 \left(y^{\frac{3}{2}} - y^3 + y^{\frac{1}{2}} - y^2\right)\, dx = \frac{29\pi}{30}.$$

Example (from classviva.org). The volume of the solid obtained by rotating the region bounded by $y = e^x$, $y = \ln x$, $x = 1$ and $x = 5$ about the line y-axis. Find the volume.
solution.
The bounded domain and the typical shell are shown below. The volume is



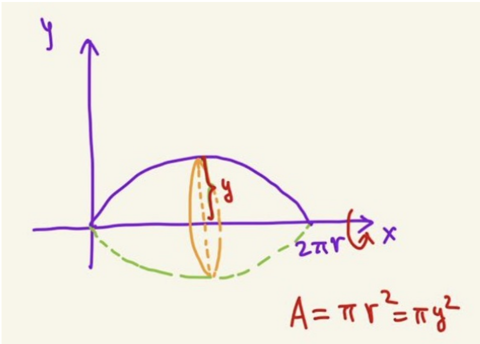
$$\begin{aligned} V &= \int_1^5 2\pi x\cdot(e^x - \ln x)\, dx = 2\pi\left(\int_1^5 xe^x\, dx - \int_1^5 x\ln x\, dx\right) \\ &= 2\pi\left(xe^x|_1^5 - \int_1^5 e^x\, dx - \frac{1}{2}\int_1^5 \ln x\, dx^2\right) = 2\pi\left((x-1)e^x|_1^5 - \frac{1}{2}(x^2\ln x|_1^5 - \int_1^5 x^2\frac{1}{x}\, dx)\right) \\ &= 2\pi\left[4e^5 - \frac{25}{2}\ln 5 + 6\right]. \end{aligned}$$

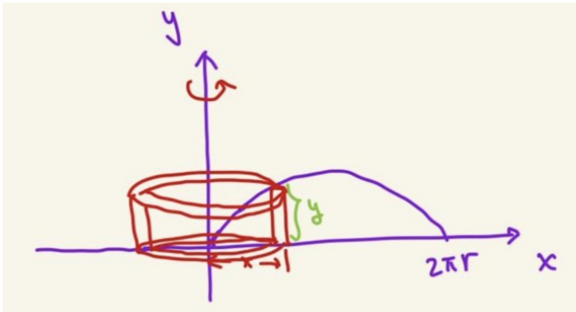
Example (solid generated by a parametric curve). Rotate the area under one arch of the cycloid

$$x = r(t - \sin t), \quad y = r(1 - \cos t), \quad 0 \le t \le 2\pi,$$

about the x-axis and y-axis, respectively. Find the volumes of the two solids.
solution. Note that

- when $t = 0$, we have $x = 0$, $y = 0$;
- when $t = 2\pi$, we have $x = 2\pi r$, $y = 0$;
- when $0 \le t \le 2\pi$, we have $1 - \cos t \ge 0 \Longrightarrow y \ge 0$.





✓ When rotating the region about the x-axis, the volume by slices is

$$
\begin{aligned}
V &= \int_0^{2\pi r} \pi [y(x)]^2 \, dx = \int_0^{2\pi} \pi \left[r(1-\cos t)\right]^2 \cdot \left[r(t - \sin t)\right]' \, dt \\
&= \pi r^3 \int_0^{2\pi} (1-\cos t)^3 \, dt = \pi r^3 \int_0^{2\pi} \left[1 - 3\cos t + 3\cos^2 t - \cos^3 t\right] \, dt \\
&= \pi r^3 \int_0^{2\pi} \left[1 - 3\cos t + \frac{3}{2}(1+\cos 2t) - (1-\sin^2 t)\cos t\right] \, dt \\
&= \pi r^3 \left[\frac{5}{2}t - 3\sin t + \frac{3}{4}\sin 2t - \sin t + \frac{1}{3}\sin^3 t\right] |_0^{2\pi} = 5\pi^2 r^3.
\end{aligned}
$$

✓ When rotating the region about the y-axis, the volume by shells is

$$
\begin{aligned}
V &= \int_0^{2\pi r} 2\pi x \cdot y(x) \, dx = \int_0^{2\pi} 2\pi r(t - \sin t) \cdot \left[r(1-\cos t)\right] \cdot \left[r(t-\sin t)\right]' \\
&= 2\pi r^3 \int_0^{2\pi} (t - \sin t)(1-\cos t)^2 \, dt.
\end{aligned}
$$

Note that (since $\sin t(1-\cos t)^2$ is a 2π periodic odd function)

$$
\int_0^{2\pi} \sin t \cdot (1-\cos t)^2 \, dt = \int_{-\pi}^{\pi} \sin t (1-\cos t)^2 \, dt = 0.
$$

Thus,

$$
V = 2\pi r^3 \int_0^{2\pi} t(1-\cos t)^2 \, dt = 2\pi r^3 \int_0^{2\pi} t\left[1 - 2\cos t + \frac{1}{2}(1+\cos 2t)\right] \, dt = 6\pi^3 r^3.
$$