Lecture 11 (Applications of integration)–Arc lengths and surface areas

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1 Arc lengths

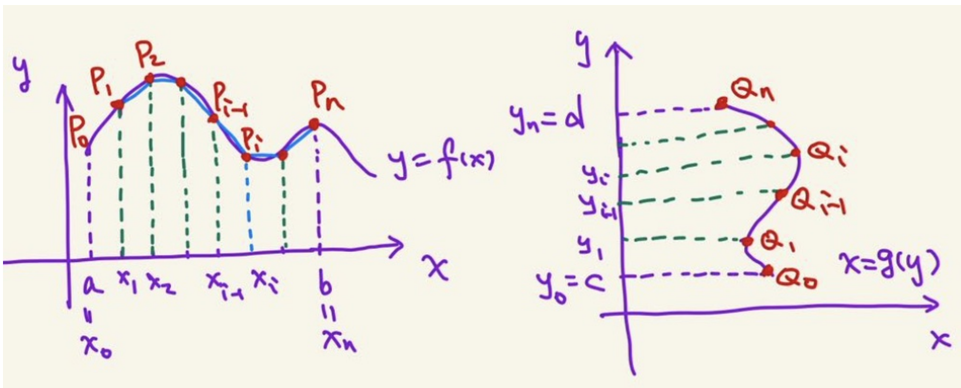
Intuition of the arc length

- taking a curve $y = f(x)$, $f(x)$ is continuous over $[a, b]$. Dividing the interval $[a, b]$ into n subintervals with end points $x_0, x_1, \cdots, x_n$ and equal width Δx. Denote the endpoints of the line segments to be $P_0, P_1, \cdots, P_n$,where P_i lines on the curve $y = f(x)$ with coordinates $(x_i, f(x_i))$. Then length $\mathcal{L}$ of the curve $y = f(x)$ is defined by

$$\mathcal{L} = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|,$$

- taking a curve $x = g(y)$, $g(y)$ is continuous over $[c, d]$. Dividing the interval $[c, d]$ into n subintervals with end points $y_0, y_1, \cdots, y_n$ and equal width Δy. Denote the endpoints of the line segments to be $Q_0, Q_1, \cdots, Q_n$,where Q_i lines on the curve $x = g(y)$ with coordinates $(y_i, g(y_i))$. Then length $\mathcal{L}$ of the curve $x = g(y)$ is defined by

$$\mathcal{L} = \lim_{n\to\infty} \sum_{i=1}^{n} |Q_{i-1}Q_i|.$$

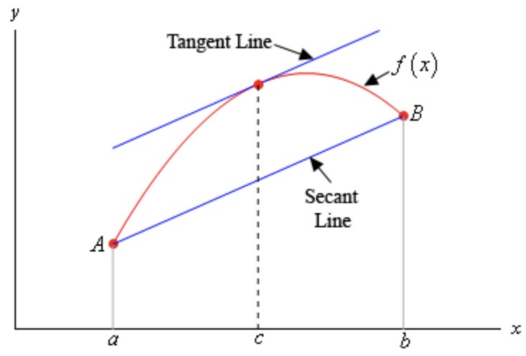


Formula of the arc length Let $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$. The distance of the straight line between P_{i-1} and P_i is given by

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

Note that the mean value theorem below.
Mean value theorem Suppose $f(x)$ is continuous on $[a, b]$, differentiable on (a, b). Then there is a number c such that $a < c < b$, and

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$



or $f(b) - f(a) = f'(c)(b-a)$. The sketch is shown below,
By the mean value theorem, we have

$$\Delta y_i = f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) = f'(x_i^*)\Delta x,$$

where $x_{i-1} < x_i^* < x_i$ and $f'(x_i^*)$ is the slope of the tangent line at point $(x_i^*, f(x_i^*))$. Thus,

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2}$$
$$= \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Thus,

$$\mathcal{L} = \lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i| = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+[f'(x_i^*)]^2}\Delta x = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

Theorem

- If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$, $x \in [a,b]$ is given by

$$\mathcal{L} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1+[f'(x)]^2}\,dx,$$

- If g' is continuous on $[c,d]$, then the length of the curve $x = g(y)$, $y \in [c,d]$ is given by

$$\mathcal{L} = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d \sqrt{1+[g'(y)]^2}\,dy,$$

Rk. If $f'(x)$ is continuous on $[a,b]$, the distance along the curve from the initial point $(a, f(a))$ to the point $(x, f(x))$ defines the **arc length function** below,

$$s(x) = \int_a^x \sqrt{1+[f'(t)]^2}\,dt.$$

Note that by FTC, $s'(x) = \sqrt{1+[f'(x)]^2}$ means that $ds = \sqrt{1+[f'(x)]^2}\,dx$.
Example. Find the arc length of the curve $y = e^x$ from $(0,1)$ to $(1,e)$.
solution.

- $y = e^x$, $y' = e^x$, the arc length is

$$\mathcal{L} = \int_0^1 \sqrt{1+(y')^2}\,dx = \int_0^1 \sqrt{1+e^{2x}}\,dx,$$

by substitution $u = e^x$, we have $x = \ln u$, $dx = \frac{1}{u}\,du$ and $x : 0 \to 1$, $u : 1 \to e$ and

$$\mathcal{L} = \int_1^e \frac{\sqrt{1+u^2}}{u}\,du,$$

- $x = \ln y$, $x' = \frac{1}{y}$, the arc length is

$$\mathcal{L} = \int_1^e \sqrt{1 + (x')^2}\, dy = \int_1^e \frac{\sqrt{1+y^2}}{y}\, dy$$

As a result, we have

$$\begin{aligned}
\mathcal{L} &= \int_1^e \frac{\sqrt{1+y^2}}{y}\, dy = \int_1^e \frac{y\sqrt{1+y^2}}{y^2}\, dy = \frac{1}{2}\int_1^e \frac{\sqrt{1+y^2}}{y^2}\, dy^2 \\
&= \frac{1}{2}\int_1^{e^2} \frac{\sqrt{1+u}}{u}\, du = \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{e^2+1}} \frac{t}{t^2-1}\cdot 2t\, dt = \int_{\sqrt{2}}^{\sqrt{e^2+1}} \frac{t^2}{t^2-1}\, dt = \int_{\sqrt{2}}^{\sqrt{e^2+1}} \frac{t^2-1+1}{t^2-1}\, dt \\
&= \left(\sqrt{e^2+1} - \sqrt{2}\right) + \int_{\sqrt{2}}^{\sqrt{e^2+1}} \frac{1}{t^2-1}\, dt = \left(\sqrt{e^2+1} - \sqrt{2}\right) + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| \Big|_{\sqrt{2}}^{\sqrt{e^2+1}} \\
&= \left(\sqrt{e^2+1} - \sqrt{2}\right) + \frac{1}{2}\left[\ln\left|\frac{\sqrt{e^2+1}-1}{\sqrt{e^2+1}+1}\right| - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right].
\end{aligned}$$

where the substitutions $u = y^2$ and $t = \sqrt{1+u}$, i.e., $u = t^2 - 1$, $du = 2t\, dt$ have been used.
Example. Find the arc length of the curve $y = x^2$ from $(0,0)$ to $(\frac{1}{2}, \frac{1}{4})$.
solution. Since that $y = x^2$, we have $y' = 2x$. Thus, the arc length of the curve from $(0,0)$ to $(\frac{1}{2}, \frac{1}{4})$ is given by

$$\mathcal{L} = \int_0^{\frac{1}{2}} \sqrt{1 + (y')^2}\, dx = \int_0^{\frac{1}{2}} \sqrt{1 + 4x^2}\, dx.$$

Using the substitution $x = \frac{1}{2}\tan\theta$, we have $x : 0 \to \frac{1}{2}$, $\theta : 0 \to \frac{\pi}{4}$, $dx = \frac{1}{2}\sec^2\theta\, d\theta$ and

$$\mathcal{L} = \int_0^{\frac{\pi}{4}} \sec\theta \cdot \frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta.$$

Note that by the integration by parts, we have

$$\begin{aligned}
I = \int \sec^3\theta\, d\theta &= \int \sec\theta\, d\tan\theta = \sec\theta\tan\theta - \int \tan^2\theta \sec\theta\, d\theta \\
&= \sec\theta\tan\theta - \int \left(\sec^2\theta - 1\right)\cdot \sec\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta - I,
\end{aligned}$$

in turn,

$$\begin{aligned}
2I &= \sec\theta\tan\theta + \int \sec\theta\, d\theta \\
&= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C,
\end{aligned}$$

say,

$$I = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

Thus, the arc length is given by

$$\mathcal{L} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \frac{1}{4}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]\Big|_0^{\frac{\pi}{4}} = \sqrt{2} + \ln(\sqrt{2}+1).$$

Rk.

- since that $(\ln|\sec x + \tan x|)' = \sec x$, we have $\int \sec x\, dx = \ln|\sec x + \tan x| + C$;

- let $u = \sin x$, we have

$$\begin{aligned}\int \sec x\, dx &= \int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{\cos^2 x}\, dx = \int \frac{1}{1-\sin^2 x}\, d\sin x \\ &= \int \frac{1}{1-u^2}\, du = \frac{1}{2}\left[\int \frac{1}{u+1}\, du - \int \frac{1}{u-1}\, du\right] = \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C \\ &= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C = \frac{1}{2}\ln\left|\frac{\tan x+\sec x}{\tan x-\sec x}\right| + C.\end{aligned}$$

Note that

$$\left(\frac{1}{2}\ln\left|\frac{\tan x+\sec x}{\tan x-\sec x}\right|\right)' = (\ln|\sec x+\tan x|)' = \sec x.$$

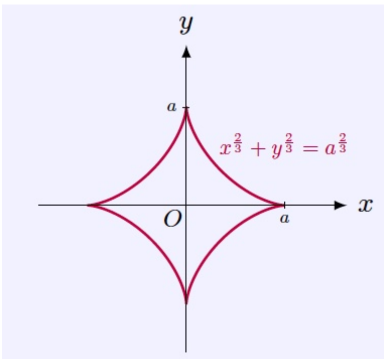
Rk. How about the arc length for the same curve $x = y^{\frac{1}{2}}$. We have $\frac{dx}{dy} = \frac{1}{2}y^{-\frac{1}{2}}$. Thus, the arc length from $(0,0)$ to $(\frac{1}{2}, \frac{1}{4})$ is

$$\mathcal{L} = \int_0^{\frac{1}{4}} \sqrt{1+(x')^2}\, dy = \int_0^{\frac{1}{4}} \sqrt{1+\frac{1}{4}y^{-1}}\, dy.$$

Evaluate this improper integral is more complicated! So, when there are more than one choices in finding a quantity (like area, volume, arc length, etc), one may need to choose a wise setup.

Example. Find the circumference of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

solution. We have $y = f(x) = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} = \sqrt[3]{\left(\sqrt[3]{a^2} - \sqrt[3]{x^2}\right)^2}$, thus, $f(-x) = f(x)$, similarly, $x = g(y) = \sqrt[3]{\left(\sqrt[3]{a^2} - \sqrt[3]{y^2}\right)^2}$, thus, $g(-y) = g(y)$. Thus, the astroid is symmetric about the x-axis and the y-axis. The graph of this curve is shown below, So, its circumference is four time the



arc length in the first quadrant. In the first quadrant. In the first quadrant, $y \geq 0$, In this region, solving the equation of the astroid gives

$$y = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}},$$

where $x \in [0, a]$. By chain rule, we have

$$\frac{dy}{dx} = \frac{3}{2}\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{1}{2}} \cdot (-1)\frac{2}{3}x^{-\frac{1}{3}} = -x^{-\frac{1}{3}}\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{1}{2}}.$$

Thus, the circumference of the astroid is

$$\begin{aligned}\mathcal{L} &= 4\int_0^a \sqrt{1+(y')^2}\, dx = 4\int_0^a \sqrt{1+\left[x^{-\frac{2}{3}}\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)\right]}\, dx \\ &= 4a^{\frac{1}{3}}\int_0^a x^{-\frac{1}{3}}\, dx,\end{aligned}$$

which is an improper integral, by the definition, we have

$$\int_0^a x^{-\frac{1}{3}}\,dx = \lim_{h\to 0^+}\int_h^a x^{-\frac{1}{3}}\,dx = \lim_{h\to 0^+}\left[\frac{3}{2}x^{\frac{2}{3}}\right]\Big|_h^a = \frac{3}{2}a^{\frac{2}{3}}.$$

Thus, we have

$$\mathcal{L} = 4a^{\frac{1}{3}}\cdot\frac{3}{2}a^{\frac{2}{3}} = 6a.$$

Theorem (arc length for parametric curve). Suppose a parametric curve is given by the parametric equations

$$\begin{cases} x = f(t), \\ y = g(t), \end{cases}$$

where $t\in[\alpha,\beta]$, f and g' are continuous and the parametric curve is traversed exactly once as t increases from α to β. Since $t:\alpha\to\beta$, we have $x: f(\alpha)\to f(\beta)$, $y: g(\alpha)\to g(\beta)$ and $dx = f'(t)\,dt$, $dy = g'(t)\,dt$.
Then the arc length is

$$\begin{aligned}\mathcal{L} &= \int_{f(\alpha)}^{f(\beta)}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{f(\alpha)}^{f(\beta)}\sqrt{(dx)^2+(dy)^2} \\ &= \int_\alpha^\beta\sqrt{[f'(t)\,dt]^2+[g'(t)\,dt]^2} = \int_\alpha^\beta\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt.\end{aligned}$$

Say,

$$\mathcal{L} = \int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_\alpha^\beta\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt.$$

Rk. Of course, we can get the similar arc length by

$$\begin{aligned}\mathcal{L} &= \int_{g(\alpha)}^{g(\beta)}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_{g(\alpha)}^{g(\beta)}\sqrt{(dy)^2+(dx)^2} \\ &= \int_\alpha^\beta\sqrt{[g'(t)\,dt]^2+[f'(t)\,dt]^2} = \int_\alpha^\beta\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt.\end{aligned}$$

Example. Find the circumference of the astroid shown above, given by the parametric equations

$$\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t, \end{cases}$$

where $t\in[0,2\pi]$.
solution. The astroid is symmetric about the x-axis and the y-axis. So, its circumference is four time the arc length in the first quadrant. Thus,

$$\begin{aligned}\mathcal{L} &= 4\int_0^{\frac{\pi}{2}}\sqrt{[3a\cos^2 t\cdot(-\sin t)]^2+\left[3a\sin^2 t\cdot(\cos t)\right]^2}\,dt \\ &= 12a\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt = 12a\left(\frac{1}{2}\sin^2 t\right)\Big|_0^{\frac{\pi}{2}} = 6a.\end{aligned}$$

Example (from classviva.org). Find the arc length of the curve $y=\frac{1}{2}(e^x+e^{-x})$ from $x=0$ to $x=3$.
solution. Note that $y=f(x)=\frac{1}{2}(e^x+e^{-x})$, we have

- when $x = 0$, $y = 1$; when $x = 3$, $y = \frac{1}{2}(e^3 + e^{-3})$
- $f(-x) = f(x)$, the curve is symmetric about y-axis;
- $f'(x) = \frac{1}{2}(e^x - e^{-x})$, when $x > 0$, $f'(x) > 0$; when $x < 0$, $f'(x) < 0$.

The arc length of the curve is

$$\mathcal{L} = \int_0^3 \sqrt{1 + (y')^2}\, dx = \int_0^3 \sqrt{1 + \frac{1}{4}(e^x - e^{-x})^2}\, dx.$$

Note that $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh^2 x - \sinh^2 x = 1$ and $\frac{d}{dx}\sinh x = \cosh x$, we have

$$\mathcal{L} = \int_0^3 \sqrt{1 + \sinh^2 x}\, dx = \int_0^3 \cosh x\, dx = \sinh x|_0^3 = \left(\frac{e^x - e^{-x}}{2}\right)|_0^3 = \frac{1}{2}(e^3 - e^{-3}).$$

Rk. Note that

$$\sqrt{1 + \frac{1}{4}(e^x - e^{-x})^2} = \sqrt{\frac{4 + e^{2x} - 2 + e^{-2x}}{4}} = \sqrt{\frac{(e^x + e^{-x})^2}{4}} = \frac{e^x + e^{-x}}{2}.$$

Thus, we have

$$\mathcal{L} = \int_0^3 \frac{e^x + e^{-x}}{2}\, dx = \frac{1}{2}\left[e^x - e^{-x}\right]|_0^3 = \frac{1}{2}(e^3 - e^{-3}).$$

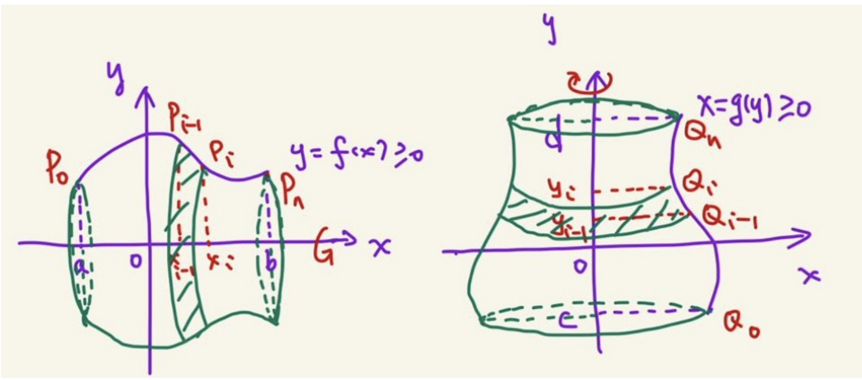
2 Surface areas

Intuition of surface area of revolution

- taking a curve $y = f(x) \geq 0$, over $[a, b]$. Dividing the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \cdots, x_n$ and equal width Δx. The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis, where $P_i(x_i, f(x_i))$ lies on the curve. The result is a band with slant height $P_{i-1}P_i$. The surface area is obtained by rotating the curve $y = f(x)$ over $[a, b]$ about the x-axis by

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} S_i,$$

where S_i is the area of the band with slant height $|P_{i-1}P_i|$.



- taking a curve $x = g(y) \geq 0$, over $[c, d]$. Dividing the interval $[a, b]$ into n subintervals with endpoints $y_0, y_1, \cdots, y_n$ and equal width Δy. The part of the surface between y_{i-1} and y_i is approximated by taking the line segment $Q_{i-1}Q_i$ and rotating it about the y-axis, where $Q_i(y_i, g(y_i))$ lies on the curve. The result is a band with slant height $Q_{i-1}Q_i$. The surface area is obtained by rotating the curve $x = g(y)$ over $[c, d]$ about the y-axis by

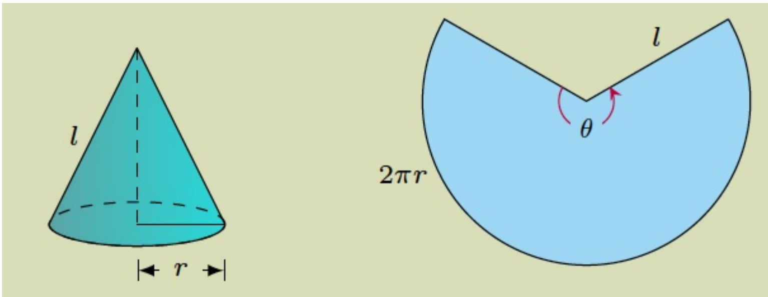
$$S = \lim_{n\to\infty} \sum_{i=1}^{n} S_i,$$

where S_i is the area of the band with slant height $|Q_{i-1}Q_i|$.

Let's look at two examples here.

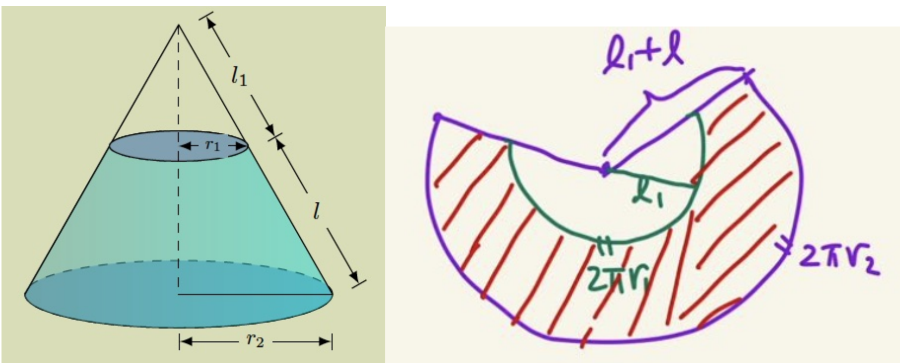
Example. Find the area of a circular cone with base radius r and slant height ℓ.

solution. We can flat the cone to form a sector with the area A of a circle with radius ℓ and central angle θ. Thus, we have



$$\frac{\theta}{2\pi} = \frac{2\pi r}{2\pi \ell} \implies \theta = \frac{2\pi r}{\ell}$$
$$\frac{A}{\pi \ell^2} = \frac{\theta}{2\pi} \implies A = \pi r \ell = \frac{1}{2}\ell^2 \cdot \left(\frac{2\pi r}{\ell}\right) = \frac{1}{2}\ell^2\theta.$$

Example. Find the area A of the band, or frustum of a cone, with slant height ℓ and upper and lower radii r_1 and r_2.



solution. We have

$$A = A_1 - A_2 = \pi r_2(\ell + \ell_1) - \pi r_1 \ell_1 = \pi\left[(r_2 - r_1)\ell_1 + r_2\ell\right].$$

From similar triangles we have

$$\frac{\ell_1}{r_1} = \frac{\ell_1 + \ell}{r_2},$$

which gives $r_2\ell_1 = r_1(\ell_1 + \ell) \implies (r_2 - r_1)\ell_1 = r_1\ell$.

Thus, we have

$$A = \pi\left[r_1\ell + r_2\ell\right] = 2\pi r\ell = \text{(average circumference)} \times \text{(slant height)},$$

where the average radius of the band is $r = \frac{1}{2}(r_1 + r_2)$.
Formula of the surface area

- Assume that f' is continuous on $[a, b]$, For a band with slant height $\ell = |P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(f(x_{i-1}) + f(x_i))$. The surface area is

$$S_i = 2\pi r\ell = 2\pi \cdot \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right] \cdot |P_{i-1}P_i|.$$

By the Pythagorean Theorem and the Mean Value Theorem, we have

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \\ &= \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} = \sqrt{1 + [f'(x_i^*)]^2}\Delta x, \end{aligned}$$

where taking x_i^*, such that $f(x_i^*) = \frac{1}{2}[f(x_{i-1}) + f(x_i)]$, where $x_i^* \in [x_{i-1}, x_i]$. we have

$$S_i = 2\pi \cdot f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Thus,

$$\begin{aligned} S &= \lim_{n\to\infty}\sum_{i=1}^{n} S_i = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi \cdot f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x \\ &= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi f(x)\, ds, \end{aligned}$$

where $ds = \sqrt{1 + [f'(x)]^2}\, dx$.

- Assume that g' is continuous on $[c, d]$, For a band with slant height $\ell = |Q_{i-1}Q_i|$ and average radius $r = \frac{1}{2}(g(y_{i-1}) + g(y_i))$. The surface area is

$$S_i = 2\pi r\ell = 2\pi \cdot \frac{1}{2}\left[g(y_{i-1}) + g(y_i)\right] \cdot |Q_{i-1}Q_i|.$$

By the Pythagorean Theorem and the Mean Value Theorem, we have

$$\begin{aligned} |Q_{i-1}Q_i| &= \sqrt{(y_i - y_{i-1})^2 + [g(y_i) - g(y_{i-1})]^2} \\ &= \sqrt{(\Delta y)^2 + [g'(y_i^*)\Delta y]^2} = \sqrt{1 + [g'(y_i^*)]^2}\Delta y. \end{aligned}$$

where taking y_i^*, such that $g(y_i*) = \frac{1}{2}[g(y_{i-1}) + g(y_i)]$, where $y_i^* \in [y_{i-1}, y_i]$. we have

$$S_i = 2\pi \cdot g(y_i^*)\sqrt{1 + [g'(y_i^*)]^2}\Delta y.$$

Thus,

$$\begin{aligned} S &= \lim_{n\to\infty}\sum_{i=1}^{n} S_i = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi \cdot g(y_i^*)\sqrt{1 + [g'(y_i^*)]^2}\Delta y \\ &= \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy = \int_c^d 2\pi g(y)\, ds, \end{aligned}$$

where $ds = \sqrt{1 + [g'(y)]^2}\, dy$.

Theorem (general formula for surface area of revolution)

- For rotations of $y = f(x)$ over $[a, b]$ about the x-axis, the surface area is given by

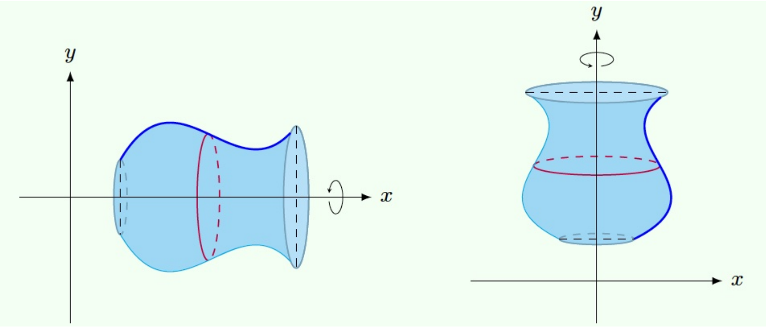
$$S = \int_a^b 2\pi y \, ds,$$

where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$.

- For rotations of $x = g(y)$ over $[c, d]$ about the y-axis, the surface area is given by

$$S = \int_c^d 2\pi x \, ds,$$

where $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$.



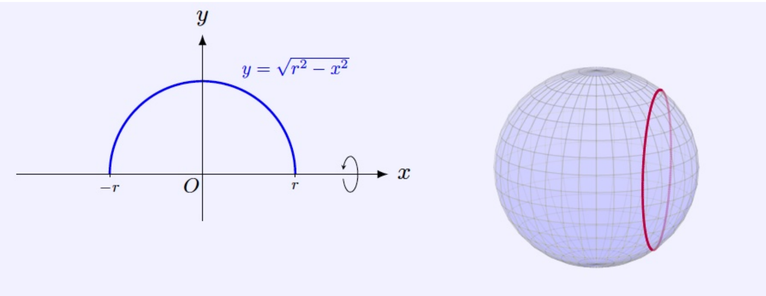
Rk. In general, the surface area of rotation is

$$S = \int (2\pi r) \cdot (ds) = \int (\text{circumference}) \cdot (\text{arc length}).$$

Example (surface area of a sphere). A sphere can be generated by rotating the semicircle

$$y = \sqrt{r^2 - x^2} \geq 0, \quad -r \leq x \leq r.$$

about the x-axis, as shown below. Find its surface area.



solution. Since that

$$\frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}},$$

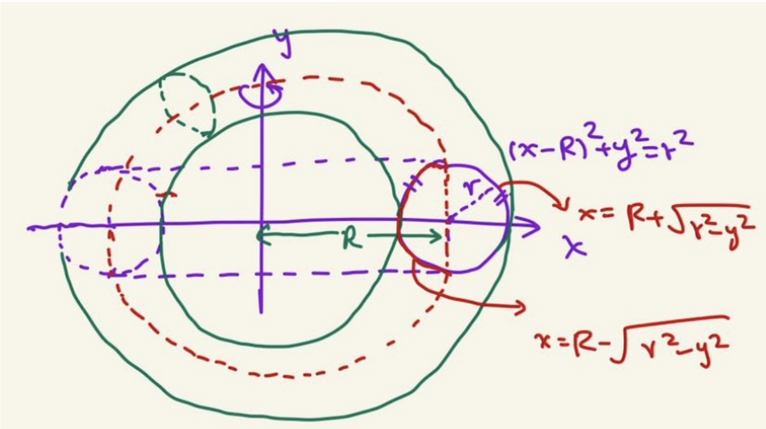
Thus, the surface area of the sphere is

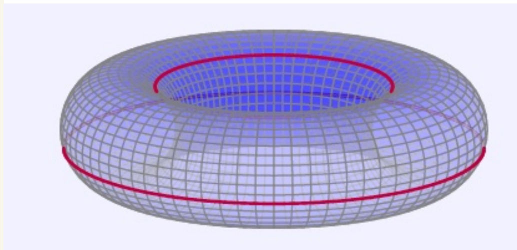
$$\begin{aligned} S &= \int_{-r}^{r} 2\pi f(x)\sqrt{1+[f']^2}\,dx \\ &= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\left[-\frac{x}{\sqrt{r^2-x^2}}\right]^2}\,dx \\ &= 2\pi\int_{-r}^{r} r\,dx = 4\pi r^2. \end{aligned}$$

Example (surface area of a torus). A torus can be generated by rotating the circle

$$(x-R)^2+y^2=r^2, \quad 0<r<R, \quad x\geq 0$$

about y-axis, as shown below. Find the surface area of this torus.





solution. The torus consists of two portions: the outer portion generated by the rightmost semicircle $x=R+\sqrt{r^2-y^2}$, $y\in[-r,r]$ and the inner portion by the leftmost semicircle $x=R-\sqrt{r^2-y^2}$, $y\in[-r,r]$

- along the rightmost semicircle:

$$\frac{dx}{dy}=-\frac{y}{\sqrt{r^2-y^2}},$$

- along the leftmost semicircle :

$$\frac{dx}{dy}=\frac{y}{\sqrt{r^2-y^2}}$$

Thus, the surface area of the torus is

$$\begin{aligned} S = S_1+S_2 &= \int_{-r}^{r} 2\pi\left(R+\sqrt{r^2-y^2}\right)\sqrt{1+\left(-\frac{y}{\sqrt{r^2-y^2}}\right)^2}\,dy \\ &+ \int_{-r}^{r} 2\pi\left(R-\sqrt{r^2-y^2}\right)\sqrt{1+\left(\frac{y}{\sqrt{r^2-y^2}}\right)^2}\,dy \\ &= 4\pi R\int_{-r}^{r}\frac{r}{\sqrt{r^2-y^2}}\,dy = 8\pi R\int_{0}^{r}\frac{r}{\sqrt{r^2-y^2}}\,dy = 8\pi Rr\int_0^{\frac{\pi}{2}}\frac{r\cos\theta}{r\cos\theta}\,d\theta \\ &= 8\pi Rr\cdot\frac{\pi}{2} = 4\pi^2 Rr. \end{aligned}$$

where the property of even function and the substitution $y = r\sin\theta$ have been used.
Exercise. Find the surface area of the rotation of the curve $y = e^x$ on $x \in [0, \ln 4]$ about x-axis or y-axis.
solution.

- For the rotation about x-axis, the area can be given by

$$S = \int_0^{\ln 4} 2\pi y \, ds = \int_0^{\ln 4} 2\pi e^x \sqrt{1 + (e^x)^2} \, dx.$$

  This area can be also written below, ($x = \ln y \Longrightarrow \frac{dx}{dy} = \frac{1}{y}$)

$$S = \int_1^4 2\pi y \, ds = \int_1^4 2\pi y \sqrt{1 + \left(\frac{1}{y}\right)^2} \, dy,$$

  by usage of the same arc length of ds w.r.t different variables.

- For the rotation about y-axis, the area can be given by

$$S = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_1^4 2\pi \ln y \sqrt{1 + \left(\frac{1}{y}\right)^2},$$

  or as an x-integral by

$$S = \int_0^{\ln 4} 2\pi x \, ds = \int_0^{\ln 4} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^{\ln 4} 2\pi x \sqrt{1 + (e^x)^2} \, dx.$$

Rk. Go wolframAlpha to see some"strange function" as anti-derivatives.
Rk.

- For general curve $y = f(x)$ over $[a, b]$ passing the x-axis, note that $y > 0$ gives $2\pi y \, ds > 0$; $y < 0$ gives $2\pi y \, ds < 0$. Rotating this curve about x-axis. However the surface is positive. Thus, the area is

$$S = \int_a^b 2\pi |f(x)| \sqrt{1 + [f'(x)]^2} \, dx.$$

- For general curve $x = g(y)$ over $[c, d]$ passing the y-axis, note that $x > 0$ gives $2\pi x \, ds > 0$; $x < 0$ gives $2\pi x \, ds < 0$. Rotating this curve about y-axis. However the surface is positive. Thus, the area is

$$S = \int_c^d 2\pi |g(y)| \sqrt{1 + [g'(y)]^2} \, dy.$$

- For the curve $y = f(x) > k > 0$ over $[a, b]$, rotating this curve about the line $y = k$. The area is

$$S = \int_a^b 2\pi \left[f(x) - k\right] \sqrt{1 + [f'(x)]^2} dx.$$

  Or the curve $0 < y = f(x) < k$ over $[a, b]$, rotating this curve about the line $y = k$. The area is

$$S = \int_a^b 2\pi \left[k - f(x)\right] \sqrt{1 + [f'(x)]^2} dx.$$

  Or the general curve $y = f(x)$ over $[a, b]$, rotating this curve about the line $y = k$. The area is

$$S = \int_a^b 2\pi \left|f(x) - k\right| \sqrt{1 + [f'(x)]^2} dx.$$

- For the curve $x = g(y) > k > 0$ over $[c, d]$, rotating this curve about the line $x = k$. The area is

$$S = \int_c^d 2\pi \left[g(y) - k\right] \sqrt{1 + [g'(y)]^2} dy.$$

Other cases are similar to derive. For the curve $y = f(x) < 0$ over $[a, b]$, rotating this curve about the x-axis. The area is

$$S = \int_a^b 2\pi |f(x)| \sqrt{1 + [f'(x)]^2}\, dx,$$

Theorem (surface area of revolution by parametric curve) Assume that a parametric curve is given by the parametric equations

$$x = f(t)$$
$$y = g(t),$$

where $\alpha \le t \le \beta$, where f' and g' are continuous and $g(t) \ge 0$. Then the surface area of revolution about the x-axis is

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta 2\pi g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Similarly, the surface area of revolution about the y-axis is

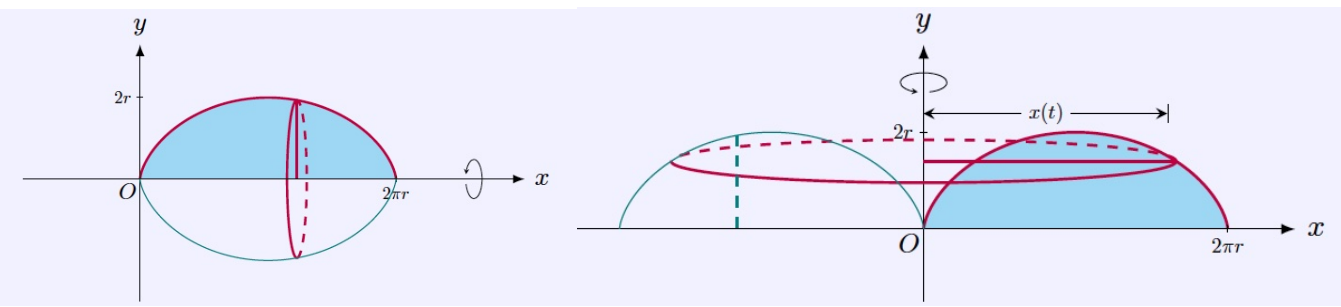
$$S = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta 2\pi f(t) \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example.
Rotate the area under one arch of the cycloid

$$x = r(t - \sin t),$$
$$y = r(1 - \cos t),$$

where $t \in [0, 2\pi]$. about the x-axis and the y-axis, respectively. Find the surface areas of the two solids.
solution.
The surface area of revolution is

$$S = \int (2\pi r) \cdot (ds) = \int (\text{circumference}) \cdot (\text{arc length})$$



1. Rotation about x-axis, a typical circle has radius $y = y(t)$, while the arc length is given by

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

Thus, the surface area of the solid is

$$\begin{aligned}
S &= \int_0^{2\pi} 2\pi y(t) \cdot \sqrt{x'(t)^2 + y'(t)^2}\ dt \\
&= 2\pi r^2 \int_0^{2\pi} (1 - \cos t)\sqrt{(1 - \cos t)^2 + \sin^2 t}\ dt \\
&= 2\pi r^2 \int_0^{2\pi} (1 - \cos t) \cdot \sqrt{2 - 2\cos t}\ dt \\
&= 2\pi r^2 \int_0^{2\pi} 2\sin^2\left(\frac{1}{2}t\right) \cdot 2\sin\left(\frac{1}{2}t\right)\ dt \\
&= 8\pi r^2 \int_0^{\pi} (1 - \cos^2 u)\sin u \cdot 2\ du = 16\pi r^2 \left[-\cos u + \frac{1}{3}\cos^3 u\right] |_0^\pi \\
&= 16\pi r^2 \cdot \frac{4}{3} = \frac{64}{3}\pi r^2,
\end{aligned}$$

where the half-angle formula and the substitution $t = 2u$ have been used.

2. Rotation about y-axis, a typical circle has radius $x = x(t)$, while the arc length is given by

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

Thus, the surface area of the solid is

$$\begin{aligned}
S &= \int_0^{2\pi} 2\pi x(t) \cdot \sqrt{x'(t)^2 + y'(t)^2}\ dt \\
&= 2\pi r^2 \int_0^{2\pi} (t - \sin t)\sqrt{(1 - \cos t)^2 + \sin^2 t}\ dt \\
&= 2\pi r^2 \int_0^{2\pi} (t - \sin t) \cdot \sqrt{2 - 2\cos t}\ dt \\
&= 2\pi r^2 \int_0^{2\pi} (t - \sin t) \cdot 2\sin\left(\frac{1}{2}t\right)\ dt \\
&= 4\pi r^2 \int_0^{\pi} (2u - \sin 2u)\sin u \cdot 2\ du \\
&= 8\pi r^2 \int_0^{\pi} (2u\sin u - \sin 2u \sin u)\ du \\
&= 8\pi r^2 \left[-2u\cos u + \frac{3}{2}\sin u + \frac{1}{6}\sin 3u\right] |_0^\pi = 8\pi r^2 \cdot 2\pi = 16\pi^2 r^2,
\end{aligned}$$

where the half-angle formula and the substitution $t = 2u$ have been used.