Lecture 12 (Applications of integration)–Polar coordinates and calculus

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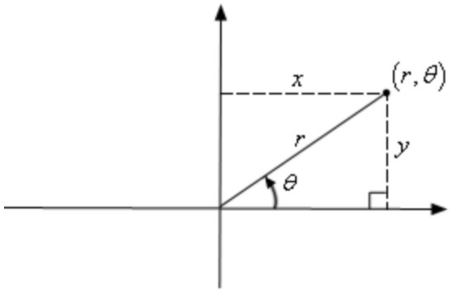
1 Basic concepts

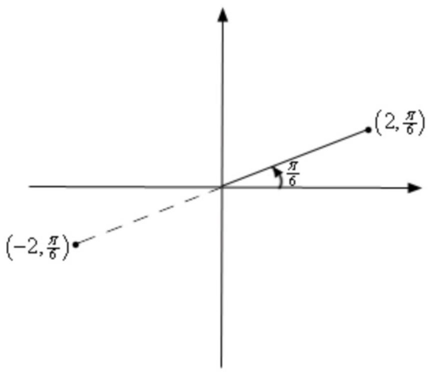
Polar coordinates We could use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point.
Polar to Cartesian conversion formulas The relation between Cartesian (or Rectangular) coordinates (x, y) and the polar coordinates (r, θ) is given by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$, which is shown below.



Example. Convert the polar coordinate $(2, \frac{\pi}{4})$ into rectangular coordinate.
solution. Note that $r = 2$ and $\theta = \frac{\pi}{4}$, we have $x = r\cos\theta = 2\cos\frac{\pi}{4} = \sqrt{2}$ and $y = r\sin\theta = 2\sin\frac{\pi}{4} = \sqrt{2}$. The rectangular coordinate is $(\sqrt{2}, \sqrt{2})$.
Rk. The above discussion may lead one to think that r must be a positive number. However, we also allow r to be negative. Below is a sketch of the two points $(2, \frac{\pi}{6})$ and $(-2, \frac{\pi}{6})$. We have

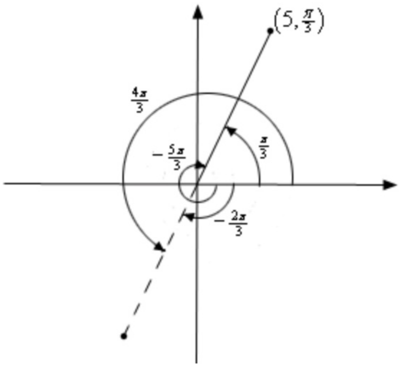


$$\left(-2, \frac{\pi}{6}\right) = \left(2, \frac{7\pi}{6}\right)$$

This leads to an important difference between Cartesian coordinates and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.

$$\left(5, \frac{\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right).$$

Here is a sketch of the angles used in these four sets of coordinates. In general, the point (r, θ) can



be represented by any of the following coordinate pairs.

$$(r, \theta) = (r, \theta + 2n\pi) = (-r, \theta + (2n+1)\pi), \quad \text{where } n \text{ is any integer.}$$

Example (from classviva.org). The Cartesian coordinates of a point are $(-1, -\sqrt{3})$. (a) Find polar coordinates (r, θ) of this point, where $r > 0$ and $0 \le \theta < 2\pi$; (b) Find polar coordinates (r, θ) of this point, where $r < 0$ and $0 \le \theta < 2\pi$.
solution. Note for (a)

$$r = \sqrt{x^2 + y^2} = 2, \quad \cos\theta = -\frac{1}{2}, \quad \sin\theta = -\frac{\sqrt{3}}{2} \Longrightarrow \theta = \frac{4\pi}{3}.$$

Note for (b),

$$r = -2, \quad \theta = \frac{\pi}{3}.$$

2 Tangents with polar coordinates

Taking $r = f(\theta)$, we have

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta. \end{cases}$$

Thus, by chain rule, we have

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta = \frac{dr}{d\theta}\sin\theta + r\cos\theta. \end{aligned}$$

The derivative of $\frac{dy}{dx}$ is then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta},$$

where $\frac{dx}{d\theta} \neq 0$.
Rk. The second derivative is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}},$$

which can be applied to determine that parametric equations is concave up and concave down.
Tangent line Recall that the tangent line to $y = F(x)$ at $x = a$ which is given by

$$y = F(a) + F'(a)(x - a).$$

Example. Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$.
solution Note that $\frac{dr}{d\theta} = 8\cos\theta$, thus we have

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3 + 8\sin\theta)\cos\theta}{8\cos^2\theta - (3 + 8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}.$$

The slope of the tangent line is,

$$\frac{dy}{dx}|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}.$$

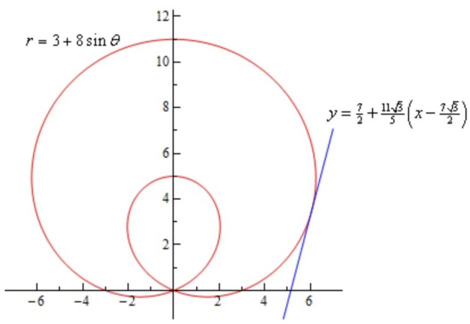
When $\theta = \frac{\pi}{6}$, we have $r = 7$. The line passes $(7, \frac{\pi}{6})$ with corresponding $x - y$ coordinate below,

$$x = 7\cos\frac{\pi}{6} = \frac{7\sqrt{3}}{2}, \quad y = 7\sin\frac{\pi}{6} = \frac{7}{2}.$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right).$$

For the sake of completeness here is a graph of the curve and the tangent line.

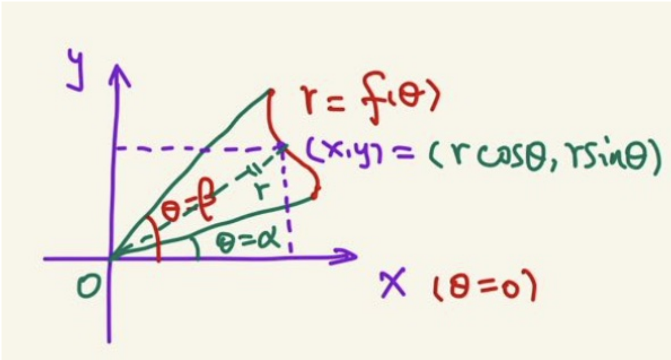


Exercise. Find the slope of the tangent to the curve $r = 7 + 8\cos\theta$ at $\theta = \frac{\pi}{2}$.

3 Arc length of polar curve

The curve in the polar coordinates is given by $r = f(\theta)$ shown below. We write its parametric equations as

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta. \end{cases}$$



Note that

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left[f'(\theta)\cos\theta-f(\theta)\sin\theta\right]^2+\left[f'(\theta)\sin\theta+f(\theta)\cos\theta\right]^2=\left[f(\theta)\right]^2+\left[f'(\theta)\right]^2,$$

the arc length on $\theta\in[\alpha,\beta]$ is

$$\mathcal{L}=\int_\alpha^\beta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_\alpha^\beta\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta.$$

The fact is given below.

Theorem (arc length of polar curve).

Suppose a polar curve is given by the equation $r=f(\theta)$. If f' is continuous, then the arc length of the polar curve from $\theta=\alpha$ to $\theta=\beta$ is

$$\mathcal{L}=\int_\alpha^\beta\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int_\alpha^\beta ds,$$

where $ds=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$.

Example. Find the arc length of the cardioid

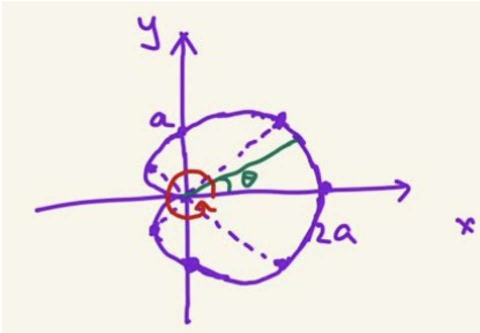
$$r(\theta)=a+a\cos\theta,\quad a>0.$$

solution. Since that

$$r(\theta+2\pi)=r(\theta),$$

thus, $r(\theta)$ is periodic of period 2π. Note that $(\theta,r(\theta))$ passing the points $(0,2a)$, $\left(\frac{\pi}{4},a+\frac{\sqrt{2}}{2}a\right)$, $\left(\frac{\pi}{2},a\right)$, $\left(\frac{3\pi}{4},a-\frac{\sqrt{2}}{2}a\right)$, $(\pi,0)$, $\left(\frac{5\pi}{4},a-\frac{\sqrt{2}}{2}a\right)$, $\left(\frac{3\pi}{2},a\right)$, $\left(\frac{7\pi}{4},a+\frac{\sqrt{2}}{2}a\right)$. The sketch is shown below. Thus the curve $r(\theta)$ go through $\theta\in[0,2\pi]$. Since $r'=-a\sin\theta$, we get the arc length

$$\begin{aligned}\mathcal{L}&=\int_0^{2\pi}\sqrt{(a+a\cos\theta)^2+(-a\sin\theta)^2}\,d\theta\\&=\int_0^{2\pi}a\sqrt{2+2\cos\theta}\,d\theta\\&=2a\int_0^{2\pi}\left|\cos\left(\frac{1}{2}\theta\right)\right|\,d\theta\\&=2a\int_0^{\pi}\cos\left(\frac{1}{2}\theta\right)\,d\theta+2a\int_\pi^{2\pi}(-1)\cos\left(\frac{1}{2}\theta\right)\,d\theta\\&=2a\left[2\sin\left(\frac{1}{2}\theta\right)\right]\Big|_0^\pi-2a\left[2\sin\left(\frac{1}{2}\theta\right)\right]\Big|_\pi^{2\pi}=8a.\end{aligned}$$



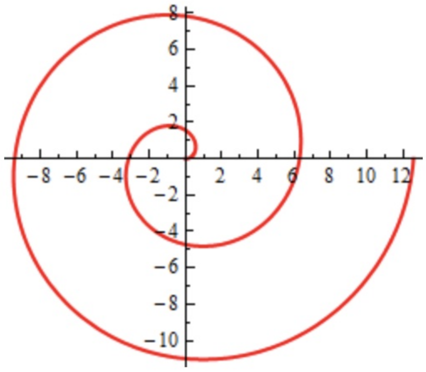
Example. Determine the length of $r = \theta$, where $0 \le \theta \le 1$.
solution. The arc length is

$$\mathcal{L} = \int_0^1 \sqrt{\theta^2 + 1}\, d\theta.$$

By the substitution $\theta = \tan x$, we have $d\theta = \sec^2 x\, dx$, $\theta : 0 \to 1$, $x : 0 \to \frac{\pi}{4}$, and

$$\mathcal{L} = \int_0^{\frac{\pi}{4}} \sec^3 x\, dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right]\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\sqrt{2} + \ln(1+\sqrt{2})\right].$$

Rk. The polar equation $r = \theta$ is the equation of a spiral. Below is a quick sketch of $r = \theta$ for $0 \le \theta \le 4\pi$.



4 Area of polar region

For the polar curve $r = f(\theta)$, we write its parametric equations as

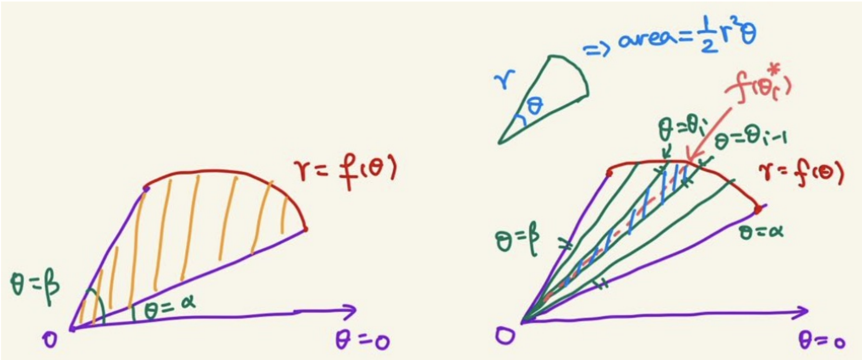
$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

Dividing the interval $[\alpha, \beta]$ into subintervals of equal width $\Delta\theta$ with endpoints $\theta_0, \theta_1, \theta_2, \cdots, \theta_n$. For the subinterval $[\theta_{i-1}, \theta_i]$, the area can be approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$ by

$$\Delta A_i \approx \frac{1}{2} r^2 \theta = \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta.$$

By the definition, the total area is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2\, d\theta.$$



The fact is given below.
Theorem (area of polar region) The area of the region bounded by the polar curve $r = f(\theta)$ and rays $\theta = \alpha$, $\theta = \beta$ $(0 < \beta - \alpha \leq 2\pi)$ is

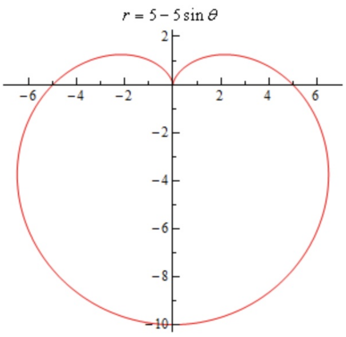
$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 \, d\theta.$$

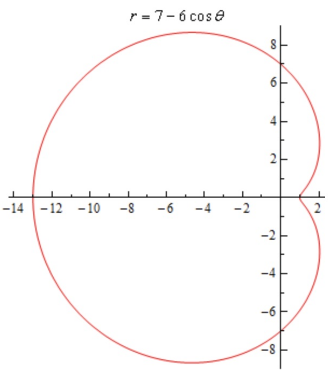
For the region bounded by two polar curves is given by the following fact.
Theorem (area between two polar curves The area of the region bounded by two polar curves $r = f(\theta)$, $r = g(\theta)$ and rays $\theta = \alpha$, $\theta = \beta$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < \beta - \alpha \leq 2\pi$ is

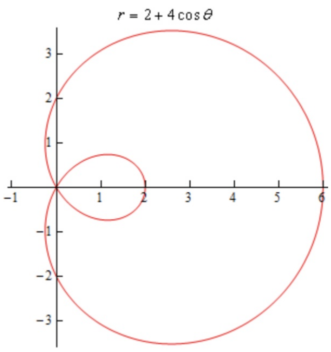
$$A = \frac{1}{2}\int_\alpha^\beta \left([f(\theta)]^2 - [g(\theta)]^2\right) \, d\theta.$$

Example. Graph $r = 5 - 5\sin\theta$, $r = 7 - 6\cos\theta$ and $r = 2 + 4\cos\theta$.
solution.
These will all graph out once in the range $0 \leq \theta \leq 2\pi$. Here is a table below of values for each followed by graphs of each. We have graphs below. For the last graph, if we take $r = 0$, say

θ	$r = 5 - 5\sin\theta$	$r = 7 - 6\cos\theta$	$r = 2 + 4\cos\theta$
0	5	1	6
$\frac{\pi}{2}$	0	7	2
π	5	13	-2
$\frac{3\pi}{2}$	10	7	2
2π	5	1	6







$2 + 4\cos\theta = 0$, we have $\cos\theta = -\frac{1}{2}$. Since $0 \leq \theta \leq 2\pi$, thus $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.
Example. Find the area of the region that lies inside the circle $r = 9\cos\theta$ outside of the cardioid $r = 3 + 3\cos\theta$.

solution. Note that $r = 9\cos\theta$ can be represented by

$$\begin{cases} x = r\cos\theta = 9\cos^2\theta = \dfrac{9}{2}(1+\cos 2\theta) \\ y = r\sin\theta = 9\sin\theta\cos\theta = \dfrac{9}{2}\sin 2\theta, \end{cases}$$

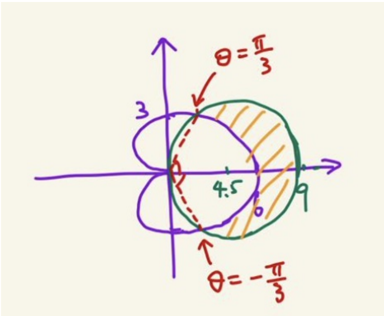
which gives

$$\left(x - \frac{9}{2}\right)^2 + y^2 = \left(\frac{9}{2}\right)^2$$

Note that $(\theta, 3+3\cos\theta)$ passing the points $(0,6)$, $\left(\frac{\pi}{4}, 3+\frac{3\sqrt{2}}{2}\right)$, $\left(\frac{\pi}{2}, 3\right)$, $\left(\frac{3\pi}{4}, 3-\frac{3\sqrt{2}}{2}\right)$, $(\pi, 0)$, $\left(\frac{5\pi}{4}, 3-\frac{3\sqrt{2}}{2}\right)$, $\left(\frac{3\pi}{2}, 3\right)$, $\left(\frac{7\pi}{4}, 3+\frac{3\sqrt{2}}{2}\right)$. To find the intersection points, we take

$$9\cos\theta = 3+3\cos\theta \quad \Longrightarrow \cos\theta = \frac{1}{2} \quad \Longrightarrow \theta = \pm\frac{\pi}{3}.$$

The graph is shown below. Since $r(-\theta) = r(\theta)$, for both polar equations, these curves are symmetric

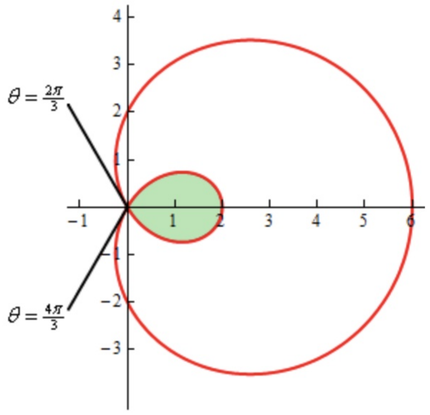


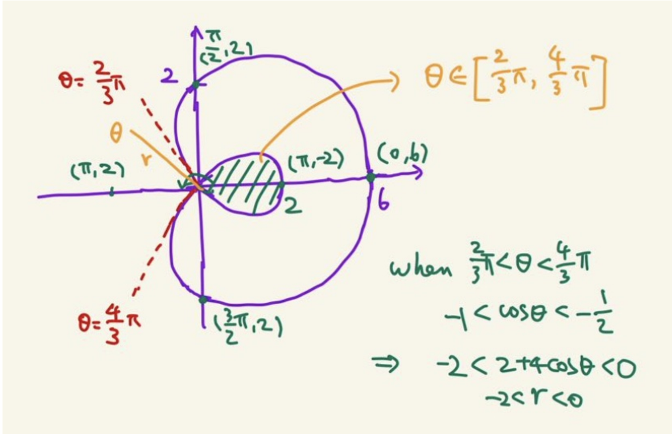
about the polar axis.
Thus, the area of the region that lies inside the circle outside of the cardioid is

$$\begin{aligned} A &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left[(9\cos\theta)^2 - (3+3\cos\theta)^2\right]\, d\theta = 2\int_0^{\frac{\pi}{3}} \frac{1}{2}\left[(9\cos\theta)^2 - (3+3\cos\theta)^2\right]\, d\theta \\ &= \int_0^{\frac{\pi}{3}} \left(-9 - 18\cos\theta + 72\cos^2\theta\right)\, d\theta = \int_0^{\frac{\pi}{3}} \left[-9 - 18\cos\theta + 36(1+\cos 2\theta)\right]\, d\theta \\ &= (27\theta - 18\sin\theta + 18\sin 2\theta)\big|_0^{\frac{\pi}{3}} = 9\pi. \end{aligned}$$

Example. Determine the area of the inner loop of $r = 2 + 4\cos\theta$.
solution.
Take $r = 0$, we have $2 + 4\cos\theta = 0$, thus, $\cos\theta = -\frac{1}{2}$. Since that $0 \le \theta \le 2\pi$, we have $\theta = \frac{2\pi}{3}$, $\frac{4\pi}{3}$. Here is the sketch below of this curve with the inner loop shaded in. Can you see why we needed to know the values of θ where the curve goes through the origin? These points define where the inner loop starts and ends and hence are also the limits of integration in the formula.
Thus, the area is then

$$\begin{aligned} A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(2+4\cos\theta)^2\, d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4 + 16\cos\theta + 16\cos^2\theta\right)\, d\theta \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2 + 8\cos\theta + 4(1+\cos(2\theta)))\, d\theta = (6\theta + 8\sin\theta + 2\sin(2\theta))\big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = 4\pi - 6\sqrt{3} \approx 2.714. \end{aligned}$$



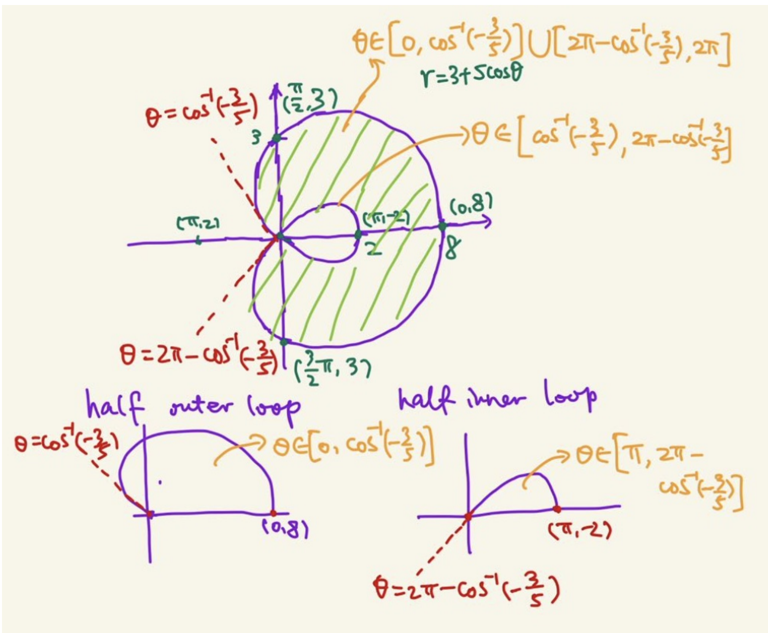


Rk. Method 2: Since that $r(2\pi + \theta) = r(\theta)$ and $r(-\theta) = r(\theta)$, thus the curve is symmetric to the polar axis and periodic of period 2π. Thus the area is double half-upper area below.

$$A = 2\int_{\pi}^{\frac{4\pi}{3}} \frac{1}{2}(2+4\cos\theta)^2 \, d\theta = 2\left[6\theta + 8\sin\theta + 2\sin(2\theta)\right]\Big|_{\pi}^{\frac{4\pi}{3}} = 4\pi - 6\sqrt{3} \approx 2.714.$$

Example. Taking $r = 3 + 5\cos\theta$, consists of an inner loop and an outer loop. Find the area between these loops.

solution. Since that $r(2\pi+\theta) = r(\theta)$ and $r(-\theta) = r(\theta)$, thus the curve symmetric to the polar axis and periodic of period 2π. take $r = 3 + 5\cos\theta = 0$, we have $\theta = \cos^{-1}\left(-\frac{3}{5}\right)$, $2\pi - \cos^{-1}\left(-\frac{3}{5}\right)$.



Note that

- for the upper-half outer loop, $0 \le \theta \le \cos^{-1}(-\frac{3}{5})$, the outer area is

$$A_1 = 2\int_0^{\cos^{-1}(-\frac{3}{5})} \frac{1}{2}\left[3 + 5\cos\theta\right]^2 \, d\theta$$

- for the upper-half inner loop, $\pi \le \theta \le 2\pi - \cos^{-1}(-\frac{3}{5})$, the outer area is

$$A_2 = 2\int_{\pi}^{2\pi - \cos^{-1}(-\frac{3}{5})} \frac{1}{2}\left[3 + 5\cos\theta\right]^2 \, d\theta.$$

Thus, the enclosed area is

$$A = A_1 - A_2 = \int_0^{\cos^{-1}(-\frac{3}{5})} [3 + 5\cos\theta]^2 \; d\theta - \int_\pi^{2\pi - \cos^{-1}(-\frac{3}{5})} [3 + 5\cos\theta]^2 \; d\theta,$$

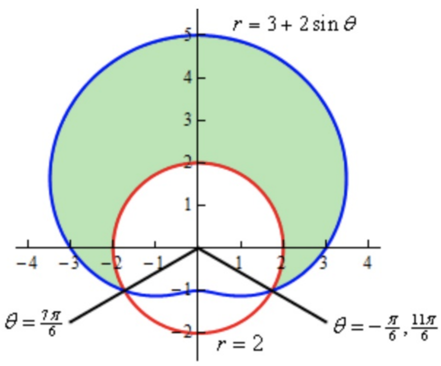
We first compute the indefinite integral below,

$$\begin{aligned} \int (3 + 5\cos\theta)^2 \; d\theta &= \int \left(9 + 30\cos\theta + 25\cos^2\theta\right) \; d\theta \\ &= \int \left[9 + 30\cos\theta + \frac{25}{2}(1 + \cos 2\theta)\right] \; d\theta \\ &= \frac{43}{2}\theta + 30\sin\theta + \frac{25}{4}\sin 2\theta + C. \end{aligned}$$

Hence

$$\begin{aligned} A = A_1 - A_2 &= \left[\frac{43}{2}\theta + 30\sin\theta + \frac{25}{4}\sin 2\theta\right] \Big|_0^{\cos^{-1}(-\frac{3}{5})} - \left[\frac{43}{2}\theta + 30\sin\theta + \frac{25}{4}\sin 2\theta\right] \Big|_\pi^{2\pi - \cos^{-1}(-\frac{3}{5})} \\ &= \left[\frac{43}{2}\cos^{-1}(-\frac{3}{5}) + 30\sin\left(\cos^{-1}(-\frac{3}{5})\right) + \frac{25}{4}\sin 2\left(\cos^{-1}(-\frac{3}{5})\right)\right] \\ &\quad - \left[\frac{43}{2}\left(2\pi - \cos^{-1}(-\frac{3}{5})\right) + 30\sin\left(2\pi - \cos^{-1}(-\frac{3}{5})\right) + \frac{25}{4}\sin 2\left(2\pi - \cos^{-1}(-\frac{3}{5})\right) - \frac{43}{2}\pi\right] \\ &= 43\cos^{-1}\left(-\frac{3}{5}\right) + \frac{25}{2}\sin 2\left(\cos^{-1}\left(-\frac{3}{5}\right)\right) - \frac{43}{2}\pi + 60\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right) \\ &= 43\cos^{-1}\left(-\frac{3}{5}\right) - 12 - \frac{43\pi}{2} + 48 \approx 63.6705. \end{aligned}$$

Example. Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.
solution.
Below is a sketch of the region.



To find the intersection of two curve, we take

$$3 + 2\sin\theta = 2, \quad \Longrightarrow \sin\theta = -\frac{1}{2}, \quad \theta \in [0, 2\pi] \Longrightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

Note that another representation of $\theta = \frac{11\pi}{6}$ is $\theta = -\frac{\pi}{6}$. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So, if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$, we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However, if we use the angles $-\frac{\pi}{6}$ to $\frac{7\pi}{6}$ we will enclose the area.

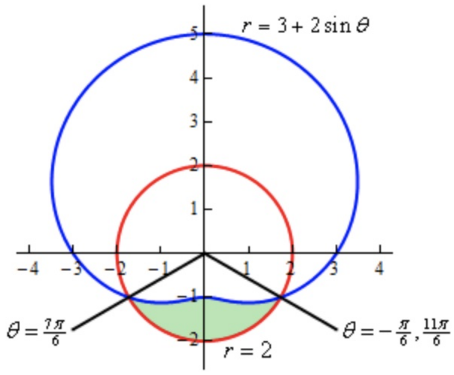
Thus, the area is then,

$$\begin{aligned} A &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left[(3+2\sin\theta)^2 - 2^2\right]\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left[5 + 12\sin\theta + 4\sin^2\theta\right]\, d\theta \\ &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left[7 + 12\sin\theta - 2\cos(2\theta)\right]\, d\theta = \frac{1}{2}\left[7\theta - 12\cos\theta - \sin(2\theta)\right]\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\ &= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} \approx 24.187. \end{aligned}$$

Let's work a slight modification of the previous example.

Example. Determine the area of the region outside $r = 3 + 2\sin\theta$, and inside $r = 2$.

solution. This time we're looking for the following region.

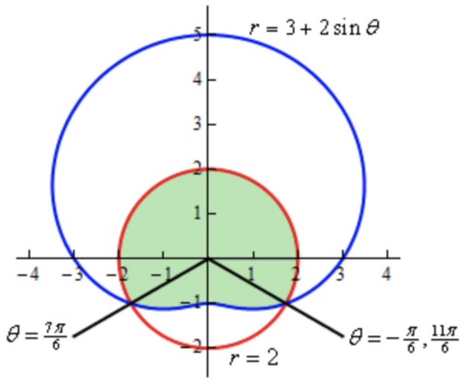


So, this is the region that we get by using the limits $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$. The area for this region is,

$$\begin{aligned} A &= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left[2^2 - (3+2\sin\theta)^2\right]\, d\theta = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left[-5 - 12\sin\theta - 4\sin^2\theta\right]\, d\theta \\ &= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left[-7 - 12\sin\theta + 2\cos 2\theta\right]\, d\theta = \frac{1}{2}\left[-7\theta + 12\cos\theta + \sin 2\theta\right]\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\ &= \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} \approx 2.196. \end{aligned}$$

Example. Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.

solution. Below is the sketch.



- **Method 1:** The area is

$$\begin{aligned} A &= \text{area of cicle} - \texttt{area of example above} \\ &= \pi(2)^2 - 2.196 \approx 10.370. \end{aligned}$$

- **Method 2:** The area is

$$\begin{aligned} A &= \text{area of Limacon} - \text{area of example above} \\ &= \int_0^{2\pi} \frac{1}{2}\left[(3+2\sin\theta)^2\right]\, d\theta - 24.187 \\ &= \frac{1}{2}\left[11\theta - 12\cos\theta - \sin 2\theta\right]\Big|_0^{2\pi} - 24.187 = 11\pi - 24.187 \approx 10.370. \end{aligned}$$

- **Method 3:** The area is

$$A = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}(2)^2\, d\theta + \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}(3+2\sin\theta)^2\, d\theta = \frac{8\pi}{3} + \frac{11\pi}{3} - \frac{11}{2}\sqrt{3} \approx 10.370.$$

Example (from classviva.org). Find the area enclosed by the curve $r^2 = 3\sin 2\theta$.
solution. Note that $r^2 = 3\sin 2\theta \geq 0 \implies 0 \leq 2\theta \leq \pi \implies 0 \leq \theta \leq \frac{\pi}{2}$, which gives $r = \sqrt{3\sin 2\theta}$. The area is

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\cdot 3\sin 2\theta\, d\theta = -\frac{3}{4}\cos 2\theta\Big|_0^{\frac{\pi}{2}} = \frac{3}{2}.$$

5 Surface area with polar coordinates

We want to find the surface area of the region found by rotating, $r = f(\theta)$ for $\theta \in [\alpha, \beta]$ about the x- or y-axis.
We can write the curve in terms of a set of parametric equations below,

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta, \\ y = r\sin\theta = f(\theta)\sin\theta. \end{cases}$$

We use the parametric formula for finding the surface area by

$$\begin{aligned} S &= \int_\alpha^\beta 2\pi y\, ds = \int_\alpha^\beta 2\pi f(\theta)\sin\theta\, ds \quad \text{rotation about } x-\text{axis} \\ S &= \int_\alpha^\beta 2\pi x\, ds = \int_\alpha^\beta 2\pi f(\theta)\cos\theta\, ds \quad \text{rotation about } y-\text{axis}, \end{aligned}$$

where

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$