Lecture 13 (Applications of integration)–Work

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1 Recap last time

One arc length formula and two surface area formulas below.

$$\mathcal{L} = \int ds$$
$$S = \int 2\pi y \, ds, \quad \text{rotation about } x-\text{axis}$$
$$S = \int 2\pi x \, ds, \quad \text{rotation about } y-\text{axis.}$$

Here is a complete listing of all the ds that we've seen and when they are used.

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx, \quad \text{if } y = f(x),\ a \le x \le b,$$
$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy, \quad \text{if } x = g(x),\ c \le y \le d,$$
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt, \quad \text{if } x = f(t),\ y = g(t)\ \alpha \le t \le \beta,$$
$$ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta, \quad \text{if } r = f(\theta),\ \alpha \le \theta \le \beta.$$

2 Average Function Value

In this section we will look at using definite integrals to determine **the average value of a function on an interval.**

Theorem The average value of a continuous function $f(x)$ over the interval $[a, b]$ is given by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

Proof. Dividing $[a, b]$ to n subintervals with equal base length $\Delta x = \frac{b-a}{n}$ and choosing the points $x_1^*, x_2^*, \cdots, x_n^*$ form each subinterval. The average of the function values

$$\frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{n} = \frac{[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)]\,\Delta x}{b-a} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Let's now increase n. Doing this will mean that we're taking the average of more and more function values in the interval and so the larger we chose n the better this will approximate the average value of the function. Thus,

$$f_{avg} = \lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

Corollary (Mean Value Theorem for Integrals). If $f(x)$ is a continuous function on $[a, b]$, then there is a number c in $[a, b]$ such that

$$\int_a^b f(x)\, dx = f(c)(b-a).$$

Example. Determine the average value of each of the following functions on the given interval.

(a) $f(x) = x^2 - 5x + 6\cos(\pi x)$ over $[-1, \frac{5}{2}]$

(b) $f(x) = \sin 2x e^{1-\cos(2x)}$ over $[-\pi, \pi]$

solution.
using the formula below,

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

For (a), we have

$$f_{avg} = \frac{1}{\frac{5}{2}-(-1)}\int_{-1}^{\frac{5}{2}} \left[x^2 - 5x + 6\cos \pi x\right]\, dx = \frac{2}{7}\left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + \frac{6}{\pi}\sin \pi x\right]|_{-1}^{\frac{5}{2}} = \frac{12}{7\pi} - \frac{13}{6}.$$

For (b), we have

$$\begin{aligned} f_{avg} &= \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} \sin 2x e^{1-\cos 2x}\, dx = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{1-\cos 2x}\, d(1-\cos 2x) \\ &= \frac{1}{4\pi} e^{1-\cos 2x}|_{-\pi}^{\pi} = 0. \end{aligned}$$

Example (from classviva.org) If a cup of coffee has temperature $95\ ^\circ C$ in a room where the temperature is $20\ ^\circ C$, then, according to Newton's Law of Cooling, the temperature of the coffee after t minutes is

$$T(t) = 20 + 75e^{-t/50}.$$

What is the average temperature (in degrees Celsius) of the coffee during the first half hour?
solution.

$$\begin{aligned} T_{avg} &= \frac{1}{30-0}\int_0^{30} T(t)\, dt = \frac{1}{30}\int_0^{30}\left[20 + 75e^{-t/50}\right]\, dt = \frac{1}{30}\left[20t - 75\cdot 50e^{-t/50}\right]|_0^{30} \\ &= 20 + 125\left(1 - e^{-\frac{3}{5}}\right) = \approx 76.40\ ^\circ C. \end{aligned}$$

3 Work Done Problems

Suppose that the object moves along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b, a force $F(x)$ acts on the object.
The work done by the force $F(x)$ (assuming that $F(x)$ is continuous) over the range $[a, b]$ is

$$W = \int_a^b F(x)\, dx.$$

Rk. If the force F is a constant, thus we have the work from $x = a$ to $x = b$ is $W = F \cdot x = F(b-a)\ (J)$.

Exercise. Think about a ball with mass m is falling down from the height h to the ground. We have the work $W = \int_0^h mg\, dx = mgh$.
Exercise. Newton's gravitational law: attraction force between two mass is $F = -\frac{GmM}{x^2}$, where G is the gravitational constant. The work for lifting the mass m to a hight h from the surface of the earth with mass M, is

$$W = \int_R^{R+h} F(x)\, dx = \int_R^{R+h} \frac{GmM}{x^2}\, dx = \left[-\frac{GmM}{x}\right]|_R^{R+h} = \frac{GmM}{R} - \frac{GmM}{R+h}.$$

intuition. Dividing $[a, b]$ into n subintervals. The work on each interval is then approximately,

$$W_i \approx F(x_i^*)\Delta x.$$

The total work over $[a, b]$ is then approximately,

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Thus, we have the exact work by

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)\, dx.$$

Example (spring stretching). A spring exerts a force $10\ N$ when stretched $10\ cm$ beyond its natural length. How much work is required to stretch the spring $20\ cm$ beyond its natural length?
solution. According to Hooke's law, the force needed to extend or compress a spring by some distance x meters away from its natural length scales linearly with respect to that distance, say,

$$f(x) = kx,$$

When the spring is stretched $10\ cm = 0.1\ m$ beyond its natural length, the force is $f(0.1) = 0.1k$, By the hypothesis, we have $10 = 0.1k$, then $k = 100$. Thus, $f(x) = 100x$.
Hence the work done in stretching the spring $20\ cm = 0.2\ m$ beyond its natural length is

$$W = \int_0^{0.2} 100x\, dx = \left(50x^2\right)|_0^{0.2} = 2\ J$$

Rk. In the work done problem, we should use the standard units of international system below,
Exercise. A spring has a natural length of $20\ cm$. A $40\ N$ force is required to stretch (and hold the spring) to a length of $30\ cm$. How much work is done in stretching the spring from $35\ cm$ to $38\ cm$?
solution.
Hooke's Law tells us that the force required to stretch a spring a distance of x meters from its natural length is,

$$F(x) = kx,$$

where $k > 0$ is called the spring constant, x in this formula is the distance the spring is stretched from its natural length and not the actual length of the spring.
A force of $40\ N$ is required to stretch the spring

$$30\ cm - 20\ cm = 10\ cm = 0.1\ m.$$

from its natural length. Using Hooke's Law we have, $40 = 0.1k \Longrightarrow k = 400$. Thus, according to Hooke's Law the force required to hold this spring x meters from its natural length is,

$$F(x) = 400x$$

Quantity	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric Current	ampere	A
Thermodynamic Temperature	kelvin	K
Amount of Substance	mole	mol
Area	square meter	m^2
Volume	cubic meter	m^3
Speed, Velocity	meter per second	m/s
Acceleration	meter per second squared	m/s^2
Density	kilogram per cubic meter	kg/m^3
Current Density	ampere per square meter	A/m^2
Frequency	hertz	$Hz = s^{-1}$
Force	newton	$N = kg \cdot m/s^2$
Pressure, Stress	pascal	$Pa = N/m^2$
Energy, Work, Amount of Heat	joule	$J = N \cdot m$
Motive Power, Electrical Power	watt	$W = J/s$

Table 1: International System of Units (SI)

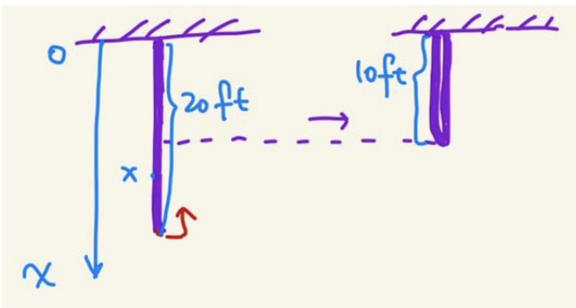
First, we need to convert these into distances from the natural length in meters. Doing that gives us x's of $0.15\ m$ and $0.18\ m$. The work is then,

$$W = \int_{0.15}^{0.18} 400x\ dx = 200x^2|_{0.15}^{0.18} = 1.98\ (J).$$

Example. A $20\ ft$ cable weighs (the force) $80\ lb_f$ and hangs from the ceiling of a building without touching the floor. Determine the work that must be done to lift the bottom end of the chain all the way up until it touches the ceiling.
solution.
First, we need to determine the weight per foot of the cable. This is easy enough to get,

$$\frac{80\ lb_f}{20\ ft} = 4\ lb_f/ft$$

Let x be the distance from the ceiling to any point on the cable.



- when $10 \le x \le 20$, the portion of the cable is lifted;
- when $0 \le x \le 10$, the portion of the cable is not lifted;

So, the force is then,

$$F(x) = (\text{distance lifted}) \times (\text{gravity per foot of cable})$$
$$= 2(x-10)(4) = 8(x-10),$$

The work required is now,

$$W = \int_{10}^{20} 8(x-10)\, dx$$
$$= (4x^2 - 80x)|_{10}^{20} = 400\ (ft * lb_f) \approx 542.33\ (J)$$

Since that $1\ ft = 0.3048\ m$, $1\ lb = 0.45359237\ kg$, the force $1\ lb_f = 0.45359237\ kg \times 9.8\ m/s^2 \approx 4.448\ N$ and $1\ ft * lb_f = 1.3558179483\ J$.

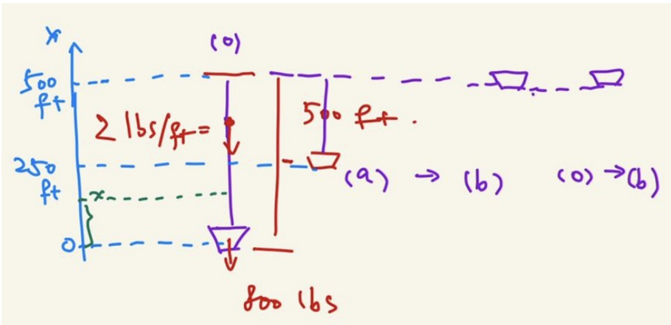
Example. We have a cable that gravity $2\ lb_f/ft$ attached to a bucket filled with coal that gravity $800\ lb_f$. The bucket is initially set at the bottom of a $500\ ft$ mine shaft. Answer each of the following about this.

(a) Determine the amount of work required to lift the bucket to the midpoint of the shaft;

(b) Determine the amount of work required to lift the bucket from the midpoint of the shaft to the top of the shaft;

(c) Determine the amount of work required to lift the bucket all the way up the shaft.

solution. For (a), the work required to move the cable and bucket/coal from $x = 0$ to $x = 250$. The work required is,



$$W = \int_0^{250} F(x)\, dx = \int_0^{250} [500 \times 2 + 800 - 2x]\ dx$$
$$= \left[1800x - x^2\right]|_0^{250} = 387500\ (ft * lb) \approx 525379\ (J).$$

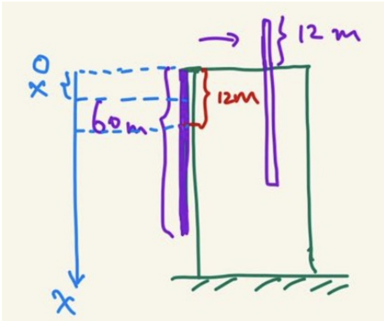
For (b), in this case we want to move the cable and bucket/coal from $x = 250$ to $x = 500$. The work required is,

$$W = \int_{250}^{500} F(x)\, dx = \int_{250}^{500} [1800 - 2x]\ dx = (1800x - x^2)|_{250}^{500} = 262500\ (ft * lb_f) \approx 355902\ (J).$$

For (c), in this case the work is,

$$W = \int_0^{500} F(x)\, dx = \int_0^{500} (1800 - 2x)\, dx = 650000\ (ft * lb_f) \approx 881282\ (J).$$

Exercise. A cable 60 meters long whose mass is 30 kilograms is hanging over the edge of a tall building and does not touch the ground. How much work is required to lift the top $12\ m$ of the cable to the top of the building?



solution.
The total work done consists two parts: (1) W_1 for moving the first top 12 m of the cable to the top of the building; (2) W_2 for moving the bottom $(60-12)=48$ m of the cable 12 m higher.
For W_1, dividing the cable into small segments with length Δx. Taking x_i^* in the i-th segment, then all points in the segment are lifted by approximately the same amount, namely x_i^*. Since the cable weights $\frac{30}{60}=\frac{1}{2}$ kg/m, thus the weight of the i-th segment is $\frac{1}{2}\Delta x$. Thus, the work done on the i-th segment is

$$\left(\frac{1}{2}\Delta x\right)\cdot x_i^*\cdot g=\frac{1}{2}gx_i^*\Delta x.$$

We have the work done for moving the first top 12 m of the cable to the top of building:

$$W_1=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}gx_i^*\Delta x=\int_0^{12}\frac{1}{2}gx\,dx=\left[\frac{1}{4}gx^2\right]\Big|_0^{12}=36g.$$

For W_2, The work done on the i-th segment is

$$\left(\frac{1}{2}\Delta x\right)\cdot 12\cdot g=6g\Delta x.$$

We have the work done for moving the bottom 48 m of the cable to the top of the building:

$$W_2=\lim_{n\to\infty}\sum_{i=1}^{n}6g\Delta x=\int_{12}^{60}6g\,dx=6g\times(60-12)=288g.$$

Thus, the total work done is

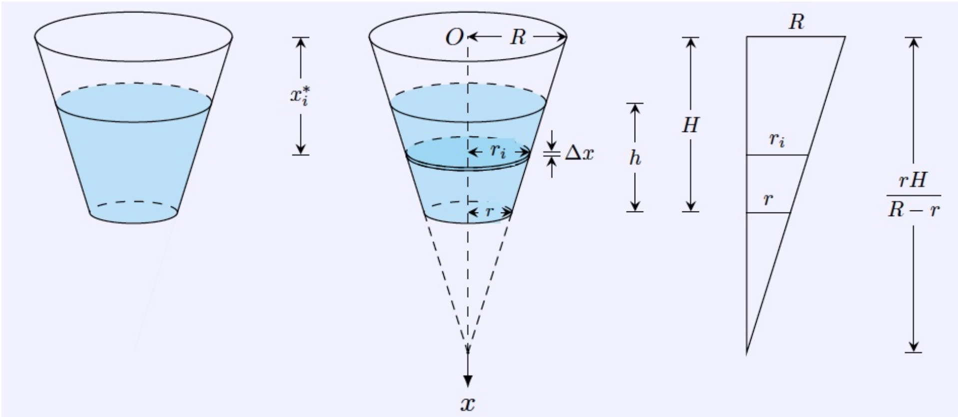
$$W=W_1+W_2=36g+288g=324g\approx 3175.2\ (J).$$

Rk. Provided we can find the force, $F(x)$, for a given problem then using the above method for determining the work is (generally) pretty simple. However, there are some problems where this approach won't easily work. Let's take a look at one of those kinds of problems.
Example (pumping liquid) A frustum of a cone of height H, top radius R and base radius r contains some liquid of density ρ Suppose that the depth of the liquid in the frustum is h. How much work does it take to pump the liquid to the top of the container?
solution. Let x be the distance to the top of the container. Then the liquid extends from $x=H-h$ to $x=H$. We sketch a diagram of the situation.
By similar triangles, the radius $r(x)$ of the disk at depth x satisfies

$$\frac{r(x)-r}{R-r}=\frac{H-x}{H},$$

which gives

$$r(x)=(R-r)\frac{H-x}{H}+r.$$



We divide the interval $[H-h, H]$ into subintervals. Then the thin layer corresponding to $[x_{i-1}, x_i]$ has approximate volume

$$\Delta V_i \approx \pi[r(x_i^*)]^2 \Delta x,$$

and the work needed to pump the layer to the top of the container is approximately

$$\Delta W_i \approx (g\rho\Delta V_i)x_i^* = \pi g\rho x_i^*[r(x_i^*)]^2 \Delta x,$$

where g is the gravitational constant. Therefore the total work needed to pump all the liquid to the top of container is approximately

$$\sum_{i=1}^{n} \Delta W_i = \pi g\rho \sum_{i=1}^{n} x_i^*[r(x_i^*)]^2 \Delta x.$$

Taking the limit as $\Delta x \to 0$, we get the total work

$$\begin{aligned} W &= \pi g\rho \int_{H-h}^{H} x[r(x)]^2\, dx = \frac{\pi g\rho}{H^2} \int_{H-h}^{H} x[(R-r)(H-x)+rH]^2\, dx \\ &= \pi g\rho H^2 R^2 \left[a^2 b + \frac{1}{2}a(2a-3b)b^2 + \frac{1}{3}(1-a)(1-3a)b^3 - \frac{1}{4}(1-a)^2 b^4\right], \end{aligned}$$

where $a = \frac{r}{R}$ and $b = \frac{h}{H}$.

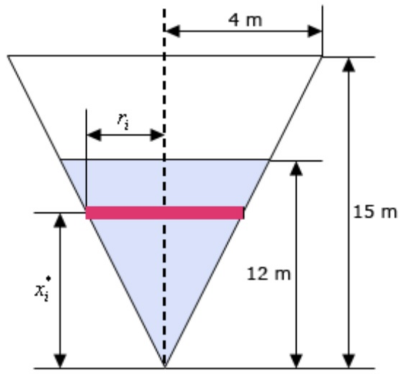
Exercise. A tank in the shape of an inverted cone has a height of $15\ m$ and a base radius of $4\ m$ and is filled with water to a depth of $12\ m$. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of the water is $1000\ kg/m^3$.

solution. In this case we cannot just determine a force function, $F(x)$ that will work for us. So, we are going to need to approach this from a different standpoint. Let's first set $x = 0$ to be the lower end of the tank/cone and $x = 15$ to be the top of the tank/cone. Thus the water in the tank will correspond to the interval $[0, 12]$. So, let's start off by dividing $[0, 12]$ into n subintervals, each of width Δx, Take x_i^* in each subinterval. Now, for each subinterval we will approximate the water in the tank corresponding to that interval as a cylinder of radius r_i and height Δx. The red strip in the sketch represents the "cylinder" of water in the i-th subinterval. A quick application of similar triangles will allow us to relate r_i to x_i^* as follows.

$$\frac{r_i}{x_i^*} = \frac{4}{15}, \quad \Longrightarrow r_i = \frac{4}{15}x_i^*.$$

The mass, m_i of the volume of water V_i for the i-th subinterval is simply,

$$m_i = \text{density} \times V_i,$$



where the volume of cylinder by

$$V_i \approx \pi(\text{radius})^2(\text{height}).$$

In turn,

$$m_i \approx 1000\left(\pi r_i^2 \Delta x\right) = 1000\pi\left(\frac{4}{15}x_i^*\right)^2 \Delta x = \frac{640}{9}\pi(x_i^*)^2\Delta x.$$

To raise this volume of water we need to overcome the force of gravity that is acting on the volume and that is, $F = m_i g$, where $g = 9.8\ m/s^2$ is the gravitational acceleration. The force to raise the volume of water in the i-th subinterval is then approximately,

$$F_i = m_i g \approx 9.8 \times \frac{640}{9}\pi(x_i^*)^2\Delta x.$$

Next, in order to reach to the top of the tank the water in the i-th subinterval will need to travel approximately $15 - x_i^*$ to reach the top of the tank.
Therefore, the work to move the volume of water in the i-th subinterval to the top of the tank, i.e. raise it a distance of $15 - x_i^*$, is then approximately,

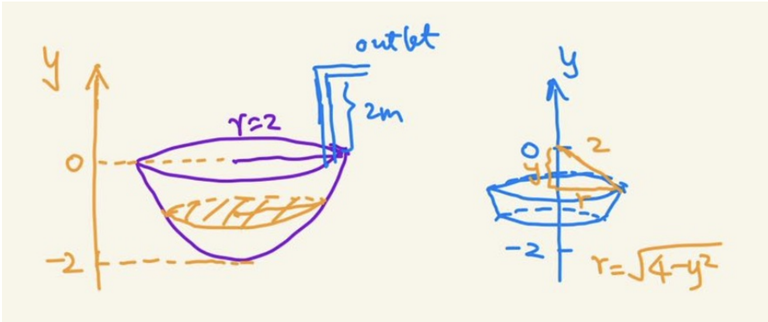
$$W_i \approx F_i(15 - x_i^*) = 9.8 \times \frac{640}{9}\pi(x_i^*)^2(15 - x_i^*)\Delta x.$$

The total amount of work required to raise all the water to the top of the tank is then approximately the sum of each of the W_i,

$$\begin{aligned} W &= \lim_{n\to\infty}\sum_{i=1}^{n} 9.8 \times \frac{640}{9}\pi(x_i^*)^2(15 - x_i^*)\Delta x \\ &= \int_0^{12} 9.8 \times \frac{640}{9}\pi x^2(15 - x)\,dx = 9.8 \times \frac{640}{9}\pi\int_0^{12}\left[15x^2 - x^3\right]\,dx \\ &= 9.8 \times \frac{640}{9}\pi\left(5x^3 - \frac{1}{4}x^4\right)\Big|_0^{12} \approx 7566362.543\ (J) \end{aligned}$$

Example. Find the work required to pump all water to an outlet $2\ m$ above the top of the semi-spherical tankwith radius $r = 2$ which is full of water (with density $\rho = 1000\ kg/m^2$). This is an example with a continuous distribution of mass.
solution. Dividing the interval $[-2, 0]$ into n subintervals, and taking the y_i^* in the each subinterval. Thus, the height to be lifted up is $2 - y_i^*$. The work in a subinterval is

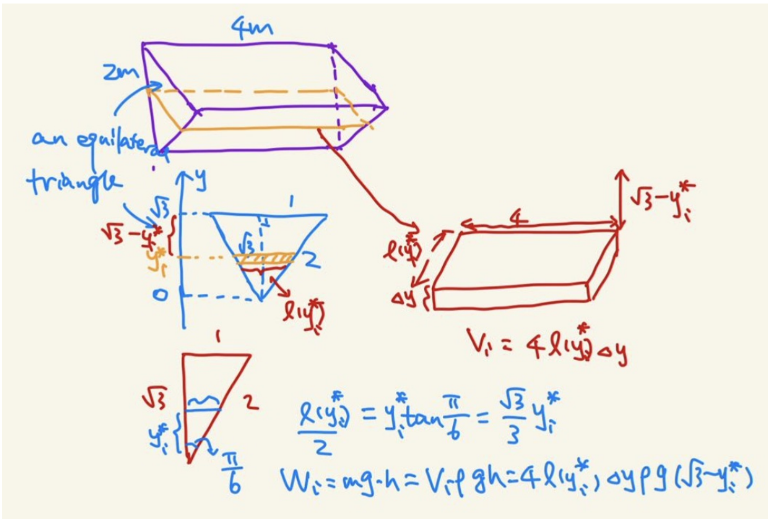
$$W_i = m \cdot g \cdot h = (\text{volume}) \times (\text{density}) \cdot g \cdot h = \pi(4 - (y_i^*)^2)\Delta y\rho \cdot g(2 - y_i^*),$$



where the tiny volume $V_i \approx \pi r^2 \Delta y$. Thus, we have the required work by

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(4-(y_i^*)^2)\Delta y \rho \cdot g(2-y_i^*) = \int_{-2}^{0} \pi \rho g(4-y^2)(2-y)\, dy$$
$$= \pi\rho g \int_{-2}^{0} \left[8-4y-2y^2+y^3\right]\, dy = \pi\rho g \left[8y-2y^2-\frac{2}{3}y^3+\frac{1}{4}y^4\right]|_{-2}^{0} = \frac{44}{3}\pi\rho g\ (J).$$

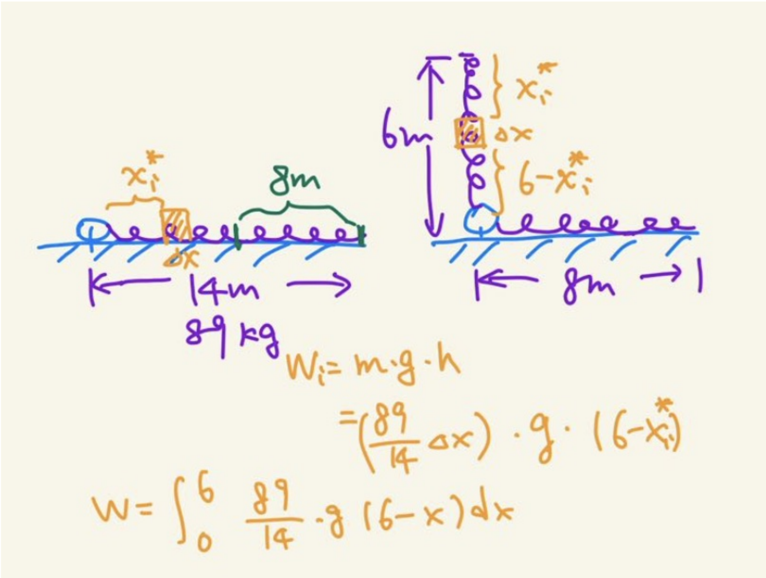
Exercise. The tank is full of water. Find the work required to pump all the water to the top of the tank of the triangular prism shape. The leftmost plane is an equilateral triangle with length $2\ m$. The top length is $4\ m$.
solution. Dividing the interval $[0, \sqrt{3}]$ into n subintervals. Taking y_i^* in each subinterval. The work



is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 4\frac{2\sqrt{3}}{3} y_i^* \Delta y \rho g(\sqrt{3}-y_i^*) = \int_{0}^{\sqrt{3}} 4\frac{2\sqrt{3}}{3} y \cdot \rho g(\sqrt{3}-y)\, dy$$
$$= \frac{8\sqrt{3}}{3}\rho g \int_{0}^{\sqrt{3}} (\sqrt{3}-y)y\, dy = \frac{8\sqrt{3}}{3}\rho g \left[\frac{\sqrt{3}}{2}y^2 - \frac{1}{3}y^3\right]|_{0}^{\sqrt{3}} = 24\rho g\ (J)$$

Example (from classviva.org) A chain lying on the ground is 14 meters long and its mass is 89 kilograms. The chain is threaded through a pulley, which is fixed to the ground, and pulled directly up so that it forms the shape of an L. How much work is required to raise one end of the chain to a height of 6 meters. Use that the acceleration due to gravity is $9.8\ m/s^2$. Assuming that the chain slides effortlessly and without friction along the ground as its end is lifted.
solution



Note that the weight per meters is $\frac{89\ kg}{14\ m}$. Diving $[0, 6]$ into n subintervals, each mass with $\frac{89}{14}\Delta x$. Taking $6 - x_i^*$ in each subinterval, we have

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{89}{14}\Delta x \cdot g \cdot (6 - x_i^*) = \int_0^6 \frac{89}{14} g(6-x)\,dx = \frac{89}{14} g\left(6x - \frac{1}{2}x^2\right)\Big|_0^6 = \frac{801}{7} g = 1121.4\ (J)$$

Rk. In general, if the total length of the chain is denoted by a, the work required to raise one end of the chain with $w\ kg/m$ to a height h, where $h < L$, is given by

$$W = \int_0^h wg(h-x)\,dx = wg(hx - \frac{1}{2}x^2)|_0^h = \frac{1}{2}wgh^2.$$

One can also use

$$W = \int_0^h wgx\,dx = wg\frac{1}{2}x^2|_0^h = \frac{1}{2}wgh^2.$$

Example ("changing mass"). $5\ kg$ of water is picked initially in the basket, but water is leaking at a constant rate when the basket is lifted up to the platform with height $4\ m$. Suppose only $3\ kg$ of water is left in the basket when it reaches the platform. Find the work done (ignore the weight of the rope and basket).

solution. Since that the leaking is taken with a constant rate, we can take $m = m(x) = kx + b$, when $x = 0\ m$, $m = 5\ kg$; when $x = 4\ m$, $m = 3\ kg$. Thus the slope of the line is $\frac{5-3}{0-4} = -\frac{1}{2}$. Thus, $m(x) = -\frac{1}{2}x + 5$. Thus the work is

$$W = \int_0^4 mg\,dx = g\int_0^4\left(-\frac{1}{2}x + 5\right)dx = g\left(-\frac{1}{4}x^2 + 5x\right)|_0^4 = 16g\ (J),$$

where $g = 9.8\ m/s^2$.