Lecture 14 Infinite sequences and series

Instructor: Dr. C.J. Xie (macjxie@ust.hk)

1 Infinite Sequences

Definition. An infinite sequence is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \cdots, a_n, \cdots\}.$$

For each n, the term a_n is called the n-th term of the sequence.
Example.

- $\{1, 2, 3, \cdots, n, \cdots\}$, $a_n = n$;
- $\{1, -1, 1, -1, 1, -1, \cdots\}$, $a_n = (-1)^{n-1}$;
- $\{3, 3.14, 3.141, 3.1415, \cdots\}$, a_n can be thought of the truncation of π to $n-1$ decimal place, but no explicit expression;
- Given $a_n = f(n)$ with $f(n) = \cos(\frac{\pi}{n})$, gives $\{\cos\frac{\pi}{1}, \cos\frac{\pi}{2}, \cos\frac{\pi}{3}, \cdots\}$;
- The sequence

  $$\left\{\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots\right\},$$

  $a_1 = \sqrt{2}$, $a_n = \sqrt{2 + a_{n-1}}$, for $n \geq 2$.

Definition (limit of sequence).

- An infinite sequence $\{a_n\}$ has the **limit** a, which is a finite number, and we write

  $$\lim_{n\to\infty} a_n = a,$$

  or $a_n \to a$ as $n \to \infty$. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

- The infinite limit $\lim_{n\to\infty} a_n = \infty$. In this case, we say the sequence **diverges** to infinity. Similarly, we can definite $\lim_{n\to\infty} a_n = -\infty$.

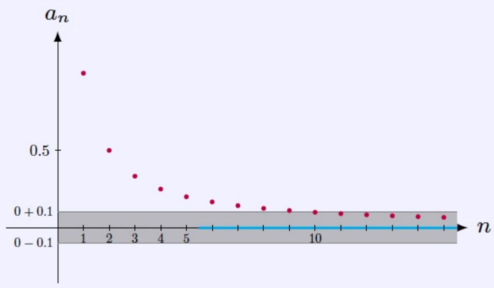
Example.

- The sequence $\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$, $a_n = \frac{1}{n}$. Since that $a_n \to 0$ as $n \to \infty$. We say that the sequence $\{a_n\}$ is convergent to 0. Note that

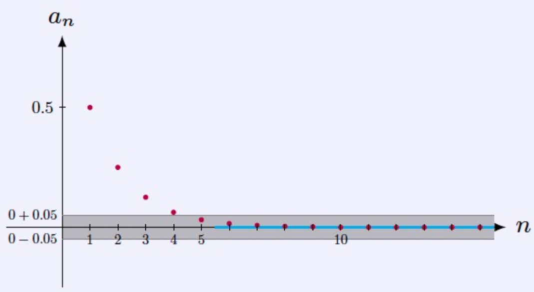
  $$|a_n - 0| < 0.1, \Longrightarrow n > 10.$$

  The variation of a_n is shown below.

n	a_n	n	a_n
1	1.0	11	0.090909
2	0.5	12	0.083333
3	0.333333	13	0.076923
4	0.25	14	0.071429
5	0.2	15	0.066667
6	0.166667	20	0.05
7	0.142857	30	0.033333
8	0.125	40	0.025
9	0.111111	50	0.02
10	0.1	60	0.016667



n	a_n
1	0.5
2	0.25
3	0.125
4	0.0625
5	0.03125
6	0.015625
7	0.0078125
8	0.00390625
9	0.00195313
10	0.000976563



- The sequence $\{\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \cdots, \left(\frac{1}{2}\right)^n, \cdots\}$, $a_n = \left(\frac{1}{2}\right)^n$. Since that $a_n \to 0$ as $n \to \infty$, we say that the sequence $\{a_n\}$ is convergent to 0. Note that

$$|a_n - 0| < 0.05, \Longrightarrow n > 4.$$

  The variation of a_n is shown below.

- The sequence $\{0.9, 0.99, 0.999, 0.9999\}$ converges to 1.

- The sequence

$$\left\{-\frac{1}{3}, \frac{8}{9}, \frac{3}{19}, \cdots, \frac{n^2 + 2 \cdot (-1)^n n}{2n^2 + 1}, \cdots\right\},$$

  $a_n = \frac{n^2+2\cdot(-1)^n}{2n^2+1}$. Since that $a_n \to \frac{1}{2}$ as $n \to \infty$, we that the sequence $\{a_n\}$ converges to $\frac{1}{2}$. Note that

$$|a_n - \frac{1}{2}| < 0.15, \Longrightarrow n > 6.$$

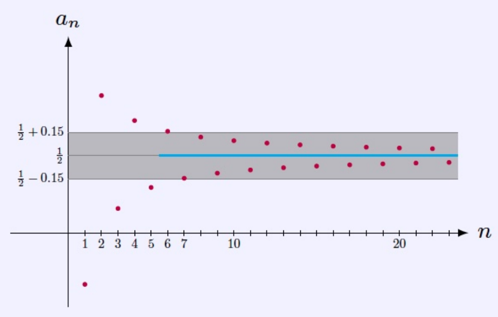
  The variation of a_n is shown below.

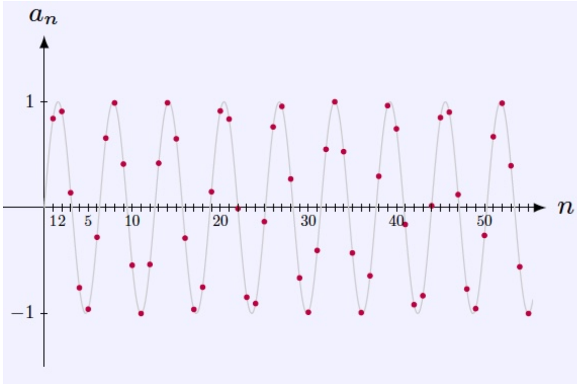
- The sequence $\{\sin 1, \sin 2, \sin 3, \cdots, \sin n, \cdots\}$ diverges, since $a_n = \sin n$ does not appear to approach any finite number as $n \to \infty$. As is shown below.

- The sequence

$$\left\{\frac{1}{\ln 2}, \frac{2}{\ln 3}, \frac{3}{\ln 4}, \cdots, \frac{n}{\ln(n+1)}, \cdots\right\}.$$

n	a_n	n	a_n
1	−0.333333	11	0.407407
2	0.888889	12	0.581315
3	0.157895	13	0.421829
4	0.727273	14	0.569975
5	0.294118	15	0.432373
6	0.657534	20	0.549313
7	0.353535	30	0.533037
8	0.620155	40	0.524836
9	0.386503	60	0.516595
10	0.597015	80	0.512460





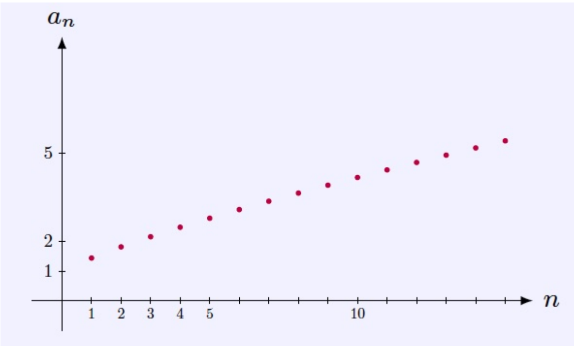
Note that by l'Hospital rule

$$\lim_{x\to\infty}\frac{x}{\ln(x+1)}=\lim_{x\to\infty}\frac{1}{\frac{1}{x+1}}=\lim_{x\to\infty}(x+1)=\infty.$$

Thus, we have

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n}{\ln(n+1)}=\infty.$$

In turn, the sequence is divergent. As is shown below.



Theorem (limit of a sequence from limit of a function). If $\lim_{x\to\infty}f(x)=a$, and $f(n)=a_n$, when n is an integer, then $\lim_{n\to\infty}a_n=a$. The same result holds when $a=\infty$ or $-\infty$.

Theorem (Limit Laws for sequences) Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then,

- $\lim_{n\to\infty}(ca_n\pm b_n)=c\lim_{n\to\infty}a_n\pm\lim_{n\to\infty}b_n$;
- $\lim_{n\to\infty}(a_nb_n)=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}b_n$;

- $\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n}$, if $\lim_{n\to\infty}b_n\neq 0$;
- $\lim_{n\to\infty}a_n^p=[\lim_{n\to\infty}a_n]^p$, if $p>0$ and $a_n>0$.

Example.

- Compute $\lim_{n\to\infty}\frac{1}{n^p}$, $(p>0)$. Since that $\lim_{x\to\infty}\frac{1}{x}=0$, we have $\lim_{n\to\infty}\frac{1}{n}=0$. Thus for $p>0$, by the Limit Laws,

$$\lim_{n\to\infty}\frac{1}{n^p}=\left(\lim_{n\to\infty}\frac{1}{n}\right)^p=0.$$

- Compute

$$\lim_{n\to\infty}\frac{n^{\frac{3}{2}}+2n-1}{n^{\frac{3}{2}}+1}.$$

Since that

$$\frac{n^{\frac{3}{2}}+2n-1}{n^{\frac{3}{2}}+1}=\frac{n^{\frac{3}{2}}\left(1+2\cdot\frac{1}{n^{\frac{1}{2}}}-\frac{1}{n^{\frac{3}{2}}}\right)}{n^{\frac{3}{2}}\left(1+\frac{1}{n^{\frac{3}{2}}}\right)}=\frac{1+2\cdot\frac{1}{n^{\frac{1}{2}}}-\frac{1}{n^{\frac{3}{2}}}}{1+\frac{1}{n^{\frac{3}{2}}}},$$

from the Limit Laws, we have

$$\lim_{n\to\infty}\frac{n^{\frac{3}{2}}+2n-1}{n^{\frac{3}{2}}+1}=\frac{1+2\cdot\lim_{n\to\infty}\frac{1}{n^{\frac{1}{2}}}-\lim_{n\to\infty}\frac{1}{n^{\frac{3}{2}}}}{1+\lim_{n\to\infty}\frac{1}{n^{\frac{3}{2}}}}=1.$$

Theorem (Squeeze Theorem). If $a_n\leq b_n\leq c_n$ for $n\geq n_0$ and $\lim_{n\to\infty}a_n=\lim_{n\to\infty}c_n=L$, then $\lim_{n\to\infty}b_n=L$.

Example. Compute $\lim_{n\to\infty}\frac{\cos n}{\sqrt{n}}$.

solution.

Since that $-1<\cos n<1$ for all n, we have

$$-\frac{1}{\sqrt{n}}\leq\frac{\cos n}{\sqrt{n}}\leq\frac{1}{\sqrt{n}}.$$

Since $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=\lim_{n\to\infty}\left(-\frac{1}{\sqrt{n}}\right)=0$, by the Squeeze Theorem, we get that $\lim_{n\to\infty}\frac{\cos n}{\sqrt{n}}=0$.

Rk.

- $\lim_{n\to\infty}|a_n|=0\Longleftrightarrow\lim_{n\to\infty}a_n=0$;
- If $\lim_{n\to\infty}a_{2n}=\lim_{n\to\infty}a_{2n+1}=a$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty}a_n=a$.

Definition (two asymptotic relations).

- $a_n\ll b_n$, as $n\to\infty$ means that a_n is much smaller than b_n as $n\to\infty$, $\Longrightarrow\lim_{n\to\infty}\frac{a_n}{b_n}=0$;
- $a_n\sim b_n$ as $n\to\infty$ means that a_n is asymptotic to b_n as $n\to\infty$, $\Longrightarrow\lim_{n\to\infty}\frac{a_n}{b_n}=1$.

Rk. For any real numbers $p>0$, $q>0$, $r>0$ and $b>1$, when $n\to\infty$, we have

$$\ln^q n\ll n^p\ll n^p\ln^r n\ll b^n\ll n!=n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 2\cdot 1\ll n^n.$$

Actually, by Stirling's formula, we have

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n,\quad\text{as } n\to\infty.$$

Theorem (geometric sequences). Let r be a real number, then

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } |r| < 1, \\ 1, & \text{if } r = 1, \\ \infty \text{ or } -\infty \text{ (diverges)}, & \text{if } r \le -1 \text{ or } r > 1. \end{cases}$$

Theorem (sequential limit for continuous function). If $\lim_{n\to\infty} a_n = a$ and the function f is continuous at a, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(a).$$

Example. Compute

$$(1) \lim_{n\to\infty} \sqrt[n]{a} \text{ for } a > 0, \quad (2) \lim_{n\to\infty} \sqrt[n]{n}, \quad (3) \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n.$$

solution. For (1), let the substitution $a_n = \ln \sqrt[n]{a} = \frac{\ln a}{n}$. Thus, $\lim_{n\to\infty} a_n = 0$ and $\sqrt[n]{a} = e^{a_n}$,

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} e^{a_n} = e^0 = 1.$$

For (2), let the substitution $a_n = \ln \sqrt[n]{n} = \frac{\ln n}{n}$. Then $\sqrt[n]{n} = e^{a_n}$ Note that $\ln n \ll n$ as $n \to \infty$, so that $\lim_{n\to\infty} a_n = 0$, we have

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{a_n} = e^0 = 1.$$

For (3), let the substitution

$$a_n = \ln\left(1 - \frac{2}{n}\right)^n = n \cdot \ln\left(1 - \frac{2}{n}\right).$$

Then $\left(1 - \frac{2}{n}\right)^n = e^{a_n}$. Note that by l'Hospital's Rule

$$\lim_{x\to 0^+} x^{-1} \cdot \ln(1-2x) = \lim_{x\to 0^+} \frac{\ln(1-2x)}{x} = \lim_{x\to 0^+} \frac{\frac{-2}{1-2x}}{1} = -2,$$

so we have $\lim_{n\to\infty} a_n = -2$. Then

$$\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n = \lim_{n\to\infty} e^{a_n} = e^{-2}.$$

Rk. Terminology for sequences below,

- $\{a_n\}$ is **increasing** if $a_{n+1} > a_n$, e.g., $\{1, 2, 3, \cdots\}$ with $a_n = n$;
- $\{a_n\}$ is **non-decreasing** if $a_{n+1} \ge a_n$, e.g., $\{1, 1, 2, 2, 3, 3, \cdots\}$;
- $\{a_n\}$ is **decreasing** if $a_{n+1} < a_n$, e.g., $\{0, -1, -2, -3, \cdots\}$;
- $\{a_n\}$ is **non-increasing** if $a_{n+1} \le a_n$, e.g., $\{0, 0, -1, -1, -2, -2, \cdots\}$;
- $\{a_n\}$ is **monotonic** if it is either non-increasing or non-decreasing;
- $\{a_n\}$ is **bounded** if there is a number M such that $|a_n| \le M$ for all n.

Rk (product of sequences). If $\{a_n\}$ is bounded and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty}(a_n b_n) = 0$.
Theorem (Monotonic Sequence Theorem). Every **bounded**, **monotonic** sequence is convergent.
Example. Compute the limit of a sequence $\{a_n\}$ with

$$a_n = \frac{1}{2}a_{n-1} + 1, \quad (n \geq 2); \quad a_1 = 1.$$

solution. From the observation, we have

$$a_{n+1} > a_n, \quad 1 \leq a_n < 2,$$

actually, this can be proved by mathematical induction. Since that $\{a_n\}$ is increasing and bounded, by the Monotonic Sequence Theorem, we have $\lim_{n\to\infty} a_n = a$. To find a, note that $a_n = \frac{1}{2}a_{n-1}+1$, we take the limit on both sides as $n \to \infty$, and have

$$a = \frac{1}{2}a + 1, \quad \Longrightarrow a = 2.$$

Example. Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

$$(a)\ \left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty} \quad (b)\ \left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty} \quad (c)\ \left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty} \quad (d)\ \{(-1)^n\}_{n=0}^{\infty}.$$

solution. For (a),

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2} = \lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)} = \lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5} = \frac{3}{5}.$$

For (b), since that by the L'Hospital rule, we have

$$\lim_{x\to\infty}\frac{e^{2x}}{x} = \lim_{x\to\infty}\frac{2e^{2x}}{1} = \infty.$$

Thus, we have

$$\lim_{n\to\infty}\frac{e^{2n}}{n} = \infty,$$

which says the sequence diverges to ∞.
For (c), Since that

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0.$$

Thus,

$$\lim_{n\to\infty}\frac{(-1)^n}{n} = 0.$$

For (d), by the exponential series using $r = -1$, the sequence is divergent.

2 Series

Definition (convergent series). For a given series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n-th partial sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{k=1}^{\infty} a_k$ is called **convergent** and we write

$$a_1 + a_2 + a_3 + \cdots = s \quad \text{or} \quad \sum_{k=1}^{\infty} a_k = s.$$

The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Example. Perform the following index shifts.

(a) Write $\sum_{n=1}^{\infty} ar^{n-1}$ as a series that starts at $n = 0$;

(b) Write $\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}}$ as a series that starts at $n = 3$.

solution.

For (a),

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{(n+1)-1} = \sum_{n=0}^{\infty} ar^n.$$

For (b),

$$\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}} = \sum_{n=3}^{\infty} \frac{(n-2)^2}{1-3^{(n-2)+1}} = \sum_{n=3}^{\infty} \frac{(n-2)^2}{1-3^{n-1}}.$$

Rk. Note that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n.$$

For example,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{4} a_n + \sum_{n=5}^{\infty} a_n.$$

Theorem (geometric series). Let $a \neq 0$ and r be a real number. Then the geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = \begin{cases} \dfrac{a}{1-r}, & \text{if } |r| < 1; \\ \text{diverges}, & \text{if } |r| \geq 1. \end{cases}$$

Proof.

- If $r = 1$, the partial sum $s_n = a + a + \cdots + a = na$, diverges either to ∞ or $-\infty$, as $n \to \infty$;
- If $r \neq 1$, the partial sum is

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1},$$

In turn,

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n,$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n,$$

so that

$$s_n = \frac{a - ar^n}{1-r}.$$

✓ if $-1 < r < 1$, since $r^n \to 0$ as $n \to \infty$, we get

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \frac{a}{1-r};$$

✓ if $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent, so $\{s_n\}$ is divergent as well.

Example. Evaluate the series

$$\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}} \right).$$

solution. The partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}} \right) = \left(\frac{1}{2^1} - \frac{1}{2^2} \right) + \left(\frac{1}{2^2} - \frac{1}{2^3} \right) + \cdots + \left(\frac{1}{2^n} - \frac{1}{2^{n+1}} \right) \\ &= \frac{1}{2} - \frac{1}{2^{n+1}}. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{2^{n+1}} \right) = \frac{1}{2}.$$

Hence,

$$\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}} \right) = \frac{1}{2}.$$

Example. Evaluate the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

solution. Note that the partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right) = 1.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

Exercise. Determine if the following series converges or diverges. If it converges determine its sum.

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}. \quad (\text{ hint: convergent, } \frac{3}{4})$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

Theorem (properties of convergent series).

- If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then

$$\sum_{k=1}^{\infty}(ca_k \pm b_k) = c\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

- the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=N}^{\infty} a_k$ either both converge or both diverge, where N is a positive integer. In general, changing a finite number of terms in a convergent series does not change its convergence, although it does change the value of the series.

Rk. Generally, it is not easy to find the sum of a series. However, it is possible to study convergence of a series without knowing its sum.
Theorem (Divergence Test).

- $\sum_{k=1}^{\infty} a_k$ converges $\Longrightarrow \lim_{k\to\infty} a_k = 0$;
- $\lim_{k\to\infty} a_k \neq 0 \Longrightarrow \sum_{k=1}^{\infty} a_k$ diverges .

Since that if $\sum a_k$ converges, we have $\lim_{n\to\infty} s_n = s$ exists, where $s_n = \sum_{k=1}^{n} a_k$. Thus

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}(s_k - s_{k-1}) = \lim_{k\to\infty} s_k - \lim_{k\to\infty} s_{k-1} = s - s = 0.$$

Example. Determine the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{k^2+1},$$

if it is convergent or divergent.
solution. Since, as $k \to \infty$,

$$\left|\frac{(-1)^k k^2}{k^2+1}\right| = \frac{k^2}{k^2+1} = \frac{k^2}{k^2\left(1+\frac{1}{k^2}\right)} = \frac{1}{1+\frac{1}{k^2}} \to 1 \neq 0.$$

In turn, $\lim_{k\to\infty} \frac{(-1)^k k^2}{k^2+1} \neq 0$. Hence, by the Divergence Test, the given series is divergent.
Rk.

- $\lim_{k\to\infty} a_k \neq 0$, then either the limit does not exist, or the limit exists but does not equal to 0;
- If $\lim_{k\to\infty} a_k = 0$, the Divergence Test is inconclusive. In other words, the zero limit of the sequence $\{a_n\}$ is not sufficient for the convergence of the series $\sum a_k$. An example of such kind of series is the so-called harmonic series $\sum \frac{1}{k}$ (diverges), since that the partial sum $\{s_n\}$ does not have a finite limit.

Exercise. Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}, \qquad (\text{ hint: divergent })$$

Since that

$$\lim_{n\to\infty} \frac{4n^2 - n^3}{10 + 2n^3} = -\frac{1}{2} \neq 0.$$