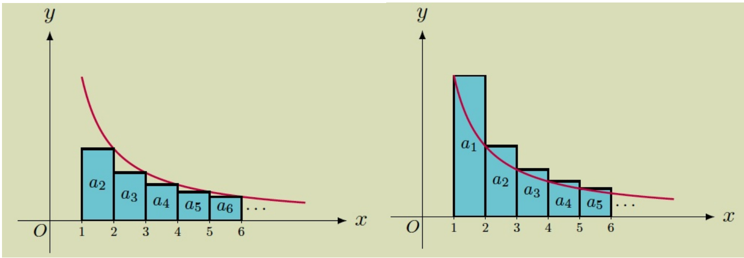
Lecture 15 Integral test

Instructor: Dr. C.J. Xie (macjxie@ust.hk)

1 Integral test

Intuition. Suppose that f is continuous, positive, decreasing function on $[1, \infty)$ and take $a_k = f(k)$,



- if $\int_1^\infty f(x)\ dx$ converges, taking the piecewise constant function $p(x) = f(k+1)$ for $x \in [k, k+1)$, which is defined on $[1, \infty)$. Since that f is decreasing, so $p \le f$. Thus

$$\begin{aligned}\sum_{k=1}^{n} f(k) &= f(1) + \sum_{k=2}^{n} f(k) \\ &\le f(1) + \int_1^n p(x)\ dx \\ f(1) + \int_1^n f(x)\ dx &\le f(1) + \int_1^\infty f(x)\ dx.\end{aligned}$$

  Note that $\int_1^\infty f(x)\ dx$ is convergent. Thus, $s_n = \sum_{k=1}^n f(k)$ is bounded. Since f is positive, thus $s_{n+1} > s_n$, say, $\{s_n\}$ is increasing monotonically. According to the Monotonic Sequence Theorem, s_n as $n \to \infty$ is convergent as well as $\sum_{k=1}^\infty f(k)$.

- if $\int_1^\infty f(x)\ dx$ is divergent. Taking the piecewise constant $q(x) = f(k)$ for $x \in [k, k+1)$, which is defined on $[1, \infty)$. Since that f is decreasing, so $q \ge f$. Thus

$$\int_1^b f(x)\ dx \le \int_1^n f(x)\ dx \le \int_1^n q(x)\ dx = \sum_{k=1}^{n-1} f(k) \le \sum_{k=1}^{\infty} f(k).$$

  Note that $\int_1^\infty f(x)$ is divergent. By contradiction, if $\sum_{k=1}^\infty f(k)$ is convergent, say it is bounded. Thus, $\int_1^b f(x)\ dx$ is also bounded. However, $\int_1^\infty f(x)\ dx$ is divergent (means that $\int_1^b f(x)\ dx$ is unbounded). Thus, $\sum_{k=1}^\infty f(k)$ is divergent.

Exercise. Take $\sum_{n=1}^\infty \frac{1}{n}$ and $\sum_{n=1}^\infty \frac{1}{n^2}$ to see that divergence or convergence.
Theorem Suppose that f is continuous, positive, decreasing function on $[1, \infty)$ and take $a_k = f(k)$, we have

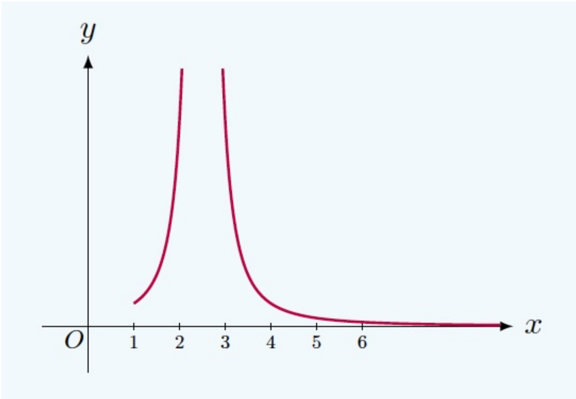
- if $\int_1^\infty f(x)\ dx$ is convergent, then $\sum_{k=1}^\infty a_k$ is convergent;

- if $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{k=1}^\infty a_k$ is divergent.

Rk If f is decreasing for $x \ge N$, we have the same facts, since changing a finite number of terms in a convergent series does not change its convergence.
Example. Determine the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{(2k-5)^2},$$

if it is convergent or divergent.
solution.
We take $f(x) = \frac{1}{(2x-5)^2}$, which is not decreasing on $[1,\infty)$ (also not continuous in this interval). However, f is continuous, positive and decreasing on $[3,\infty)$. Since that $\int_3^\infty \frac{1}{(2x-5)^2}\,dx$ is convergent,



so $\sum_{k=3}^\infty \frac{1}{(2k-5)^2}$ is convergent. Note that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-5)^2} = \sum_{k=1}^{2} \frac{1}{(2k-5)^2} + \sum_{k=3}^{\infty} \frac{1}{(2k-5)^2}.$$

Thus, $\sum_{k=1}^\infty \frac{1}{(2k-5)^2}$ is convergent.
Example (harmonic series). Determine the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k},$$

if it is convergent or divergent.
solution. Take $f(x) = \frac{1}{x}$ and note that f is continuous, positive, decreasing on $[1,\infty)$. Since that

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x|_1^b = \lim_{b\to\infty} \ln b = \infty,$$

which indicates $\int_1^\infty$ is divergent. By the integral test, the series $\sum_{k=1}^\infty \frac{1}{k}$ is divergent.
Example. Determine whether the series $\sum_{k=1}^\infty \frac{k}{k^2+1}$ converges.
solution.
Take $f(x) = \frac{x}{x^2+1}$. It is clear that f is a continuous, positive function on $[1,\infty)$. Note that

$$f'(x) = \left(\frac{x}{x^2+1}\right)' = \frac{1\cdot(x^2+1) - x\cdot 2x}{(x^2+1)^2} = -\frac{x^2-1}{(x^2+1)^2} < 0,$$

thus, f is decreasing. Note that

$$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\frac{1}{2}\ln(x^2+1)|_1^b$$
$$= \lim_{b\to\infty}\frac{1}{2}\left[\ln(b^2+1)-\ln 2\right] = \infty,$$

so the improper integral $\int_1^\infty \frac{x}{x^2+1}\,dx$ is divergent. Hence, by the Integral Test, the series $\sum_{k=1}^\infty \frac{k}{k^2+1}$ is divergent.

Example. Determine whether the series $\sum_{k=1}^\infty \frac{1}{k^2+2}$ converges.

solution.

Take $f(x) = \frac{1}{x^2+2}$. It is clear that f is continuous, positive and decreasing on $[1,\infty)$. Note that

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2+2}\,dx = \lim_{b\to\infty}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}x\right)|_1^b$$
$$= \lim_{b\to\infty}\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{1}{\sqrt{2}}b\right)-\tan^{-1}\frac{1}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{\sqrt{2}}\right),$$

which indicates the improper integral $\int_1^\infty \frac{1}{x^2+1}\,dx$ is convergent. Hence, by the Integral Test, the series $\sum_{k=1}^\infty \frac{1}{k^2+2}$ is convergent.

Example. Determine whether the series $\sum_{k=1}^\infty \frac{1}{\sqrt{2k-1}}$ converges.

solution. Take $f(x) = \frac{1}{\sqrt{2x-1}}$. It is clear that f is continuous, positive, decreasing on $[1,\infty)$. Note that

$$\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty}\sqrt{2x-1}|_1^b = \lim_{b\to\infty}\left(\sqrt{2b-1}-1\right) = \infty,$$

which indicates the improper integral $\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx$ is divergent. Hence, by the Integral Test, the series $\sum_{k=1}^\infty \frac{1}{\sqrt{2k-1}}$ is divergent.

Example. Determine whether the series $\sum_{k=1}^\infty \frac{\ln k}{k}$ converges.

solution. Take $f(x) = \frac{\ln x}{x}$. It is clear that f continuous, positive on $[1,\infty)$. Note that

$$f'(x) = \left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - \ln x\cdot 1}{x^2} = -\frac{\ln x - 1}{(x^2+1)^2}.$$

When $\ln x > 1$, say $x > e \approx 2.718$, so $f'(x) < 0$. Thus, f is decreasing on $(3,\infty)$. Note that

$$\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_3^b \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\frac{1}{2}\ln^2 x|_3^b = \lim_{b\to\infty}\frac{1}{2}\left(\ln^2 b - \ln^2 3\right) = \infty,$$

which indicates that the improper integral $\int_3^\infty \frac{\ln x}{x}\,dx$ is divergent. Thus, by the Integral Test, the series $\sum_{k=3}^\infty \frac{\ln k}{k}$ is divergent as well as $\sum_{k=1}^\infty \frac{\ln k}{k}$.

Exercise. Determine whether the series $\sum_{k=1}^\infty \frac{1}{\sqrt{k^3}}$ converges.

solution. Take $f(x) = \frac{1}{\sqrt{x^3}}$, which is continuous, positive, decreasing on $[1,\infty)$. Since $\int_1^\infty \frac{1}{\sqrt{x^3}}\,dx$ is convergent, thus by the Integral Test, the series $\sum_{k=1}^\infty \frac{1}{\sqrt{k^3}}$ is convergent.

Example. Determine if the following series is convergent or divergent,

$$\sum_{n=2}^\infty \frac{1}{n\ln n}.$$

solution. Taking $f(x) = \frac{1}{x \ln x}$ which is clearly positive, continuous on $[2, \infty)$. If we make x larger, the denominator will get larger and so the function is decreasing. Note that

$$\int_2^\infty \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \ln(\ln(x))|_2^t = \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 2)] = \infty,$$

where the substitution $u = \ln x$ is used. The integral is divergent and so the series is also divergent by the Integral Test.

Example. Determine if the following series is convergent or divergent,

$$\sum_{n=0}^{\infty} ne^{-n^2}.$$

solution. Taking $f(x) = xe^{-x^2}$ which is clearly positive and continuous on $[0, \infty)$. Note that $f'(x) = e^{-x^2}(1 - 2x^2)$. Let $f'(x) = 0$, we have $x = \pm\frac{1}{\sqrt{2}}$. Since that

- $0 \le x \le \frac{1}{\sqrt{2}}$, $f'(x) \ge 0 \Longrightarrow f$ is increasing;
- $x \ge \frac{1}{\sqrt{2}}$, $f'(x) \le 0 \Longrightarrow f$ is decreasing;

which means f is decreasing on $[1, \infty)$. Note that

$$\int_1^\infty xe^{-x^2}\, dx = \lim_{t\to\infty} \int_0^t xe^{-x^2}\, dx = \lim_{t\to\infty} \left(-\frac{1}{2}e^{-x^2}\right)|_0^t = \lim_{t\to\infty} \left(\frac{1}{2} - \frac{1}{2}e^{-t^2}\right) = \frac{1}{2}.$$

The integral is convergent. Note that

$$\sum_{n=0}^{\infty} ne^{-n^2} = \sum_{n=1}^{\infty} ne^{-n^2},$$

so, the series is convergent.

Theorem (convergence of the p-series). The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$.

Rk. In other words, if $k > 0$, then $\sum_{n=k}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$. (This fact follows directly from the Integral Test and a similar fact we saw in the Improper Integral section. This fact says that the integral $\int_k^\infty \frac{1}{x^p}\, dx$ converges for $p > 1$ and diverges for $p \le 1$.)

Example. Determine if the following series are convergent or divergent.

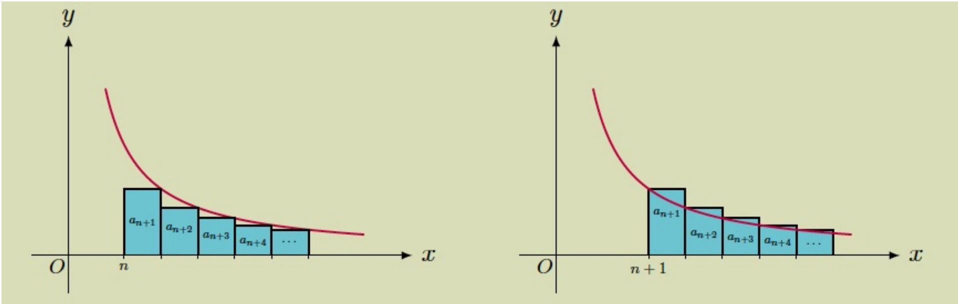
$$(a) \sum_{n=4}^{\infty} \frac{1}{n^7}, \quad (b) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

solution. For (a), take $p = 7 > 1$, so by this fact the series is convergent.
For (b), take $p = \frac{1}{2} \le 1$, so the series is divergent by the fact.

Theorem (remainder estimate for the Integral Test). Suppose f is continuous, positive, decreasing on $[1, \infty)$. Let $a_k = f(k)$ and suppose $\sum_{k=1}^{\infty} a_k = s$ is convergent. Denote $R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k$. Then

- $\int_{n+1}^{\infty} f(x)\, dx \le R_n \le \int_n^\infty f(x)\, dx$;
- $s_n + \int_{n+1}^{\infty} f(x)\, dx \le s \le s_n + \int_n^\infty f(x)\, dx$.

Proof. Since that f is decreasing on $[n, \infty)$, comparing the areas of the rectangles with the area under $y = f(x)$, for $x > n$, we have



$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)\, dx,$$

and similarly,

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^\infty f(x)\, dx.$$

□

Example. For the series

$$\sum_{k=1}^\infty \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

(a) How many terms of the series must be used to obtain an approximation that is within 10^{-4} of the exact value of the series;

(b) Find an approximation to the series using 50 terms of the series.

solution. For (a), taking $f(x) = \frac{1}{x^2}$, which is continuous, positive and decreasing on $[1, \infty)$. Note that

$$R_n < \int_n^\infty f(x)\, dx = \int_n^\infty \frac{1}{x^2}\, dx = \frac{1}{n}.$$

Thus, to ensure that $R_n < 10^{-4}$, we take $\frac{1}{n} < 10^{-4} \implies n > 10000$. Thus, we can take $n = 10001$ at least.
For (b), using the bounds on the series in the Integral Test, we have

$$s_{50} + \int_{51}^\infty \frac{1}{x^2}\, dx < s < s_{50} + \int_{50} \frac{1}{x^2}\, dx,$$

say

$$s_{50} + \frac{1}{51} < s < s_{50} + \frac{1}{50},$$

where $s_{50} = \sum_{k=1}^{50} \frac{1}{k^2}$. By using a calculator, we have $s_{50} \approx 1.62513273$. Thus

$$1.64474057 < s < 1.64513273.$$

Taking the average of these two bounds as our approximation of s, we have $s \approx 1.64493665$. For comparison, $\frac{\pi^2}{6} = 1.64493406\ldots$. We see that our approximation matches first five decimal places.
Rk. In this section for integral test,

- we have used the fact that that a **bounded** and **monotonic** sequence was guaranteed to be **convergent** ($\implies$ the sequence of partial sums is convergent $\implies$ the series must then also be convergent);
- once again we can relate a series to an improper integral, the series and the integral have the same convergence.