Lecture 16 Comparison test

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1 Comparison test

Intuition. Suppose $\sum a_k$ and $\sum b_k$ are series with *nonnegative* terms.

- Suppose $\sum b_k$ is **convergent**. Let

$$s_n = \sum_{k=1}^{n} a_k, \quad t_n = \sum_{k=1}^{n} b_k, \quad t = \sum_{k=1}^{\infty} b_k.$$

  Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing. Since $t_n \to t$, we have $t_n \le t$ for all n. Since $a_k \le b_k$, we have $s_n \le t_n$. Thus $s_n \le t$ for all n which means $\{s_n\}$ is increasing and bounded above and so converges by the Monotonic Sequence Theorem. Thus $\sum a_k$ **converges**.

- Suppose $\sum b_k$ is **divergent**. We show that $\sum a_k$ is **divergent** by contradiction. In fact, if $\sum a_k$ is convergent, since $b_k \le a_k$, from the deviation above, we have that $\sum b_k$ is convergent, a contradiction.

We have the fact below,
Theorem (Comparison Test). Suppose $\sum a_k$ and $\sum b_k$ are series with *nonnegative* terms.

- if $a_k \le b_k$ for all k and $\sum b_k$ is **convergent** $\Longrightarrow \sum a_k$ is **convergent**;
- if $a_k \ge b_k$ for all k and $\sum b_k$ is **divergent** $\Longrightarrow \sum a_k$ is **divergent**.

Example. Determine whether the series converges below,

$$\sum_{k=1}^{\infty} \frac{k}{3k^3 - k^2 + 1}.$$

solution. Take $a_k = \frac{k}{3k^3-k^2+1}$, clearly, $a_k = \frac{k}{k^2(3k-1)+1} > 0$ for all $k \ge 1$. Note that

$$a_k = \frac{k}{2k^3 + (k^3 - k^2) + 1} \le \frac{k}{2k^3+1} < \frac{k}{2k^3} < \frac{1}{k^2}.$$

Since that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, by the Comparison Test, the series $\sum_{k=1}^{\infty} a_k$ is convergent.
Example. Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2 n}.$$

solution. Since the cosine term in the denominator doesn't get too large we can assume that the series terms will behave like $\frac{n}{n^2} = \frac{1}{n}$, which, as a series, will diverge. So, from this we can guess that the series will probably diverge and so we'll need to find a smaller series that will also diverge. Note that

$$\frac{n}{n^2 - \cos^2 n} > \frac{n}{n^2} = \frac{1}{n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the Comparison Test, our original series must also diverge.
Example. Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n+\cos^2 n}.$$

solution. Since that the exponential goes to zeros very fast. So, let's guess that this series will converge and we'll need to find a larger series that will also converge. Note that

$$\frac{e^{-n}}{n+\cos^2 n} \le \frac{e^{-n}}{n} \le \frac{e^{-n}}{1} = e^{-n},$$

for all $n \ge 1$. Take $f(x) = e^{-x}$ which is positive and decreasing on $[1, \infty)$. Note that

$$\int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty} (-e^{-x})|_1^t = \lim_{t\to\infty} (-e^{-t} + e^{-1}) = e^{-1}.$$

Thus, by the integral test, we have that $\sum_{n=1}^{\infty} e^{-n}$ is convergent as well as the original series.
Example. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}.$$

solution. Note that

$$\frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4},$$

and

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4} + \sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$ are both convergent, thus, $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4}$ is convergent as well as the original series by the Comparison Test.
Example. Determine whether the series converges below,

$$\sum_{k=1}^{\infty} \frac{\ln k}{k^2}.$$

solution. Take $a_k = \frac{\ln k}{k^2}$, clearly, $a_k \ge 0$ for all $k \ge 1$. Since that

$$(\ln x - \sqrt{x})' = \frac{1}{x} - \frac{1}{2\sqrt{x}} = \frac{2-\sqrt{x}}{2x}$$

We have

- $0 < x < 4$, $2 - \sqrt{x} > 0$, $\ln x - \sqrt{x}$ is increasing;
- $x \ge 4$, $2 - \sqrt{x} \le 0$, $\ln x - \sqrt{x}$ is decreasing;
- when $x = 4$, the maximum $\ln(4) - \sqrt{4} = 2\ln 2 - 2 = 2(\ln 2 - 1) < 0$.

Thus, $\ln x - \sqrt{x} < 0 \Longrightarrow \ln x < \sqrt{x}$. Note that

$$a_k < \frac{\sqrt{k}}{k^2} = \frac{1}{k^{\frac{3}{2}}}.$$

and $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ is convergent, by the Comparison Test, $\sum_{k=1}^{\infty} a_k$ is convergent.

Example (decimal series). Show that any decimal series of the form $0.a_1a_2a_3\cdots$, where a_i is an integer satisfying $0 \le a_i \le 9$, is convergent. In other words, it always represents a real number.

solution. Note that

$$0.a_1a_2a_3\cdots = a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + a_3 \cdot 10^{-3} + \cdots = \sum_{k=1}^{\infty} a_k \cdot 10^{-k}.$$

and

$$0 \le a_k \cdot 10^{-k} \le 9 \cdot 10^{-k},$$

for all $k \ge 1$. Note that the geometric series $\sum_{k=1}^{\infty} 9 \cdot 10^{-k}$ converges, by the Comparison Test, the decimal series always converges.

Theorem (Limit Comparison Test). Suppose that $\sum a_k$ and $\sum b_k$ are series with $a_k \ge 0$ and $b_k > 0$. Assume

$$\lim_{k\to\infty} \frac{a_k}{b_k} = c.$$

- If $0 < c < \infty$ (say, c is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge;
- If $c = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges;
- If $c = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Proof.

- If $c \in (0, \infty)$, let m and M be positive numbers such that $m < c < M$. Since $\frac{a_k}{b_k} \to c$ as $k \to \infty$, there is an integer K, such that

  $$m < \frac{a_k}{b_k} < M, \quad \text{for } k > K,$$

  or equivalently, $mb_k < a_k < Mb_k$, for $k > K$. By the comparison test, $\sum b_k$ has the same behavior of convergence or divergence as with $\sum a_k$.

- If $c = 0$, there is a integer K, such that

  $$a_k < Mb_k, \quad k > K.$$

  By the Comparison Test, we have that if $\sum b_k$ converges, then $\sum a_k$ converges.

- If $c = \infty$, there is an integer K, such that

  $$mb_k < a_k, \quad \text{for } k > K.$$

  By the Comparison Test, we have that if $\sum b_k$ diverges, then $\sum a_k$ diverges.

□

Example. Determine whether the series converges below.

$$\sum_{k=1}^{\infty} \frac{k}{3k^3 - k^2 + 1}.$$

solution.
Taking $a_k = \frac{k}{3k^3-k^2+1}$ which is positive for all $k \geq 1$. We take $b_k = \frac{1}{k^2}$ and note that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{k}{3k^3-k^2+1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^3}{3k^3-k^2+1} = \lim_{k\to\infty} \frac{1}{3-\frac{1}{k}+\frac{1}{k^3}} = \frac{1}{3}.$$

Since that $\sum b_k$ is convergent. By the Limit Comparison Test, the series $\sum a_k$ is convergent as well.
Example. Determine whether the series converges below

$$\sum_{k=1}^{\infty} \frac{k^2}{3k^3 - k^2 + 1}.$$

solution.
Taking $a_k = \frac{k^2}{3k^3-k^2+1}$ which is positive for all $k \geq 1$. We take $b_k = \frac{1}{k}$ and note that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{k^2}{3k^3-k^2+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^3}{3k^3-k^2+1} = \frac{1}{3}.$$

Since that $\sum b_k$ is divergent. By the Limit Comparison Test, the series $\sum a_k$ is divergent as well.
Example. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}.$$

solution. Fractions involving only polynomials or polynomials under radicals will behave in the same way as the largest power of n will behave in the limit. So, the terms in this series should behave as,

$$\frac{n^2}{\sqrt[3]{n^7}} = \frac{n^2}{n^{\frac{7}{3}}} = \frac{1}{n^{\frac{1}{3}}}.$$

and as a series this will diverge by the p-series test. In fact, this would make a nice choice for our second series in the limit comparison test so let's use it.

$$\lim_{n\to\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}} \cdot \frac{n^{\frac{1}{3}}}{1} = \lim_{n\to\infty} \frac{4n^{\frac{7}{3}}+n^{\frac{4}{3}}}{\sqrt[3]{n^7\left(1+\frac{1}{n^4}\right)}} = \lim_{n\to\infty} \frac{n^{\frac{7}{3}}\left(4+\frac{1}{n}\right)}{n^{\frac{7}{3}}\sqrt[3]{1+\frac{1}{n^4}}} = \frac{4}{\sqrt[3]{1}} = 4.$$

Since that $\sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ diverges, we have that the original series is divergent.
Example. Determine whether the series converges below,

$$\sum_{k=1}^{\infty} \frac{\ln k}{k^2}.$$

solution.
Taking $a_k = \frac{\ln k}{k^2}$ which is positive for all $k \geq 1$. We take $b_k = \frac{1}{k^p}$ and note that

$$0 < \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln k}{k^2}}{\frac{1}{k^p}} = \lim_{k\to\infty} \frac{\ln k}{k^{2-p}} < \lim_{k\to\infty} \frac{k^{\frac{1}{2}}}{k^{2-p}} = \lim_{k\to\infty} \frac{1}{k^{\frac{3}{2}-p}}.$$

- when $p > \frac{3}{2}$, $\lim_{k\to\infty} \frac{1}{k^{\frac{3}{2}-p}} = \infty$;
- when $p < \frac{3}{2}$, $\lim_{k\to\infty} \frac{1}{k^{\frac{3}{2}-p}} = 0$;
- when $p = \frac{3}{2}$, $\lim_{k\to\infty} \frac{1}{k^{\frac{3}{2}-p}} = \left(\lim_{k\to\infty} \frac{1}{k}\right)^0 = 1$.

Thus, we could choose $p \in (1, \frac{3}{2}]$, since $\sum b_k$ is convergent, by the Limit Comparison Test, the series $\sum a_k$ is convergent. For simplicity, we can take $b_k = \frac{1}{k^{\frac{3}{2}}}$.

Example. Determine whether the series converges below.

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}.$$

solution. Taking $a_k = \frac{\ln k}{k}$ which is positive for all $k \geq 1$. We take $b_k = \frac{1}{k^p}$ and note that

$$0 < \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln k}{k}}{\frac{1}{k^p}} = \lim_{k\to\infty} \frac{\ln k}{k^{1-p}} < \lim_{k\to\infty} \frac{k^{\frac{1}{2}}}{k^{1-p}} = \lim_{k\to\infty} \frac{1}{k^{\frac{1}{2}-p}}.$$

- when $p > \frac{1}{2}$, $\lim_{k\to\infty} \frac{1}{k^{\frac{1}{2}-p}} = \infty$;
- when $p < \frac{1}{2}$, $\lim_{k\to\infty} \frac{1}{k^{\frac{1}{2}-p}} = 0$;
- when $p = \frac{1}{2}$, $\lim_{k\to\infty} \frac{1}{k^{\frac{1}{2}-p}} = \left(\lim_{k\to\infty} \frac{1}{k}\right)^0 = 1$.

Thus, we could choose $p \in [\frac{1}{2}, 1]$, since $\sum b_k$ is divergent, by the Limit Comparison Test, the series $\sum a_k$ is divergent. For simplicity, we can take $b_k = \frac{1}{k}$.

Theorem (Ratio Test) Suppose that $\sum a_k$ is a series with *positive* terms.

- If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L < 1 \Longrightarrow \sum a_k$ converges;
- If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L > 1$ or $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \infty \Longrightarrow \sum a_k$ diverges;
- If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$, the test is inconclusive.

Proof.

- If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L < 1$, we take any number r, such that $r \in (L, 1)$. Thus, there is K such that when $k \geq K$,

$$\left|\frac{a_{k+1}}{a_k} - L\right| < r - L \Longrightarrow -(r-L) < \frac{a_{k+1}}{a_k} - L < r - L, \Longrightarrow \frac{a_{k+1}}{a_k} < r,$$

  we have $a_{k+1} < ra_k$ for all $k \geq K$. Thus $a_{K+k} < r^k \cdot a_K$ for all $k \geq 1$. Since that $|r| = r < 1$, the geometry series $\sum_{k=1}^{\infty} a_K r^k$ converges. By the Comparison Test, the series $\sum_{k=K+1}^{\infty} a_k$ converges. Thus, $\sum_{k=1}^{\infty} a_k$ converges.

- To show $\sum a_k$ diverges, by the Divergence Test, it is sufficient to show that $\lim_{k\to\infty} a_k \neq 0$. Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L > 1$, there is K such that when $k > K$,

$$\left|\frac{a_{k+1}}{a_k} - L\right| < L - 1, \Longrightarrow -(L-1) < \frac{a_{k+1}}{a_k} - L \Longrightarrow \frac{a_{k+1}}{a_k} > 1, \Longrightarrow a_{k+1} > a_k > \cdots > a_K.$$

  Thus, $a_k > a_K$ for $k \geq K$. We have $\lim_{k\to\infty} a_k \neq 0$.

- Take $a_k = \frac{1}{k}$ and $a_k = \frac{1}{k^2}$ for $\sum a_k$. The former is divergent and the latter is convergent. For both series, we have

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1.$$

Hence, the Ratio Test is inconclusive if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$.

□

Example. Determine whether the series converge below, where $b > 0$,

$$(a) \sum_{k=1}^{\infty} \frac{b^k}{k!} \quad (b) \sum_{k=1}^{\infty} \frac{k!}{k^k}.$$

solution. Note that $k! = k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1$.
For (a), take $a_k = \frac{b^k}{k!}$ which is positive. Since that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{b^{k+1}}{(k+1)!}}{\frac{b^k}{k!}} = \lim_{k\to\infty} \frac{b}{k+1} = 0,$$

by the Ratio Test, the series $\sum a_k$ is convergent for any $b > 0$.
For (b), take $a_k = \frac{k!}{k^k}$, which is positive. Since that

$$\frac{a_{k+1}}{a_k} = \frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}} = \frac{1}{\frac{(k+1)^k}{k^k}} = \frac{1}{\left(1+\frac{1}{k}\right)^k}.$$

Since that $e = \lim_{k\to\infty} \left(1+\frac{1}{x}\right)^x$, we have $e = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k$, such that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{e} < 1.$$

By the Ratio Test, the series $\sum a_k$ is convergent.
Theorem (Root Test). Suppose $\sum a_k$ is a series with *nonnegative* terms.

- If $\lim_{k\to\infty} \sqrt[k]{a_k} = L < 1, \Longrightarrow \sum a_k$ converges;
- If $\lim_{k\to\infty} \sqrt[k]{a_k} = L > 1$ or $\lim_{k\to\infty} \sqrt[k]{a_k} = \infty, \Longrightarrow \sum a_k$ diverges;
- If $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$, the test is inconclusive.

Proof.

- If $\lim_{k\to\infty} \sqrt[k]{a_k} = L < 1$, we take $r \in (L, 1)$. Thus, there is K such that when $k \geq K$,

$$|\sqrt[k]{a_k} - L| < r - L \Longrightarrow -(r-L) < \sqrt[k]{a_k} - L < r - L, \Longrightarrow \sqrt[k]{a_k} < r \Longrightarrow a_k < r^k.$$

When $|r| < 1$, the geometric series $\sum r^k$ converges. By the Comparison Test, the series $\sum_{k=K}^{\infty} a_k$ converges. Thus, $\sum_{k=1}^{\infty} a_k$ converges.

- To show $\sum a_k$ diverges, by the Divergence Test, it is sufficient to show that $\lim_{k\to\infty} a_k \neq 0$. Since $\lim_{k\to\infty} \sqrt[k]{a_k} = L > 1$, there is K, such that when $k \geq K$,

$$|\sqrt[k]{a_k} - L| < (L-1) \Longrightarrow -(L-1) < \sqrt[k]{a_k} - L \Longrightarrow \sqrt[k]{a_k} > 1 \Longrightarrow a_k > 1.$$

We have $\lim_{k\to\infty} a_k \neq 0$. Similarly as above, if $\lim_{k\to\infty} \sqrt[k]{a_k} = \infty$, we also have $\lim_{k\to\infty} a_k \neq 0$.

- Take $a_k = \frac{1}{k}$ and $a_k = \frac{1}{k^2}$ for the series $\sum a_k$. The former is divergent and the latter is convergent. For both series, we have $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$.

□

Example. Determine whether the series converge below,

$$(a) \sum_{k=1}^{\infty} \left(\frac{3k-1}{4k+1}\right)^k, \quad (b) \sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{-k^2}.$$

solution. For (a), taking $a_k = \left(\frac{3k-1}{4k+1}\right)^k$ which is positive. Note that

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{3k-1}{4k+1}\right)^k} = \lim_{k\to\infty} \frac{3k-1}{4k+1} = \lim_{k\to\infty} \frac{3-\frac{1}{k}}{4+\frac{1}{k}} = \frac{3}{4} < 1.$$

By the Root Test, the series $\sum a_k$ is convergent.

For (b), take $a_k = \left(\frac{k}{k+1}\right)^{-k^2}$ which is positive. Note that

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{k}{k+1}\right)^{-k^2}} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e > 1.$$

By the Root Test, the series $\sum a_k$ is divergent.