Lecture 17 Alternating series

Instructor: Dr. C.J. Xie (macjxie@ust.hk)

The last two tests (integral test and comparison test) that we looked at for series convergence have required that all the terms in the series be positive. Of course there are many series out there that have negative terms in them and so we now need to start looking at tests for these kinds of series. The test that we are going to look into in this section will be a test for alternating series.

1 Alternating series

Definition An alternating series is a series whose terms are alternately positive and negative. In general, an alternating series $\sum_{k=1}^{\infty} a_k$ has its k-th term in the form

$$a_k = (-1)^{k-1} b_k, \quad \text{or} \quad a_k = (-1)^k b_k,$$

where $b_k > 0$ for all k.
Example.

$$\sum_{k=1}^{\infty} a_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k},$$

$$\sum_{k=1}^{\infty} a_k = -1 + \sqrt[2]{2} - \sqrt[3]{3} + \sqrt[4]{4} - \sqrt[5]{5} + \sqrt[6]{6} - \cdots = \sum_{k=1}^{\infty} (-1)^k \sqrt[k]{k}.$$

Alternating Series Test. If $\{b_k\}$ is a positive and non-increasing (decreasing) sequence, say $b_k \geq b_{k+1} > 0$ and $\lim_{k\to\infty} b_k = 0$, the alternating series below

$$\sum_{k=1}^{\infty} (-1)^{k-1} b_k = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \quad b_k > 0,$$

converges.

Proof. Note that the even partial sums:

$$\begin{aligned} s_2 &= b_1 - b_2 \geq 0, \\ s_4 &= s_2 + (b_3 - b_4) \geq s_2, \\ &\vdots \\ s_{2n} &= s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2}, \end{aligned}$$

and

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \leq b_1.$$

Thus, $\{s_{2n}\}$ is increasing and bounded. By the Monotonic Sequence Theorem, $\{s_{2n}\}$ is convergent. Denote $\lim_{n\to\infty} s_{2n} = s$. For the odd partial sums, we have

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} (s_{2n} + b_{2n+1}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = s + 0 = s.$$

Since both the even and odd partial sums converge to s, we have $\lim_{n\to\infty} s_n = s$. As a result, the alternating series is convergent. ☐

Rk. There are a couple of things to note about this test.

- First, unlike the Integral Test and the Comparison/Limit Comparison Test, this test will only tell us when a series converges and not if a series will diverge;
- Secondly, in the second condition all that we need to require is that the series terms, b_n will be eventually decreasing. It is possible for the first few terms of a series to increase and still have the test be valid. All that is required is that eventually we will have $b_n \geq b_{n+1}$, for all n after some point. For example, let us take the alternating series below,

$$\sum_{n=1}^{\infty}(-1)^n b_n,$$

and suppose that $\{b_n\}$ is not decreasing for $1 \leq n \leq N$ and decreasing for $n \geq N+1$. The series can the be written as

$$\sum_{n=1}^{\infty}(-1)^n b_n = \sum_{n=1}^{N}(-1)^n b_n + \sum_{n=N+1}^{\infty}(-1)^n b_n,$$

where $\sum_{n=1}^{\infty}(-1)^n b_n$ has the same behavior of convergence or divergence as $\sum_{n=N+1}^{\infty}(-1)^n b_n$. The point of all this is that we don't need to require that the series terms be decreasing for all n.

Example. The alternating harmonic series

$$\sum_{k=1}^{\infty} a_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} = \sum_{k=1}^{\infty}(-1)^{k-1} b_k,$$

where $b_k = \frac{1}{k}$ is non-increasing and $\lim_{k\to\infty} b_k = 0$. Thus by the Alternating Series Test, the series converges.

Example. Determine if the series is convergent or divergent below.

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{n^2+5}.$$

solution. First, identify the b_n for the test,

$$\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{n^2+5} = \sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^2+5} \quad \Longrightarrow \quad b_n = \frac{n^2}{n^2+5}.$$

Since that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^2}{n^2+5} = 1 \neq 0.$$

So, the condition isn't met and so there is no reason to check the monotonicity. Since this condition isn't met we'll need to use another test to check convergence. In this case where the condition isn't met it is usually best to use the divergence test. Note that

$$a_{2n} = \frac{(2n)^2}{(2n)^2+5}, \quad a_{2n+1} = \frac{-(2n+1)^2}{(2n+1)^2+5}$$

and

$$\lim_{n\to\infty} a_{2n} = 1 \neq 0, \quad \lim_{n\to\infty} a_{2n+1} = -1 \neq 0.$$

By the divergence test, Both series $\sum a_{2n}$ and $\sum a_{2n+1}$ are divergent. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$ is divergent.

Example. Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}.$$

solution. Taking $b_n = \frac{\sqrt{n}}{n+4}$, clearly which is positive and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0.$$

Then taking

$$f(x) = \frac{\sqrt{x}}{x+4}, \quad \Longrightarrow \quad f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}.$$

Now, there are two critical points for this function, $x = 0$, and $x = 4$. Note that $x = -4$ is not a critical point because the function is not defined at $x = -4$. The first is outside the bound of our series so we won't need to worry about that one. Since

- if $0 \le x \le 4$, $f'(x) \ge 0, \Longrightarrow f(x)$ is increasing;
- if $x \ge 4$, $f'(x) \le 0, \Longrightarrow f(x)$ is decreasing;

We take $b_n = f(n)$ and then have that b_n is also increasing on $0 \le n \le 4$ and decreasing on $n \ge 4$. The b_n are then eventually decreasing and so the condition is met. Thus, by the Alternating Series Test, the series is convergent.

Example. Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}.$$

solution. Note that $\cos(n\pi) = (-1)^n$ and

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \Longrightarrow b_n = \frac{1}{\sqrt{n}}.$$

Checking the two condition gives,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$
$$b_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = b_{n+1}.$$

The two conditions of the test are met and so by the Alternating Series Test, the series is convergent.

Remainder estimate for alternating series. If $s = \sum_{k=1}^{\infty} (-1)^k b_k$ is the sum of an alternating series that satisfies $\{b_k\}$ is non-increasing, $(b_k \ge b_{k+1} > 0)$ and $\lim_{k\to\infty} b_k = 0$, then

$$|R_n| = |s - s_n| \le b_{n+1}.$$

Proof. From the proof of alternating test, we have known that the sequence $\{s_{2n}\}$ is increasing. Also, note that for the odd partial sums:

$$\begin{aligned} s_1 &= b_1 > 0 \\ s_3 &= b_1 - (b_2 - b_3) \le s_1 \\ s_5 &= s_3 - (b_4 - b_5) \le s_3 \\ &\vdots \\ s_{2n+1} &= s_{2n-1} - (b_{2n} - b_{2n+1}) \le s_{2n-1}. \end{aligned}$$

Thus, $\{s_{2n+1}\}$ is decreasing. Since we know that both $\{s_{2n}\}$ and $\{s_{2n+1}\}$ converge to s, we get

$$s_{2n} \leq s \leq s_{2n+1}, \qquad \text{for all } n.$$

Thus,

$$|R_n| = |s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}.$$

□

Example. Consider the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots .$$

(a) How many terms of the series must be used to obtain an approximation that is within 10^{-4} of the exact value of the series.

(b) What is the approximation value.

solution. For (a), clearly, this alternating series is convergent. Then

$$s = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} + R_n,$$

Using the bound on the remainder in the Alternating Series Test, we have

$$|R_n| \leq \frac{1}{(n+1)!}.$$

Thus, to ensure that $|R_n| < 10^{-4}$, we need to take n such that $\frac{1}{(n+1)!} < 10^{-4} \Longrightarrow n > 7$. Hence, the finite series $\sum_{k=0}^{7} \frac{(-1)^k}{k!}$ gives an approximation that is w 10^{-4} of the exact value of the series. For (b), it follows from $|R_n| \leq b_{n+1} \Longrightarrow -b_{n+1} \leq R_n \leq b_{n+1} \Longrightarrow s_n - b_{n+1} \leq s_n + R_n = s \leq s_n + b_{n+1}$. We have

$$s_7 - \frac{1}{8!} \leq s \leq s_7 + \frac{1}{8!},$$

where $s_7 = \sum_{k=0}^{7} \frac{(-1)^k}{k!}$. By using a calculator, we have $s_7 \approx 0.36786$. Thus, $0.36784 < s < 0.36789$. Taking the average of these two bounds as our approximation of s, we find that $s \approx 0.36787$.