Lecture 18 Absolute and conditional convergence

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1 Absolute Convergence and Conditional Convergence

Definition.

- A series $\sum a_k$ is called **absolutely convergent**, if the series of the absolute values $\sum |a_k|$ is convergent;
- A series $\sum a_k$ is called **conditionally convergent**, if $\sum a_k$ is convergent, but $\sum |a_k|$ is divergent.

Intuition. Clearly, we have

$$0 \leq a_k + |a_k| \leq 2|a_k|, \qquad \text{for all } k.$$

If $\sum a_k$ is absolutely convergent, then $\sum 2|a_k|$ is convergent. By the Comparison Test, $\sum(a_k + |a_k|)$ is convergent. Thus,

$$\sum a_k = \sum(a_k + |a_k|) - \sum |a_k|,$$

is convergent.
Theorem. If a series $\sum a_k$ is **absolutely convergent**, then it is **convergent**.
Rk. This fact is one of the ways in which absolute convergence is a "stronger" type of convergence. Series that are absolutely convergent are guaranteed to be convergent. However, series that are convergent may or may not be absolutely convergent.
Example. The series

$$(a) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}, \quad (b) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}, \quad (c) \sum_{k=1}^{\infty} \frac{\sin k}{k^3}.$$

solution. For (a), since that

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{k^2},$$

is convergent. Thus, this series is absolutely convergent.
For (b), the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ is convergent by the Alternating Series Test, but the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} |a_k|$ is divergent. Thus, the alternating harmonic series is conditionally convergent.
For (c), since that

$$|a_k| = \frac{|\sin k|}{k^3} \leq \frac{1}{k^3},$$

and

$$\sum_{k=1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

Note that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is convergent, by the Comparison Test, we have that $\sum_{k=1}^{\infty} |a_k|$ is convergent. Thus $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (and hence convergent).

2 Recall the ratio rest, root test for finding the absolute convergence

In this section we are going to take a look at a test that we can use to see if a series is absolutely convergent or not. Recall that if a series is absolutely convergent then we will also know that it's convergent and so we will often use it to simply determine the convergence of a series.
Ratio test Suppose we have the series $\sum a_n$. Define

$$c = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then

- if $c < 1$, the series $\sum a_n$ is absolutely convergent (and hence convergent);
- if $c > 1$, the series $\sum a_n$ is divergent;
- if $c = 1$, the series $\sum a_n$ may be divergent, conditionally convergent, or absolutely convergent.

Proof. First note that we can assume without loss of generality that the series will start at $n = 1$.

- If $c < 1$, there is some number r, such that $c < r < 1$.

Thus, for some constant N, when $n > N$, we have

$$\left| \frac{a_{n+1}}{a_n} \right| < r, \quad \Longrightarrow \quad |a_{n+1}| < r|a_n|,$$

and

$$\begin{aligned}
|a_{N+1}| &< r|a_N| \\
|a_{N+2}| &< r|a_{N+1}| < r^2|a_N| \\
|a_{N+3} &< r|a_{N+2}|| < r^3|a_N| \\
&\vdots \\
|a_{N+k}| &< r|a_{N+k-1}| < r^k|a_N|.
\end{aligned}$$

Thus, for $k = 1, 2, 3, \cdots$, we have $|a_{N+k}| < r^k|a_N|$. Note that the series below

$$\sum_{k=0}^{\infty} |a_N| r^k$$

which is a geometric series with $0 < r < 1$. Thus, $\sum_{k=0}^{\infty} |a_N| r^k$ is convergent. Also since $|a_{N+k}| < r^k|a_N|$ by the Comparison test, the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}|,$$

is convergent. Note that

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|,$$

we know that $\sum_{n=1}^{\infty} |a_n|$ is also convergent since the first term on the right is a finite sum of finite terms and hence finite. Therefore $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $c > 1$, there is a constant N, such that $n \geq N$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| > 1, \quad \Longrightarrow \quad |a_{n+1}| > |a_n|.$$

Thus, we have

$$\lim_{n\to\infty} |a_n| \neq 0.$$

Thus, by the Divergence Test, $\sum a_n$ is divergent.

If $c = 1$, we can take some examples to see below,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{absolutely convergent}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \text{conditionally convergent}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{divergent .}$$

□

Example. Determine if the series below is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}.$$

solution.
Let's take

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)},$$

and note that

$$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)} = \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}.$$

In turn, we have

$$\begin{aligned} c = \lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}\cdot\frac{4^{2n+1}(n+1)}{(-10)^n}\right| \\ = \lim_{n\to\infty}\left|\frac{(-10)(n+1)}{4^2(n+2)}\right| &= \frac{10}{16}\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{10}{16} < 1. \end{aligned}$$

By the Ratio Test, the series converges absolutely and hence will converge.

Example. Determine if the series below is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{n!}{5^n}.$$

solution. Note that the definition of the n factorial by

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1,$$

and

$$c = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{5^{n+1}}\cdot\frac{5^n}{n!}\right| = \lim_{n\to\infty}\frac{(n+1)!}{5n!} = \lim_{n\to\infty}\frac{(n+1)n!}{5n!} = \lim_{n\to\infty}\frac{n+1}{5} = \infty > 1.$$

So, by the Ratio Test, this series diverges.

Rk.

- $0! = 1,\ 1! = 1,\ 2! = 2 \cdot 1 = 2,\ 3! = 3 \cdot 2! = 6,\ 4! = 4 \cdot 3! = 24,\ 5! = 5 \cdot 4! = 120;$
- $n! = n(n-1)(n-2)\cdots(n-k)\cdot(n-k-1)!$
- $(2n)! \neq 2 \cdot n!$, since that

$$\begin{aligned}(2n)! &= (2n)\cdot(2n-1)\cdot(2n-2)\cdots 3\cdot 2\cdot 1,\\ 2n! &= 2\left[n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\right].\end{aligned}$$

Example. Determine if the series below is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}.$$

solution. Note that

$$\begin{aligned}c = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2(n+1)-1)!}\cdot\frac{(2n-1)!}{n^2}\right|\\ &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+1)!}\cdot\frac{(2n-1)!}{n^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n)(2n-1)!}\cdot\frac{(2n-1)!}{n^2}\\ &= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n)(n^2)} = 0 < 1.\end{aligned}$$

Thus, by the Ratio Test, this series converges absolutely and so converges.

Example. Determine if the series below is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{9^n}{(-2)^{n+1}n}.$$

solution.

Note that

$$\begin{aligned}c = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{9^{n+1}}{(-2)^{n+2}(n+1)}\cdot\frac{(-2)^{n+1}n}{9^n}\right|\\ &= \lim_{n\to\infty}\left|\frac{9n}{-2(n+1)}\right| = \frac{9}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{9}{2} > 1,\end{aligned}$$

thus, by the Ratio Test, this series is divergent.

Example. Determine if the series below is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}.$$

solution. Let's first get c below,

$$c = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(-1)^n}\right| = \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1} = 1,$$

which means the ratio test is not good for determining the convergence of this series. We will need to resort to another test for this series. This series is an alternating series and so let's check the two conditions from that test.

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{1}{n^2+1} = 0,\\ b_n &= \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} = b_{n+1}.\end{aligned}$$

The two conditions are met and so by the Alternating Series Test this series is convergent. Note that $\sum |a_n|$ by the comparison test with $\sum \frac{1}{n^2}$, this series is also absolutely convergent.
Example. Determine if the series below is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{n+2}{2n+7}.$$

solution. Note that

$$c = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+3}{2(n+1)+7} \cdot \frac{2n+7}{n+2}\right| = \lim_{n\to\infty} \frac{(n+3)(2n+7)}{(2n+9)(n+2)} = 1.$$

Again, the ratio test tells us nothing here. We can however, quickly use the divergence test on this. In fact that probably should have been our first choice on this one anyway.

$$\lim_{n\to\infty} \frac{n+2}{2n+7} = \frac{1}{2} \neq 0.$$

By the Divergence Test this series is divergent.
Root Test. Suppose that we have the series $\sum a_n$. Define

$$c = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Then

- if $c < 1$, the series $\sum a_n$ is absolutely convergent (and hence convergent);
- if $c > 1$, the series $\sum a_n$ is divergent;
- if $c = 1$, the series $\sum a_n$ may be divergent, conditionally convergent, or absolutely convergent.

Proof. First note that we can assume without loss of generality that the series will start at $n = 1$.

- If $c < 1$, there is a number r, such that $c < r < 1$. Thus, there is some N, when $n > N$, we have

$$|a_n|^{\frac{1}{n}} < r, \quad \Longrightarrow \quad |a_n| < r^n.$$

  Note that $\sum_{n=0}^{\infty} r^n$ is a convergent geometric series with $0 < r < 1$. By the comparison test, the series $\sum_{n=N}^{\infty} |a_n|$ is convergent. On the other hand,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|,$$

  which indicates that $\sum_{n=1}^{\infty} |a_n|$ si convergent as well, since the first term on the right is a finite sum of finite terms and hence finite. Thus, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

- If $c > 1$, there must be some N, such that $n \geq N$, we have

$$|a_n|^{\frac{1}{n}} > 1, \quad \Longrightarrow \quad |a_n| > 1.$$

  As a result,

$$\lim_{n\to\infty} |a_n| \neq 0 \Longrightarrow \lim_{n\to\infty} a_n \neq 0.$$

  Thus, by the Divergence Test, $\sum a_n$ is divergent.

- If $c = 1$, we can take some examples below,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{absolutely convergent,}$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \text{conditionally convergent,}$$
$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{divergent .}$$

□

Fact.

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

solution. We can take $a_n = \ln\left(n^{\frac{1}{n}}\right) = \frac{\ln n}{n} \Longrightarrow n^{\frac{1}{n}} = e^{a_n}$. Since that by the L'Hospital's rule, we have

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

Thus,

$$\lim_{n\to\infty} a_n = 0, \Longrightarrow \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{a_n} = e^0 = 1.$$

Example. Determine if the series below is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}.$$

solution. Note that

$$c = \lim_{n\to\infty} \left|\frac{n^n}{3^{1+2n}}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} = \frac{\infty}{3^2} = \infty > 1.$$

Thus, by the Root Test, this series is divergent.

Example. Determine if the series below is convergent or divergent.

$$\sum_{n=0}^{\infty} \left(\frac{5n - 3n^3}{7n^3 + 2}\right)^n.$$

solution. Note that

$$c = \lim_{n\to\infty} \left|\left(\frac{5n - 3n^3}{7n^3 + 2}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{5n - 3n^3}{7n^3 + 2}\right| = \left|\frac{-3}{7}\right| = \frac{3}{7} < 1.$$

Thus, by the Root Test this series converges absolutely and hence converges.

Example. Determine if the series below is convergent or divergent.

$$\sum_{n=3}^{\infty} \frac{(-12)^n}{n}.$$

solution. Note that

$$c = \lim_{n\to\infty} \left|\frac{(-12)^n}{n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{12}{n^{\frac{1}{n}}} = \frac{12}{1} = 12 > 1.$$

By the Root Test, this series is divergent.

3 Summary: Strategy For Series

The guidelines for determining the convergence of a series below. Again, remember that these are only a set of guidelines and not a set of hard and fast rules to use when trying to determine the best test to use on a series.

- With a quick glance does it look like the series terms don't converge to zero in the limit, i.e., does $\lim_{n\to\infty} a_n \neq 0$? If so, use the Divergence Test. Note that you should only do the Divergence Test if a quick glance suggests that the series terms may not converge to zero in the limit;
- Is the series a p-series $\sum \frac{1}{n^p}$ or a geometric series ($\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$)? If so use the fact that p-series will only converge if $p > 1$ and a geometric series will only converge if $|r| < 1$. Remember as well that often some algebraic manipulation is required to get a geometric series into the correct form.
- Is the series similar to a p-series or a geometric series? If so, try the Comparison Test.
- Is the series a rational expression involving only polynomials or polynomials under radicals (i.e. a fraction involving only polynomials or polynomials under radicals)? If so, try the Comparison Test and/or the Limit Comparison Test. Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive.
- Does the series contain factorials or constants raised to powers involving n? If so, then the Ratio Test may work. Note that if the series term contains a factorial then the only test that we've got that will work is the Ratio Test.
- Can the series terms be written in the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$? If so, then the Alternating Series Test may work.
- Can the series terms be written in the form $a_n = (b_n)^n$? If so, then the Root Test may work.
- If $a_n = f(n)$ for some positive, decreasing function and $\int_a^{\infty} f(x)\, dx$ is easy to evaluate then the Integral Test may work.

In summary,
Divergence Test. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ will diverge.
Integral Test. Suppose that $f(x)$ is a positive, decreasing function on the interval $[k, \infty)$ and that $f(n) = a_n$ then,

- If $\int_k^{\infty} f(x)\, dx$ is convergent then so is $\sum_{n=k}^{\infty} a_n$;
- If $\int_k^{\infty} f(x)\, dx$ is divergent then so is $\sum_{n=k}^{\infty} a_n$.

Comparison Test. Suppose that we have two series $\sum a_n$ and $\sum b_n$ with a_n, $b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then,

- If $\sum b_n$ is convergent then so is $\sum a_n$;
- If $\sum a_n$ is divergent then so is $\sum b_n$.

Limit Comparison Test. Suppose that we have two series $\sum a_n$ and $\sum b_n$ with a_n, $b_n \geq 0$ for all n. Define

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

If $0 < c < \infty$, then either both series converge or both series diverge.
Alternating Series Test. Suppose that we have series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$, where $b_n \geq 0$ for all n. Then if

- $\lim_{n\to\infty} b_n = 0$;
- $\{b_n\}$ is eventually a decreasing sequence;

then the series $\sum a_n$ is convergent.
Ratio Test. Suppose we have series $\sum a_n$. Define

$$c = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then,

- if $c < 1$, the series is absolutely convergent (and hence convergent);
- if $c > 1$, the series is divergent;
- if $c = 1$, the series may be divergent, conditionally convergent, or absolutely convergent.

Root Test. Suppose that we have the series $\sum a_n$. Define

$$c = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Then,

- if $c < 1$, the series is absolutely convergent (and hence convergent);
- if $c > 1$, the series is divergent;
- if $c = 1$, the series may be divergent, conditionally convergent, or absolutely convergent.

Rk. Common comparison object below,

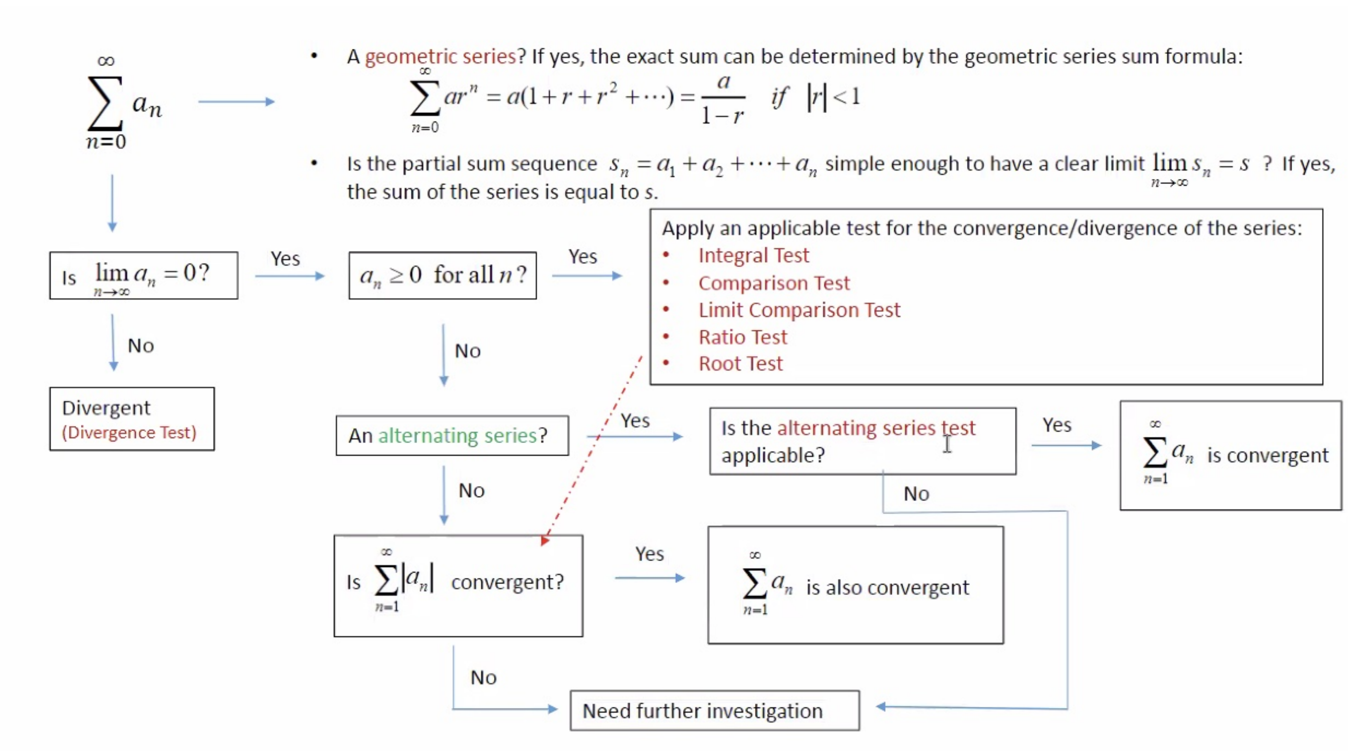
- p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \leq 1. \end{cases}$$

- geometric series:

$$\sum_{r=1}^{\infty} ar^{k-1} = \begin{cases} \dfrac{a}{1-r}, & \text{if } |r| < 1, \\ \text{diverges}, & \text{if } |r| \geq 1, \end{cases}$$

where $a \neq 0$ and r be a real number.


$\sum_{n=0}^{\infty} a_n$
A geometric series? If yes, the exact sum can be determined by the geometric series sum formula:
$\sum_{n=0}^{\infty} ar^n = a(1+r+r^2+\cdots) = \frac{a}{1-r}$ if $|r|<1$
Is the partial sum sequence $s_n = a_1 + a_2 + \cdots + a_n$ simple enough to have a clear limit $\lim_{n\to\infty} s_n = s$? If yes, the sum of the series is equal to s.
Is $\lim_{n\to\infty} a_n = 0$?
Yes
No
Divergent
(Divergence Test)
$a_n \geq 0$ for all n?
Yes
No
Apply an applicable test for the convergence/divergence of the series:
Integral Test
Comparison Test
Limit Comparison Test
Ratio Test
Root Test
An alternating series?
Yes
No
Is the alternating series test applicable?
Yes
No
$\sum_{n=1}^{\infty} a_n$ is convergent
Is $\sum_{n=1}^{\infty} |a_n|$ convergent?
Yes
$\sum_{n=1}^{\infty} a_n$ is also convergent
No
Need further investigation