Lecture 2– Substitution method

Instructor: Dr. C.J. Xie (macjxie@ust.hk)

1 Recap last time

Riemann sum to define the Definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Example (from the classviva.org). Represent

$$\int_2^6 \frac{x}{1+x^5}\,dx.$$

by a limit of Riemann sums.
solution
$\Delta x = \frac{6-2}{n} = \frac{4}{n}$, the sub-intervals are

$$\left[2, 2+\frac{4}{n}\right], \quad \left[2+\frac{4}{n}, 2+2\cdot\frac{4}{n}\right], \quad \cdots, \quad \left[2+(n-2)\cdot\frac{4}{n}, 2+(n-1)\cdot\frac{4}{n}\right], \left[2+(n-1)\cdot\frac{4}{n}, 2+n\cdot\frac{4}{n}\right].$$

the function is evaluated at $x = 2+\frac{4i}{n}$, $i = 1, \cdots, n$. Thus,

$$\int_2^6 \frac{x}{1+x^5}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}.$$

Fundamental Theorem of Calculus (FTC) Let f be a continuous function on $[a, b]$,

- $A(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$, differentiable in (a, b), and $A'(x) = f(x)$;
- If F is a differentiable function on $[a, b]$, and $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$$

Definite and Indefinite integral Find an anti-derivative $F(x)$ of $f(x)$,

- for the definite integral, use FTC, to get

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a);$$

- for the indefinite integral, we have

$$\int f(x)\,dx = F(x) + C, \quad C \text{ is a constant.}$$

Example (from the classviva.org). Evaluate the indefinite integral

$$\int \frac{5}{x^5} - 7\sqrt[3]{x^2}\, dx.$$

solution

$$\int \frac{5}{x^5} - 7\sqrt[3]{x^2}\, dx = 5\int \frac{1}{x^5}\, dx - 7\int \sqrt[3]{x^2}\, dx = 5 \cdot \left(-\frac{1}{4}\right) x^{-4} - 7 \cdot \frac{3}{5} x^{\frac{5}{3}} + C$$
$$= -\frac{5}{4x^4} - \frac{21}{5} x\sqrt[3]{x^2} + C.$$

2 Substitution method for indefinite integral

Theorem for the indefinite integral

Let $u = g(x)$ be a differentiable function and f be a continuous function. Suppose the range of g is contained in the domain of f. Then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du.$$

proof.

Let $F'(x) = f(x)$, thus, we have

$$\int f(u)\, du = F(u) + C.$$

Then by the **chain rule of differentiation**, we have

$$\frac{d}{dx} F(g(x)) = \frac{dF(u)}{du} \cdot g'(x) = f(u) \cdot g'(x) = f(g(x)) \cdot g'(x),$$

say inversely,

$$\int f(g(x)) \cdot g'(x)\, dx = \int \frac{dF(g(x))}{dx}$$
$$= F(g(x)) + C = F(u) + C = \int f(u)\, du.$$

Example. Evaluate the integral $\int 2xe^{x^2}\, dx$.

solution. Find F, such that $F'(x) = f(x)$. Note that by chain rule $(e^{x^2})' = 2xe^{x^2}$. Let $u = x^2$, we have $du = (x^2)'\, dx = 2x\, dx$ and

$$\int 2xe^{x^2}\, dx = e^{x^2} + C,$$

or by the substitution method,

$$\int 2xe^{x^2}\, dx = \int e^{x^2} 2x\, dx = \int e^u\, du = e^u + C = e^{x^2} + C.$$

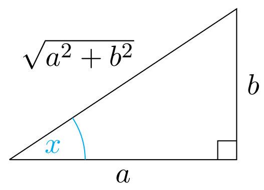
Example. Evaluate

$$\int \cot x\, dx.$$

solution

From last time, we have known that

$$\int \tan x \, dx = \ln|\sec x| + C$$



If we let $\tan x = \frac{b}{a}$, then $\cot x = \frac{a}{b} = \tan\left(\frac{\pi}{2} - x\right)$, $\cos x = \frac{a}{\sqrt{a^2+b^2}}$,

$$\sec\left(\frac{\pi}{2} - x\right) = \frac{1}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{\sqrt{a^2+b^2}}{b} = \csc x.$$

Thus,

$$\int \cot x \, dx = \int \tan\left(\frac{\pi}{2} - x\right) \, dx.$$

Let $u = \frac{\pi}{2} - x$, we then have $du = -dx$ and

$$\begin{aligned} \int \cot x \, dx &= \int \tan\left(\frac{\pi}{2} - x\right) \, dx = \int -\tan u \, du \\ &= -\ln|\sec u| + C = -\ln\left|\sec\left(\frac{\pi}{2} - x\right)\right| + C = -\ln|\csc x| + C. \end{aligned}$$

Rk. There might be more than one way to make substitution.
Example. Evaluate the integral

$$\int \frac{x}{\sqrt{x+1}} \, dx.$$

solution.

substitution 1: Let $u = x + 1$. Then $du = dx$ and

$$\begin{aligned} \int \frac{x}{\sqrt{x+1}} \, dx &= \int \frac{u-1}{\sqrt{u}} \, du = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) \, du \\ &= \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C \\ &= \frac{2}{3}\sqrt{(x+1)^3} - 2\sqrt{x+1} + C. \end{aligned}$$

substitution 2: Let $u = \sqrt{x+1}$. Then $x = u^2 - 1$, $dx = 2u \, du$ and

$$\begin{aligned} \int \frac{x}{\sqrt{x+1}} \, dx &= \int \frac{u^2-1}{u} 2u \, du = \int (2u^2 - 2) \, du \\ &= \frac{2}{3} u^3 - 2u + C = \frac{2}{3}\sqrt{(x+1)^3} - 2\sqrt{x+1} + C. \end{aligned}$$

Ex.

$$\int x^2 \sqrt{x^3+4} \, dx, \qquad \int \frac{\cos\sqrt{t}}{\sqrt{t}} \, dt.$$

solution

- Let $u = x^3 + 4$, we then have $\frac{du}{dx} = 3x^2$, $\frac{1}{3}du = x^2dx$ and

$$\int x^2\sqrt{x^3+4}\,dx = \int \frac{1}{3}\sqrt{u}\,du = \frac{1}{3}\int u^{\frac{1}{2}}\,du$$
$$= \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{9}(x^3+4)^{\frac{3}{2}} + C.$$

- Let $u = \sqrt{t}$, we then have $\frac{du}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$, $2du = \frac{1}{\sqrt{t}}\,dt$ and

$$\frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = \int 2\cos u\,du = 2\sin u + C = 2\sin\sqrt{t} + C.$$

3 Substitution method for definite integral

Theorem for the definite integral
Let $u = g(x)$, where $g'(x)$ is continuous on $[a, b]$, and let f be continuous on an interval containing the range of g. Then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_a^b f(g(x))\cdot\,dg(x) = \int_{g(a)}^{g(b)} f(u)\,du.$$

Note. $x : a \to b$, then $u = g(x) : g(a) \to g(b)$. The changes are made by three points below,

- the **variable** of integral;
- the **lower and upper limit** of integral;
- the **function** of integral.

Example. Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x\cos x\,dx$.
solution
Let $u = \sin x$, then $du = (\sin x)'\,dx = \cos x\,dx$.

$$\begin{cases} x: & 0 \to \dfrac{\pi}{2} \\ u: & \sin 0 = 0 \to \sin\dfrac{\pi}{2} = 1. \end{cases}$$

Thus, we have

$$\int_0^{\frac{\pi}{2}} \sin^3 x\cos x\,dx = \int_0^{\frac{\pi}{2}} \sin^3 x\,d\sin x = \int_0^1 u^3\,du = \frac{1}{4}u^4|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}.$$

4 Technical substitutions

4.1 Trigonometric substitutions

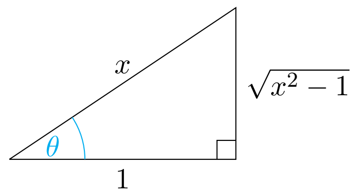
See the trigonometric substitutions below,

- If the integral involving: $\sqrt{a^2 - x^2}$, let $x = a\sin\theta$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, using $1 - \sin^2\theta = \cos^2\theta$;
- If the integral involving: $\sqrt{a^2 + x^2}$, let $x = a\tan\theta$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, using $1 + \tan^2\theta = \sec^2\theta$;
- If the integral involving: $\sqrt{x^2 - a^2}$, let $x = a\sec\theta$, $x \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3}{2}\pi)$, using $\sec^2\theta - 1 = \tan^2\theta$.

Example (involving $\sqrt{x^2-a^2}$). Evaluate

$$\int \frac{1}{\sqrt{(x^2-1)^3}}\,dx.$$

solution
Let $x = \sec\theta = \frac{1}{\cos\theta}$, note that the relation of the right triangle below,



we have

$$\cos\theta = \frac{1}{x}, \quad \sin\theta = \frac{\sqrt{x^2-1}}{x}.$$

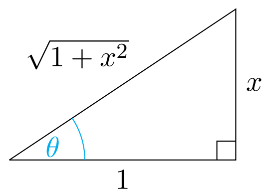
We then have $x^2-1 = \sec^2\theta - 1 = \tan^2\theta$, $dx = \sec\theta\tan\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{\sqrt{(x^2-1)^3}}\,dx &= \int \frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta \\ &= \int \frac{1}{\sin^2\theta}\,d\sin\theta \overset{u=\sin\theta}{=} \int \frac{1}{u^2}\,du \\ &= -\frac{1}{u} + C = -\frac{1}{\sin\theta} + C = -\frac{x}{\sqrt{x^2-1}} + C.\end{aligned}$$

Example (involving $\sqrt{a^2+x^2}$). Evaluate

$$\int \frac{1}{(1+x^2)^2}\,dx.$$

solution
Let $x = \tan\theta$, note that the relation of the right triangle below,



we have

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}, \quad \cos\theta = \frac{1}{\sqrt{1+x^2}},$$

and

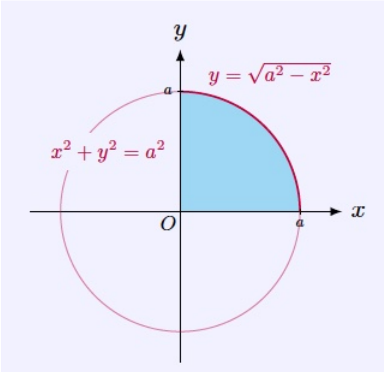
$$\theta = \arctan x, \quad \sin 2\theta = 2\sin\theta\cos\theta = \frac{2x}{1+x^2}.$$

We then have $1+x^2 = 1+\tan^2\theta = \sec^2\theta$, $dx = (\tan\theta)'\,d\theta = \sec^2\theta\,d\theta$ and

$$\begin{aligned}\int \frac{1}{(1+x^2)^2}\,dx &= \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \cos^2\theta\,d\theta \\ &= \int \frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \\ &= \frac{1}{2}\arctan x + \frac{x}{2(1+x^2)} + C.\end{aligned}$$

Example (involving $\sqrt{a^2 - x^2}$). Find the area of a circle of radius a.

solution



We have

$$\text{Area} = 4 \int_0^a \sqrt{a^2 - x^2}\, dx.$$

Let $x = a \sin\theta$, we then have $dx = a\cos\theta\, d\theta$, $x : 0 \to a$, then $\theta : 0 \to \frac{\pi}{2}$. Thus,

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2\theta} \cdot a\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1 + \cos 2\theta\right)\, d\theta = \frac{1}{2} a^2 \left(\theta + \sin 2\theta\right)|_0^{\frac{\pi}{2}} = \frac{1}{2} a^2 \left(\frac{\pi}{2} + 0\right) = \frac{1}{4}\pi a^2,$$

and

$$\text{Area} = 4 \int_0^a \sqrt{a^2 - x^2}\, dx = 4 \cdot \frac{1}{4}\pi a^2 = \pi a^2.$$

Rk: the homeworks can be found here https://www.classviva.org.

Example (from classviva.org).

Evaluate

$$\int_0^1 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\, dx.$$

solution

Let $u = e^{2x} + e^{-2x}$, we the have $x : 0 \to 1$, $u : 2 \to e^2 + e^{-2}$ and

$$du = \left(e^{2x} + e^{-2x}\right)'\, dx = 2\left(e^{2x} - e^{-2x}\right).$$

Thus,

$$\int_0^1 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\, dx = \frac{1}{2} \int_2^{e^2 - e^{-2}} \frac{1}{u}\, du = \frac{1}{2} \ln u|_2^{e^2 - e^{-2}} = \frac{1}{2} \ln(e^2 - e^{-2}) - \ln 2$$

$$= \frac{1}{2} \ln \frac{e^4 - 1}{e^2} - \ln 2 = \frac{1}{2} \ln(e^4 - 1) - \ln 2 - 1.$$