Lecture 20 Power Series And Functions

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1 Power Series And Functions

In this section we will start talking about how to represent functions with power series.
Recall that the geometric series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad \text{provided } |r| < 1.$$

Also, the series diverges if $|r| \geq 1$. Now, if we take $a = 1$ and $r = x$, this becomes

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad \text{provided } |x| < 1.$$

Turning this around we can see that we can represent the function

$$f(x) = \frac{1}{1-x}. \tag{1}$$

with the power series $\sum_{n=0}^{\infty} x^n$, provided $|x| < 1$. We can clearly plug any number other than $x = 1$ into the function, however, we will only get a convergent power series if $|x| < 1$. Note as well that we can also use this to acknowledge that the radius of convergence of this power series is $R = 1$ and the interval of convergence is $|x| < 1$. This idea of convergence is important here. We will be representing many functions as power series and it will be important to recognize that the representations will often only be valid for a range of x's and that there may be values of x that we can plug into the function that we can't plug into the power series representation.
In this section we are going to concentrate on representing functions with power series where the functions can be related back to (1).
So, let's jump into a couple of examples.
Example. Find a power series representation for the following function and determine its interval of convergence.

$$f(x) = \frac{1}{1+x^3}.$$

solution. What we need to do here is to relate this function back to (1). Note that

$$f(x) = \frac{1}{1-(-x^3)},$$

so the $-x^3$ in $g(x)$ holds the same place as the x in (1). Therefore, all we need to do is replace the x in the series and we've got a power series representation for $g(x)$.

$$f(x) = \sum_{n=0}^{\infty} (-x^3)^n, \qquad \text{provided } |-x^3| < 1.$$

Notice that we replaced both the x in the power series and in the interval of convergence.

Thus, we have

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{3n} \quad \text{provided } |x|^3 < 1 \quad \Longrightarrow \quad |x| < 1.$$

Rk. In this case the interval of convergence is the same as the original power series. This usually won't happen. More often than not the new interval of convergence will be different from the original interval of convergence.

Example. Find a power series representation for the following function and determine its interval of convergence.

$$f(x) = \frac{2x^2}{1+x^3}.$$

solution. This function is similar to the previous function. The difference is the numerator and at first glance that looks to be an important difference. Since (1) doesn't have an x in the numerator it appears that we can't relate this function back to that. Note that

$$f(x) = 2x^2 \cdot \frac{1}{1+x^3}.$$

Now, from the first example we've already got a power series for the second term so let's use that to write the function as,

$$fx) = 2x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n}, \quad \text{provided } |x| < 1.$$

Notice that the presence of x's outside of the series will not affect its convergence and so the interval of convergence remains the same.

The last step is to bring the coefficient into the series and we'll be done. When we do this make sure and combine the x's as well. We typically only want a single x in a power series.

$$f(x) = \sum_{n=0}^{\infty} 2(-1)^n x^{3n+2}, \quad \text{provided } |x| < 1.$$

Rk. As we saw in the previous example we can often use previous results to help us out. This is an important idea to remember as it can often greatly simplify our work.

Example. Find a power series representation for the following function and determine its interval of convergence.

$$f(x) = \frac{x}{5-x}.$$

solution. Note that

$$f(x) = x \cdot \frac{1}{5-x}.$$

If we had a power series representation for

$$g(x) = \frac{1}{5-x},$$

we could get a power series representation for $f(x)$. We first notice that in order to use (1), we'll need the number in the denominator to be a one. That's easy enough to get

$$g(x) = \frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}}.$$

Now all we need to do to get a power series representation is to replace the x in the series with $\frac{x}{5}$. We have

$$g(x) = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n, \quad \text{provided } \left|\frac{x}{5}\right| < 1.$$

Say,

$$g(x) = \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n} = \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}.$$

The interval of convergence for this series is,

$$\left|\frac{x}{5}\right| < 1 \quad \Longrightarrow \quad \frac{1}{5}|x| < 1 \quad \Longrightarrow \quad |x| < 5.$$

As a result, we have

$$f(x) = x \cdot \frac{1}{5-x} = x \cdot \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}}.$$

The interval of convergence doesn't change and so it will be $|x| < 5$.
So, hopefully we now have an idea on how to find the power series representation for some functions. Admittedly all of the functions could be related back to (1), but it's a start.
We now need to look at some further manipulation of power series that we will need to do on occasion. We need to discuss differentiation and integration of power series.
Let's start with differentiation of the power series,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Now, we know that if we differentiate a finite sum of terms all we need to do is differentiate each of the terms and then add them back up.
Rk. Nicely enough for us however, it is known that if the power series representation of $f(x)$ has a radius of convergence of $R > 0$ then the term by term differentiation of the power series will also have a radius of convergence of R and (more importantly) will in fact be the power series representation of $f'(x)$ provided we stay within the radius of convergence.
Thus, we have

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

We can now find formulas for higher order derivatives as well now.

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$$
$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n(x-a)^{n-3}.$$

and so on and so forth.
Let's now briefly talk about integration. Just as with the differentiation, when we've got an infinite series we need to be careful about just integration term by term. Much like with derivatives it turns

out that as long as we're working with power series we can just integrate the terms of the series to get the integral of the series itself. In other words,

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n(x-a)^n\,dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

Notice that we pick up a constant of integration, C, that is outside the series here.
Let's summarize the differentiation and integration ideas.
Fact. If $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence of $R > 0$ then the function $f(x)$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$
$$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1},$$

and both of these also have a radius of convergence of R.
Now, let's see how we can use these facts to generate some more power series representations of functions.
Example. Find a power series representation for the following function and determine its radius of convergence.

$$f(x) = \frac{1}{(1-x)^2}.$$

solution. Note that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

Then since we have got a power series representation for $\frac{1}{1-x}$, all that we'll need to do is differentiate that power series to get a power series representation for $f(x)$.

$$f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^{n-1}.$$

Then since the original power series had a radius of convergence of $R = 1$, the derivative, and hence $f(x)$, will also have a radius of convergence of $R = 1$.
Example. Find $f(x)$, such that $f(x) = 1 + 3x^2 + 5x^4 + 7x^6 + \cdots = \sum_{n=1}^{\infty}(2n-1)x^{2(n-1)}$.
solution. Note that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad x \in (-1,1).$$

Substituting x by $-x$, gives

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots, \quad x \in (-1,1).$$

Thus, differentiating these two equalities, respectively, gives

$$\left(\frac{1}{1-x}\right)' = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}, \quad x \in (-1,1),$$
$$\left(\frac{1}{1+x}\right)' = -1 + 2x - 3x^2 + 4x^3 - \cdots = \sum_{n=1}^{\infty} (-1)^n nx^{n-1}, \quad x \in (-1,1).$$

The difference of the last two equalities is

$$2(1 + 3x^2 + 5x^4 + 7x^6 + \cdots) = 2\sum_{n=1}^{\infty}(2n-1)x^{2(n-1)} = \left(\frac{1}{1-x}\right)' - \left(\frac{1}{1+x}\right)'$$
$$= \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} = \frac{2(1+x^2)}{(1-x^2)^2}.$$

Thus, $f(x) = \frac{1+x^2}{(1-x^2)^2}, \quad x \in (-1,1).$

Example. Find a power series representation for the following function and determine its radius of convergence.

$$f(x) = \ln(5-x).$$

solution. Note that

$$\int \frac{1}{5-x}\,dx = -\ln(5-x),$$

and then recall that we have a power series representation for $\frac{1}{5-x}$. Thus,

$$\ln(5-x) = -\int \frac{1}{5-x}\,dx = -\int \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}\,dx = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}.$$

We can find the constant of integration, C, by plugging in a value of x. A good choice is $x = 0$ since that will make the series easy to evaluate

$$\ln(5-0) = C - \sum_{n=0}^{\infty} \frac{0^{n+1}}{(n+1)5^{n+1}} \quad \Longrightarrow \quad \ln 5 = C.$$

So, the final answer is,

$$\ln(5-x) = \ln 5 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}.$$

Note that it is okay to have the constant sitting outside of the series like this. In fact, there is no way to bring it into the series. Finally, because the power series representation had a radius of convergence of $R = 5$, this series will also have a radius of convergence of $R = 5$.

Exercise. How about the a power series for $f(x) = \ln(1+x)$.

solution. Note that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots, \quad \text{for } x \in (-1,1).$$

Integrating the equality gives

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad \text{for } x \in (-1,1).$$

By putting $x = 0$ on both of the equality, we get $C = 0$. Hence, for $x \in (-1, 1)$, we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n},$$

We now need to check out the endpoints below.

- when $x = -1$, the series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n},$$

  by p-series test, we know that $\sum \frac{1}{n}$ is divergent.

- when $x = 1$, the series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n},$$

  by the alternating series test, the series is convergent.

We get the power series for the function $\ln(1 + x)$ below,

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad \text{for } x \in (-1, 1].$$

The following result can be applied to justify the equality at $x = 1$: If $\sum c_n$ converges and $f(x) = \sum_{n=0}^{\infty} c_n x^n (-1 < x < 1)$, then $\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} c_n$.
Example. Find a power series of $f(x) = \tan^{-1} x$.
solution. Note that

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C,$$

and

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad x \in (-1, 1).$$

Substituting x by x^2 gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad x \in (-1, 1).$$

Integrating the equality gives

$$\tan^{-1} x = C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots = C + \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}, \quad x \in (-1, 1).$$

By putting $x = 0$ on both of the equality, we get $C = 0$. Hence for $x \in (-1, 1)$, we have

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$$

We now need to check out the endpoints below. At each of the endpoints $x = \pm 1$, the power series is a convergent alternating series, by Alternating Series Test. We can get the power series for the function $\tan^{-1} x$ below,

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}, \quad \text{for } x \in [-1, 1].$$

Again, the equality holds at both endpoints due to the same result stated above, say,

$$\lim_{x \to (-1)^+} \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} (-1)^{2n+1} = \sum_{n=0}^{\infty} (-1)^{3n+1} \frac{1}{2n+1}$$

$$\lim_{x \to 1^-} \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}.$$

Rk. How to find power series for arbitrary functions, e.g., e^x, $\sin x$. $\cos x$. We need to learn more about the method of Taylor expansion.