Lecture 21 Taylor and Maclaurin Series

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1 Recall last time

Substitution The basic geometric series is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad \text{if } |x| < 1.$$

We could replace x by $-x$, x^2, $-x^2$, and get

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \text{if } |x| < 1,$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots + x^{2n} + \cdots = \sum_{n=0}^{\infty} x^{2n}, \quad \text{if } |x| < 1,$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \text{if } |x| < 1.$$

Example.

$$\frac{1}{2+3x} = \frac{1}{2\left(1+\frac{3x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{1+\frac{3x}{2}} = \frac{1}{2}\left[1 - \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 - \left(\frac{3x}{2}\right)^3 + \cdots\right] = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{2^{n+1}},$$

where $\left|\frac{3x}{2}\right| < 1$, say $|x| < \frac{2}{3} \Longrightarrow -\frac{2}{3} < x < \frac{2}{3}$. At the endpoint $x = -\frac{2}{3}$, the series is $\sum_{n=0}^{\infty} \frac{1}{2} = \infty$; at the endpoint $x = \frac{2}{3}$, the series is $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2}$, which is divergent by the Divergent Test. Thus, the interval of convergence is $I = (-\frac{2}{3}, \frac{2}{3})$. Note that this series is centered at $x = 0$.
To represent below at center $x = 1$,

$$\frac{1}{2+3x} = \sum_{n=0}^{\infty} c_n (x-1)^n$$

and note that

$$\frac{1}{2+3x} = \frac{1}{3(x-1)+1} = \frac{1}{5+3(x-1)} = \frac{1}{5} \cdot \frac{1}{1+\frac{3}{5}(x-1)}, \quad \text{if } \left|\frac{3}{5}(x-1)\right| < 1$$

$$= \frac{1}{5}\left[1 - \frac{3}{5}(x-1) + \frac{3^2}{5^2}(x-1)^2 - \frac{3^3}{5^3}(x-1)^3 + \cdots\right] = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n \cdot (x-1)^n}{5^{n+1}} \quad \text{if } |x-1| < \frac{5}{3},$$

say $-\frac{2}{3} < x < \frac{8}{3}$. At the endpoint $x = -\frac{2}{3}$, the series is $\sum_{n=0}^{\infty} \frac{1}{5}$; at $x = \frac{8}{3}$, the series is $\sum_{n=0}^{\infty} (-1)^n \frac{1}{5}$. By the divergence test, both cases are divergent. Thus, the interval of convergence is $I = \left(-\frac{2}{3}, \frac{8}{3}\right)$.
For power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, we consider the operation below,
Differentiation

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=1}^{\infty} c_n n \cdot (x-a)^{n-1},$$

which is differentiation term by term.
Integration

$$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n \, dx,$$

which is integration term by term.
Rk. Definite integral below

$$\int_a^b \left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) dx = \sum_{n=0}^{\infty} c_n \int_a^b (x-a)^n \, dx, \quad b \in (a-R, a+R).$$

Example. We have known that the convergent geometric series below,

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}, \quad \text{if } |x| < 1,$$

by differentiating on both sides, we have

$$0 + 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdot = \frac{d}{dx}(1-x)^{-1} = \frac{1}{(1-x)^2}, \quad \text{if } |x| < 1,$$

which gives a power series representation of $\frac{1}{(1-x)^2}$. We can specify some series, e.g., taking $x = \frac{1}{2}$, we have

$$1 + 2 \cdot \frac{1}{2} + 3 \cdot \left(\frac{1}{2}\right)^2 + \cdots + n \cdot \left(\frac{1}{2}\right)^{n-1} + \cdot = \frac{1}{(1-\frac{1}{2})^2} = 4,$$

say,

$$\sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^{n-1} = 4,$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

If we differentiate more, we have

$$0 + 2 + 3 \cdot 2x + 4 \cdot 3x^2 + \cdots + n(n-1)x^{n-2} = \frac{d}{dx}(1-x)^{-2} = \frac{2}{(1-x)^3},$$

say

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}.$$

Applications of differentiation and integration of power series for other function

$$1 + t + t^2 + t^3 + \cdots + t^n + \cdots = \frac{1}{1-t}, \quad \text{if } |t| < 1.$$

If we take integration over $[0, x]$, where $|x| < 1$,

$$\int_0^x (1 + t + t^2 + t^3 + \cdots + t^n + \cdots) \, dt = \int_0^x \frac{1}{1-t} \, dt = -\ln(1-t)|_0^x = -\ln(1-x),$$

say,

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^{n+1}}{n+1}+\cdots=-\ln(1-x)\Longrightarrow-\ln(1-x)=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}\frac{x^n}{n}, \quad (1)$$

where $x \in (-1,1)$. Then replace x by $-x$, we have

$$-\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n} \quad \Longrightarrow \quad \ln(1+x)=-\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}, \quad (2)$$

where $x \in (-1,1)$. At the endpoint $x=-1$, the series is $\sum\frac{(-1)^{2n-1}}{n}=-\sum\frac{1}{n}$ is divergent by p-series test ($p=1$). At the endpoint $x=1$, the series is $\sum\frac{(-1)^{n-1}}{n}$ is convergent by the alternating series test. Thus, the interval of convergence if $I=(-1,1]$, the radius $R=1$.
If we take $x=\frac{1}{2}$ for (1), we have

$$-\ln\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n \quad \Longrightarrow \quad \ln 2=\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n.$$

If we take $x=1$ for (2), we have

$$\ln 2=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

Actually, we can consider the n-th partial sum and see the remainder term. Note first that

$$1-x+x^2+\cdots+(-1)^n x^n=\frac{1-(-x)^{n+1}}{1-(-x)}=\frac{1+(-1)^n x^{n+1}}{1+x}, \quad \text{if } x\neq -1, \quad (3)$$

where the decomposition of $1-a^{n+1}=(1-a)(1+a+a^2+a^3+\cdots+a^n)$ with $a=-x$ has been used. Integrating (3) on both sides over $[0,1]$, we have

$$\int_0^1\left[1-x+x^2+\cdots+(-1)^n x^n\right]\,dx=1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^n}{n+1}$$
$$=\int_0^1\frac{1}{1+x}\,dx+(-1)^n\int_0^1\frac{x^{n+1}}{1+x}\,dx=\ln 2+(-1)^n\int_0^1\frac{x^{n+1}}{1+x}\,dx.$$

It turns out to be

$$0\le\left|\ln 2-\left[1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^n}{n+1}\right]\right|=\left|(-1)^{n+1}\int_0^1\frac{x^{n+1}}{1+x}\,dx\right|\le\int_0^1 x^{n+1}\,dx=\frac{1}{n+2}\to 0,$$

as $n\to\infty$, we have $\lim_{n\to\emptyset\infty}R_n=0$, where R_n is remainder term, which gives exactly

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{(-1)^n}{n+1}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}.$$

Of course, we could use the substitution to get the expression of power series for functions, still starting with

$$1+t+t^2+t^3+\cdots+t^n+\cdots=\frac{1}{1-t}, \quad \text{if } |t|<1,$$

by substituting t by $-t$, we have

$$1-t+t^2-t^3+\cdots+(-1)^n t^n+\cdots=\frac{1}{1+t}, \quad \text{if } |t|<1.$$

If then substituting t by t^2, we have

$$1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots = \frac{1}{1+t^2}, \quad |t^2| < 1 \Longrightarrow |t| < 1.$$

Integrating on both sides, then

$$\int_0^x 1\, dt - \int_0^x t^2\, dt + \int_0^x t^4\, dt - \int_0^x t^6\, dt + \cdots = \int_0^x \frac{1}{1+t^2}\, dt = \tan^{-1}(x),$$

say,

$$x - \frac{1}{3}x^3 + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots = \tan^{-1} x, \Longrightarrow \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1},$$

where $x \in (-1, 1)$, at the endpoint $x = -1$, the series is $\sum \frac{(-1)^{3n+1}}{2n+1}$; at the endpoint $x = 1$, the series is $\sum \frac{(-1)^n}{2n+1}$. Both cases are convergent by the alternating series test. The interval of convergence is $I = [-1, 1]$ and the radius of convergence $R = 1$. We have

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Actually, note that

$$1 + (-x^2) + (-x^2)^2 + \cdots + (-x^2)^n = \frac{1 - (-x)^{n+1}}{1 - (-x^2)},$$

where the decomposition of $1 - a^{n+1} = (1-a)(1 + a + a^2 + a^3 + \cdots + a^n)$ with $a = -x^2$ has been used. Integrating on both sides over $[0, 1]$, we have

$$\int_0^1 \left[1 - x^2 + x^4 + \cdots + (-1)^n x^{2n}\right] dx = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1}$$
$$= \int_0^1 \frac{1}{1+x^2}\, dx + \int_0^1 \frac{(-1)^n x^{2n+2}}{1+x^2}\, dx = \tan^{-1} 1 + \int_0^1 \frac{(-1)^n x^{2n+2}}{1+x^2}\, dx,$$

which gives

$$0 \le \left|\tan^{-1} 1 - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1}\right]\right| = \left|\int_0^1 \frac{(-1)^n x^{2n+2}}{1+x^2}\, dx\right| \le \int_0^1 x^{2n+2}\, dx = \frac{1}{2n+3} \to 0,$$

as $n \to \infty$ (since $0 < \frac{1}{1+x^2} \le 1$), thus $\lim_{n\to\infty} R_n = 0$, which gives exactly

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Thus,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad \text{if } x \in [-1, 1].$$

2 Taylor and Maclaurin Series

In the previous section we started looking at writing down a power series representation of a function. The problem with the approach in that section is that everything came down to needing to be able to relate the function in some way to $\frac{1}{1-x}$, and while there are many functions out there that can be related to this function there are many more that simply can't be related to this.
So, without taking anything away from the process we looked at in the previous section, what we need to do is come up with a more general method for writing a power series representation for a function.
Intuition (how to find the power series of e^x). Note first the FTC that

$$\begin{aligned} e^x - 1 = [e^t]|_0^x = \int_0^x e^t\,dt = \int_0^x (-e^t)\,d(x-t) = -e^t(x-t)|_0^x + \int_0^x (x-t)e^t\,dt \\ = x + \int_0^x (-e^t)\,d\frac{(x-t)^2}{2!} = x - e^t \cdot \frac{(x-t)^2}{2!}|_0^x + \int_0^x \frac{(x-t)^2}{2!}e^t\,dt \\ = x + \frac{x^2}{2!} + \int_0^x (-e^t)\,d\frac{(x-t)^3}{3!} = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \int_0^x (-e^t)\,d\frac{(x-t)^4}{4!} = \cdots \end{aligned}$$

where the integration by parts has been used. Thus,

$$e^x - \left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right] = \int_0^x \frac{(x-t)^n}{n!} \cdot e^t\,dt =: R_n(x),$$

For the remainder term,

$$0 \le \frac{(x-t)^n}{n!}e^t \le \frac{e^x(x-t)^n}{n!} \quad \Longrightarrow \quad R_n(x) = \int_0^x \frac{(x-t)^n}{n!}e^t\,dt \le \frac{e^x}{n!}\int_0^x (x-t)^n\,dt$$
$$= \frac{e^x}{n!}\left[\frac{(x-t)^{n+1}}{n+1}\right]|_0^x = \frac{e^x x^{n+1}}{(n+1)!}$$

Since that $\lim_{n\to\infty} \frac{e^x x^{n+1}}{(n+1)!} = 0$ for all $x \Longrightarrow \lim_{n\to\infty} R_n(x) = 0$, turn outs

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in (-\infty, \infty).$$

Using the idea of integration by parts for the power series of function f at $x = a$,

$$\begin{aligned} f(x) - f(a) = \int_a^x f'(t)\,dt = \int_a^x (-f'(t))\,d(x-t) = -f'(t)(x-t)|a^x + \int_a^x (x-t)f''(t)\,dt \\ = f'(a)(x-a) + \int_a^x (-f''(t))\,d\frac{(x-t)^2}{2!} \\ = f'(a)(x-a) - f''(t) \cdot \frac{(x-t)^2}{2!}|_a^x + \int_a^x \frac{(x-t)^2}{2!}f'''(t)\,dt \\ = f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt. \end{aligned}$$

Thus,

$$\left| f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a^2) + \cdots + + \frac{f^{(n)}(a)}{n!}(x-a)^n \right) \right|$$
$$= \left| \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \right| =: R_n(x)$$

If $\lim_{n\to\infty} R_n(x) = 0$, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a^2) + \cdots + + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Another idea of differentiation: So, for the time being, let's make two assumptions.

- Assume that $f(x)$ does in fact have a power series representation about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

- Assume that $f(x)$ has derivatives of every order and that we can in fact find them all.

Now that we've assumed that a power series representation exists we need to determine what the coefficients, c_n, are.
Let's first just evaluate everything at $x = a$. This gives

$$f(a) = c_0.$$

Let's take derivative of function $f(x)$ and then plug in $x = a$, we have

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^3 + 4c_4(x-a)^3 + \cdots$$
$$f'(a) = c_1,$$

and we now know c_1. Let's continue with this idea and find the second derivative.

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \cdots$$
$$f''(a) = 2c_2,$$

so $c_2 = \frac{f''(a)}{2}$. Using the third derivative gives,

$$f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + \cdots$$
$$f'''(a) = 3\cdot 2c_3 \quad \Longrightarrow \quad c_3 = \frac{f'''(a)}{3\cdot 2}.$$

Using the fourth derivative gives,

$$f^{(4)}(x) = 4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5(x-a) + \cdots$$
$$f^{(4)}(a) = 4\cdot 3\cdot 2c_4, \quad \Longrightarrow \quad c_4 = \frac{f^{(4)}(a)}{4\cdot 3\cdot 2}.$$

Hopefully by this time you have seen the pattern here. It looks like, in general, we've got the following formula for the coefficients.

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

This even works for $n = 0$ if you recall that $0! = 1$ and define $f^{(0)}(x) = f(x)$.
So, provided a power series representation for the function $f(x)$ about $x = a$ exists the **Taylor Series** for $f(x)$ about $x = a$ is
Fact (Taylor Series).

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

If we use $a = 0$, so we are talking about the Taylor Series about $x = 0$, we call the series a **Maclaurin Series** for $f(x)$ below.
Fact (Maclaurin Series).

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots
\end{aligned}
$$

Before working any examples of Taylor Series we first need to address the assumption that a Taylor Series will in fact exist for a given function. Let's start out with some notation and definitions that we'll need.
To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the n^{th} **degree Taylor polynomial** of $f(x)$ as,

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Note that this really is a polynomial of degree at most n. If we were to write out the sum without the summation notation this would clearly be an n^{th} degree polynomial. Notice as well that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

the n^{th} degree Taylor polynomial is just the partial sum for the series.
Next, the **remainder** is defined to be,

$$R_n(x) = f(x) - T_n(x),$$

So, the remainder is really just the *error* between the function $f(x)$ and the n^{th} degree Taylor polynomial for a given n. Thus,

$$f(x) = T_n(x) + R_n(x).$$

We now have the following Theorem.
Theorem. Suppose that $f(x) = T_n(x) + R_n(x)$. Then if

$$\lim_{n\to\infty} R_n(x) = 0,$$

for $|x-a| < R$ then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

on $|x-a| < R$.
Rk. In general, showing that $\lim_{n\to\infty} R_n(x) = 0$ is a somewhat difficult process and so we will be assuming that this can be done for some R in all of the examples that we'll be looking at.
We give the remainder formulas below,
Theorem (integral form of the remainder). If $f^{(n+1)}$ is continuous over an open interval I that contains a, then

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt, \quad x \in I.$$

Proof. This remainder formula could be proved by induction.
For $n = 0$, the formula gives

$$\frac{1}{0!}\int_a^x (x-t)^0 f'(t)\, dt = f(x) - f(a) = f(x) - T_0(x) = R_0(x).$$

So the formula is true for $n = 0$. Suppose the formula holds for $n = k$:

$$R_k(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\, dt, \quad x \in I.$$

Then, by integration by parts, we get

$$\frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\, dt$$
$$= \left[\frac{1}{(k+1)!}(x-t)^{k+1} f^{(k+1)}(t)\right]\Big|_a^x + \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\, dt$$
$$= R_k(x) - \frac{f^{(k+1)}(a)}{(k+1)}(x-a)^{k+1} = f(x) - T_k(x) - \frac{f^{(k+1)}(a)}{(k+1)}(x-a)^{k+1} = f(x) - T_{k+1} = R_{k+1}(x).$$

So the formula is also true for $n = k+1$. Hence, the remainder formula holds for all $n \geq 0$. □

By the Intermediate Value Theorem, we can easily get the variation below,
Theorem (Lagrange remainder of Taylor series.) If $f^{(n+1)}$ is continuous over I that contains a, then there is $c \in [a, x]$, such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Theorem (remainder estimate). If $|f^{(n+1)}(x)| \leq M$ over I that contains a, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}, \quad x \in I.$$

Thus, for each $x \in I$, since $\lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1!)} = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad x \in I.$$

Theorem (convergence theorem of Taylor series) If $\lim_{n\to\infty} R_n(x) = 0$ over I that contains a, then

$$f(x) = \lim_{n\to\infty} T_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad x \in I.$$

Now let's look at some examples.
Example. Find the Taylor Series for $f(x) = e^x$ about $x = 0$.
solution. Note that

$$f^{(n)}(x) = e^x \quad n = 0, 1, 2, 3, \cdots$$

and so, $f^{(n)}(0) = e^0 = 1$. Therefore, the Taylor series for $f(x) = e^x$ about $x = 0$ is,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Rk. To find the interval of convergence, we use the ratio test below,

$$L = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1, \quad \text{for all } x.$$

Thus, the interval of convergence is $I = (-\infty, \infty)$, the radius of convergence $R = \infty$. Of course, $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is convergent to e^x for all x, since that the remainder term (different form, let's take the Lagrange form remainder) is

$$\lim_{n\to\infty}|R_n(x)| = \lim_{n\to\infty}\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| = e^c\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} \le e^x\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0, \quad \text{for all } x,$$

where $c \in [0, x] \subset (-R, R) \implies e^c \le e^x$, and $f^{(n+1)}(c) = e^c$, note that $|x|^{n+1} \ll (n+1)!$ as $n \to \infty$.

Example. Find the Taylor Series for $f(x) = e^{-x}$ about $x = 0$.

solution. **Method 1**. We need to replace the x in the Taylor Series with $-x$ below,

$$e^{-x} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}.$$

Method 2. Let's first take some derivatives and evaluate them at $x = 0$.

$$\begin{aligned}
f^{(0)}(x) &= e^{-x}, &&\implies f^{(0)}(0) = 1,\\
f^{(1)}(x) &= -e^{-x}, &&\implies f^{(1)}(0) = -1,\\
f^{(2)}(x) &= e^{-x}, &&\implies f^{(2)}(0) = 1,\\
f^{(3)}(x) &= -e^{-x}, &&\implies f^{(3)}(0) = -1\\
\cdots & \cdots\\
f^{(n)}(x) &= (-1)^n e^{-x}, &&\implies f^{(n)}(0) = (-1)^n.
\end{aligned}$$

Thus, we have

$$e^{-x} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}.$$

Rk. To find the interval of convergence, we use the ratio test below,

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|-\frac{1}{n+1}x\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1, \quad \text{for all } x.$$

Thus, the interval of convergence is $I = (-\infty, \infty)$; the radius of convergence is $R = \infty$. Of course, $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ is convergent to e^{-x} for all x, since that the Lagrange remainder term as $n \to \infty$ would be

$$\lim_{n\to\infty}|R_n(x)| = \lim_{n\to\infty}\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n e^{-c}}{(n+1)!}x^{n+1}\right| \le \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0, \quad \text{for all } x$$

where $c \in [0, x] \implies e^{-c} \le 1$ and note that $|x|^{n+1} \ll (n+1)!$ as $n \to \infty$.

Example. Find the Taylor Series for $f(x) = x^4 e^{-3x^2}$ about $x = 0$.

solution. For this example, we will take advantage of the fact that we already have a Taylor Series for e^x about $x = 0$.

$$\begin{aligned}
x^4 e^{-3x^2} &= x^4\sum_{n=0}^{\infty}\frac{(-3x^2)^n}{n!} = x^4\sum_{n=0}^{\infty}\frac{(-3)^n x^{2n}}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(-3)^n x^{2n+4}}{n!}.
\end{aligned}$$

Rk. By the ratio test, we have

$$L = \lim_{n\to\infty} \left| \frac{(-3)^{n+1}x^{2(n+1)+4}}{(n+1)!} \cdot \frac{n!}{(-3)^n x^{2n+4}} \right| = x^2 \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1, \quad \text{for all } x.$$

Thus, the interval of convergence is $I = (-\infty, \infty)$; The radius $R = \infty$. Of course, $\sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+4}}{n!}$ is convergent to $x^4 e^{-3x^2}$ for all x, since that the Lagrange remainder for e^{-x} goes to 0 as $n \to \infty$. Thus, when we do the substitution and multiplication of some polynomial, this gives us the convergent function.

To this point we've only looked at Taylor Series about $x = 0$ (also known as Maclaurin Series) so let's take a look at a Taylor Series that isn't about $x = 0$.

Example. Find the Taylor Series for $f(x) = e^{-x}$ about $x = -4$.

solution. Finding a general formula for $f^{(n)}(-4)$ is fairly simple.

$$f^{(n)}(x) = (-1)^n e^{-x}, \quad f^{(n)}(-4) = (-1)^n e^4.$$

The Taylor Series is then,

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n e^4}{n!}(x+4)^n.$$

We now need to work some examples that don't involve the exponential function since these will tend to require a little more work.

Rk. By the ratio test, we have

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}e^4}{(n+1)!}(x+4)^{n+1} \cdot \frac{n!}{(-1)^n e^4 (x+4)^n} \right| = |x+4| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1, \quad \text{for all } x.$$

Thus, the interval of convergence is $I = (-\infty, \infty)$; the radius $R = \infty$. Of course, the series is convergent to e^{-x} for all x, since that $\lim_{n\to\infty} |R_n(x)| = 0$ over I below for the Lagrange remainder.

$$\begin{aligned}\lim_{n\to\infty} |R_n(x)| &= \lim_{n\to\infty} \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-(-4))^{n+1} \right| \\ &= e^{-c} \lim_{n\to\infty} \frac{|x+4|^{n+1}}{(n+1)!} \le e^{-x} \lim_{n\to\infty} \frac{|x+4|^{n+1}}{(n+1)!} = 0, \quad \text{for all } x,\end{aligned}$$

where $c \in [-4, x]$, and $|x+4|^{n+1} \ll (n+1)!$.

Example. Find the Taylor Series for $f(x) = \cos(x)$ about $x = 0$.

solution. First, we'll need to take some derivatives of the function and evaluate them at $x = 0$.

$$\begin{aligned}
&f^{(0)}(x) = \cos x, \quad f^{(0)}(0) = 1 \\
&f^{(1)}(x) = -\sin x, \quad f^{(1)}(0) = 0 \\
&f^{(2)}(x) = -\cos x, \quad f^{(2)}(0) = -1 \\
&f^{(3)}(x) = \sin x, \quad f^{(3)}(0) = 0 \\
&f^{(4)}(x) = \cos x, \quad f^{(4)}(0) = 1 \\
&f^{(5)}(x) = -\sin x, \quad f^{(5)}(0) = 0 \\
&f^{(6)}(x) = -\cos x, \quad f^{(6)}(0) = -1 \\
&\cdots \quad \cdots \quad \cdots
\end{aligned}$$

In this example, unlike the previous ones, there is not an easy formula for either the general derivative or the evaluation of the derivative. However, there is a clear pattern to the evaluations. So, let's plug what we've got into the Taylor series and see what we get,

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\ &= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 - \frac{1}{6!}x^6 + \cdots\end{aligned}$$

Thus, we have

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

we can actually come up with a general formula for the Taylor Series below,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad x \in (-\infty, \infty).$$

Rk. By the Ratio Test, we have

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| = |x^2| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0, \quad \text{for all } x.$$

Thus, the interval of convergence is $I = (-\infty, \infty)$; the radius $R = \infty$. Of course, the series is convergent to $\cos x$ for all x, since that the Lagrange remainder term as $n \to \infty$ would be

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \right| \le \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0, \quad \text{for all } x,$$

where $c \in [0, c]$ and $|f^{(n+1)}(c)| \le 1$, note that $|x|^{n+1} \ll (n+1)!$ as $n \to \infty$.

Example. Find the Taylor Series for $f(x) = \sin(x)$ about $x = 0$.

solution. As with the last example we'll start off in the same manner.

$$\begin{aligned} &f^{(0)}(x) = \sin x \quad f^{(0)}(0) = 0 \\ &f^{(1)}(x) = \cos x \quad f^{(1)}(0) = 1 \\ &f^{(2)}(x) = -\sin x \quad f^{(2)}(0) = 0 \\ &f^{(3)}(x) = -\cos x \quad f^{(3)}(0) = -1 \\ &f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0 \\ &f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1 \\ &f^{(6)}(x) = -\sin x \quad f^{(6)}(0) = 0 \\ &\cdots \quad \cdots \quad \cdots \end{aligned}$$

So, we get a similar pattern for this one. Let's plug the numbers into the Taylor Series.

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\end{aligned}$$

In this case we only get terms that have an odd exponent on x and as with the last problem once we ignore the zero terms there is a clear pattern and formula. So renumbering the terms as we did in the previous example we get the following Taylor Series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We saw the remainder term given below (note that $|f^{(k)}| \le 1$),

$$|R_n(x)| = \left|\sin x - \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right]\right| = \left|\int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} f^{(2n+2)}(t)\, dt\right|$$
$$\le \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!}\, dt = \left[-\frac{(x-t)^{2n+2}}{(2n+2)!}\right]\Big|_0^x = \frac{x^{2n+2}}{(2n+2)!},$$

thus, $\lim_{n\to\infty} R_n(x) = 0$ for all x (since that $x^{2n+2} \ll (2n+2)!$ as $n \to \infty$ for fixed x), say the interval of convergence is $I = (-\infty, \infty)$ and the radius $R = \infty$ (this could be done by the ratio test as well).

We really need to work another example or two in which $f(x)$ isn't about $x = 0$.

Exercise. Find the Taylor Series for $f(x) = \sin(x)$ at $x = \pi$.

Example. For $f(x) = \tan^{-1}(x)$, find the higher order derivative $f^{(2019)}(0)$.

solution. Note that the Maclaurin series below,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad x \in [-1, 1].$$

By the uniqueness of Taylor expansion, we have

$$\frac{f^{(2n+1)}(0)}{(2n+1)!} = \frac{(-1)^n}{2n+1}, \quad n \ge 0,$$

Thus,

$$f^{(2019)}(0) = (-1)^{1009} \cdot 2018! = -2018!.$$

Example. Find the Taylor Series for $f(x) = \ln x$ about $x = 2$.

solution. Here are the first few derivatives and the evaluations.

$$\begin{aligned}
&f^{(0)}(x) = \ln x && f^{(0)}(2) = \ln 2 \\
&f^{(1)}(x) = \frac{1}{x} && f^{(1)}(2) = \frac{1}{2} \\
&f^{(2)}(x) = -\frac{1}{x^2} && f^{(2)}(2) = -\frac{1}{2^2} \\
&f^{(3)}(x) = \frac{2}{x^3} && f^{(3)}(2) = \frac{2}{2^3} \\
&f^{(4)}(x) = -\frac{2\cdot 3}{x^4} && f^{(4)}(2) = -\frac{2\cdot 3}{2^4} \\
&f^{(5)}(x) = \frac{2\cdot 3\cdot 4}{x^5} && f^{(5)}(2) = \frac{2\cdot 3\cdot 4}{2^5} \\
&\cdots \quad \cdots && \cdots \\
&f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} && f^{(n)}(2) = \frac{(-1)^{n+1}(n-1)!}{2^n}
\end{aligned}$$

Note that while we got a general formula here it doesn't work for $n = 0$. This will happen on occasion so don't worry about it when it does. In order to plug this into the Taylor Series formula we'll need to strip out the $n = 0$ term first.

$$\begin{aligned}\ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= f(2) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!2^n}(x-2)^n \\ &= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n\end{aligned}$$

Rk. By the ratio test, we have

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^{n+1}(x-2)^n} \right| = |x-2| \cdot \frac{1}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-2|}{2} < 1,$$

say $|x-2| < 2 \implies -2 < x-2 < 2 \implies 0 < x < 4$. At the endpoint $x = 0$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(-1)^n 2^n = \sum_{n=1}^{\infty} \left(-\frac{1}{n}\right),$$

which is divergent by the p-series test. At the endpoint $x = 4$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} 2^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n},$$

which is convergent alternating series by the alternating series test. Thus, the interval of convergence is $I = (0, 4]$. The series is convergent to $\ln x$ over I, since that the Lagrange remainder term as $n \to \infty$ would be

$$\begin{aligned}\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-2)^{n+1} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2} n!}{c^{n+1}}}{(n+1)!}(x-2)^{n+1} \right| \\ &= \lim_{n\to\infty} \left| \frac{x-2}{c} \right|^{n+1} \frac{1}{n+1} \le \lim_{n\to\infty} \frac{1}{n+1} = 0, \quad \text{for } x \in (0,4],\end{aligned}$$

where $c \in [2, x]$, $x \in (0, 4]$, and

$$|x-2| < 2 \implies \frac{1}{2} \le \frac{|x-2|}{c} \le 1 \implies \left(\frac{1}{2}\right)^{n+1} \le \left| \frac{x-2}{c} \right|^{n+1} \le 1.$$

Rk. About the term sitting in front of the series, sometimes we need to do that when we can't get a general formula that will hold for all values of n.
Example. Find the Taylor Series for $f(x) = \frac{1}{x^2}$ about $x = -1$.

solution. Again, here are the derivatives and evaluations.

$$
\begin{aligned}
&f^{(0)}(x)=\frac{1}{x^2} \quad f^{(0)}(-1)=\frac{1}{(-1)^2}=1\\
&f^{(1)}(x)=-\frac{2}{x^3} \quad f^{(1)}(-1)=-\frac{2}{(-1)^3}=2\\
&f^{(2)}(x)=\frac{2\cdot 3}{x^4} \quad f^{(2)}(-1)=\frac{2\cdot 3}{(-1)^4}=2\cdot 3\\
&f^{(3)}(x)=-\frac{2\cdot 3\cdot 4}{x^5} \quad f^{(3)}(-1)=-\frac{2\cdot 3\cdot 4}{(-1)^5}=2\cdot 3\cdot 4\\
&\cdots \quad \cdots \quad \cdots\\
&f^{(n)}(x)=\frac{(-1)^n(n+1)!}{x^{n+2}} \quad f^{(n)}(-1)=\frac{(-1)^n(n+1)!}{(-1)^{n+2}}=(n+1)!
\end{aligned}
$$

Notice that all the negative signs will cancel out in the evaluation. Also, this formula will work for all n, unlike the previous example.
Here is the Taylor Series for this function.

$$
\begin{aligned}
\frac{1}{x^2}&=\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n\\
&=\sum_{n=0}^{\infty}\frac{(n+1)!}{n!}(x+1)^n=\sum_{n=0}^{\infty}(n+1)(x+1)^n.
\end{aligned}
$$

Rk. By the ratio test, we have

$$L=\lim_{n\to\infty}\left|\frac{(n+2)(x+1)^{n+1}}{(n+1)(x+1)^n}\right|=|x+1|\lim_{n\to\infty}\frac{n+2}{n+1}=|x+1|<1,$$

say $-1<x+1<1 \implies -2<x<0$. At the endpoint $x=-2$, the series $\sum_{n=0}^{\infty}(n+1)(-1)^n$ which is divergent by the Divergent test; at the endpoint $x=0$, the series $\sum_{n=0}^{\infty}(n+1)$ is divergent by the divergent test. Thus, the interval of convergence is $I=(-2,0)$; the radius $R=1$.
Example. Find the Taylor Series for $f(x)=x^3-10x^2+6$ about $x=3$.
solution. Here are the derivatives for this problem.

$$
\begin{aligned}
&f^{(0)}(x)=x^3-10x^2+6 \quad f^{(0)}(3)=-57\\
&f^{(1)}(x)=3x^2-20x \quad f^{(1)}(3)=-33\\
&f^{(2)}(x)=6x-20 \quad f^{(2)}(3)=-2\\
&f^{(3)}(x)=6 \quad f^{(3)}(3)=6\\
&f^{(n)}(x)=0 \quad f^{(4)}(3)=0, \quad n\geq 4
\end{aligned}
$$

This Taylor series will terminate after $n=3$. This will always happen when we are finding the Taylor Series of a polynomial. Here is the Taylor Series for this one.

$$
\begin{aligned}
x^3-10x^2+6&=\sum_{n=0}^{\infty}\frac{f^{(n)}(3)}{n!}(x-3)^n\\
&=f(3)+f'(3)(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3+0\\
&=-57-33(x-3)-(x-3)^2+(x-3)^3.
\end{aligned}
$$

Rk. When finding the Taylor Series of a polynomial we don't do any simplification of the right-hand side. We leave it like it is. In fact, if we were to multiply everything out we just get back to the original polynomial! While it's not apparent that writing the Taylor Series for a polynomial is useful there are times where this needs to be done.
Rk. So, we've seen quite a few examples of Taylor Series to this point and in all of them we were able to find general formulas for the series. This won't always be the case. To see an example of one that doesn't have a general formula, we will introduce some examples in the next section.
There are three important Taylor Series that we've derived in this section that we should summarize up

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Summary

Theorem (common Maclaurin series and their radii of convergence)

1. $\displaystyle \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \quad x \in (-1,1), \quad R = 1,$
2. $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \quad x \in (-\infty,\infty), \quad R = \infty,$
3. $\displaystyle \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots, \quad x \in (-\infty,\infty), \quad R = \infty,$
4. $\displaystyle \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots, \quad x \in (-\infty,\infty), \quad R = \infty$
5. $\displaystyle \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \quad x \in [-1,1], \quad R = 1$
6. $\displaystyle \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots, \quad x \in (-1,1], \quad R = 1$
7. $\displaystyle (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots,$
   at least for $x \in (-1,1)$, $R = 1$.