Lecture 22 Applications of Taylor Series

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1 Applications of Series

Limits by Taylor Series

Example. Find

$$\lim_{x\to 0}\frac{x^3-6x\cos(x^2)+6\sin x}{x^5}.$$

solution. The limit has the indeterminate form $\frac{0}{0}$. Although we may apply l'Hospital's Rule, alternatively we use the technique of Taylor series. In fact, near $x=0$, we have

$$\begin{aligned}\sin x &= x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots,\\ \cos(x^2) &= 1-\frac{1}{2!}(x^2)^2+\cdots,\end{aligned}$$

then as $n\to\infty$,

$$\begin{aligned}\sin x &= x-\frac{1}{6}x^3+\frac{1}{120}x^5=o(|x|^5),\\ \cos(x^2) &= 1-\frac{1}{2}x^4+o(|x|^4),\end{aligned}$$

thus,

$$\begin{aligned}\lim_{x\to 0}\frac{x^3-6x\cos(x^2)+6\sin x}{x^5} &= \lim_{x\to 0}\frac{x^3-6x\left[1-\frac{1}{2}x^4+o(|x|^4)\right]+6\left[x-\frac{1}{6}x^3+\frac{1}{120}x^5+o(|x|^5)\right]}{x^5}\\ &= \lim_{x\to 0}\frac{\frac{61}{20}x^5+o(|x|^5)}{x^5}=\frac{61}{20}.\end{aligned}$$

Example. Find

$$\lim_{x\to\infty}\left[x^4\sin(x^{-4})-x^2e^{-1/x^2}+x^2\right].$$

solution. If making the substitution $x=t^{-1}$, we see that $x\to\infty \iff t\to 0^+$. Thus,

$$\begin{aligned}\lim_{x\to\infty}\left[x^4\sin(x^{-4})-x^2e^{-1/x^2}+x^2\right] &= \lim_{t\to 0^+}\left[t^{-4}\sin(t^4)-t^{-2}e^{-t^2}+t^{-2}\right]\\ &= \lim_{t\to 0^+}\frac{\sin(t^4)-t^2e^{-t^2}+t^2}{t^4}.\end{aligned}$$

The last limit has the indeterminate form $\frac{0}{0}$. We use the technique of Taylor series. In fact, near $t=0$, we have

$$\begin{aligned}\sin(t^4) &= t^4+\cdots\\ e^{-t^2} &= 1+\frac{1}{1!}(-t^2)+\cdots,\end{aligned}$$

then as $t \to 0^+$,

$$\sin(t^4) = t^4 + o(|t|^4),$$
$$e^{-t^2} = 1 - t^2 + o(|t|^2),$$

thus,

$$\lim_{t\to 0^+} \frac{\sin(t^4) - t^2e^{-t^2} + t^2}{t^4} = \lim_{t\to 0^+} \frac{[t^4 + o(|t|^4)] - t^2[1 - t^2 + o(|t|^2)] + t^2}{t^4} = \lim_{t\to 0^+} \frac{2t^4 + o(|t|^4)}{t^4} = 2.$$

In turn,

$$\lim_{x\to\infty} \left[x^4 \sin(x^{-4}) - x^2e^{-1/x^2} + x^2\right] = 2.$$

Approximation of Definite Integrals

Example (approximation of a definite integral). Approximate $\int_0^1 e^{-x^2}\ dx$ within 10^{-4} of the exact value.

solution. Starting with the Maclaurin series,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad x \in (-\infty, \infty).$$

By replacing x with $-x^2$, we have

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots, \quad x \in (-\infty, \infty).$$

Thus,

$$\int_0^1 e^{-x^2}\ dx = \left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right)\Big|_0^1$$
$$= 1 - \frac{1}{3\cdot 1!} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cdot n!}.$$

This shows that the definite integral can be expressed as a convergent alternating series, with $b_n = \frac{1}{(2n+1)\cdot n!}$ satisfying

$$0 < b_{n+1} < b_n, \quad \lim_{n\to\infty} b_n = 0.$$

Using the bound on the remainder in the Alternating Series Test, we have

$$|R_n| = \left|\int_0^1 e^{-x^2}\ dx - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)\cdot k!}\right| < b_{n+1} = \frac{1}{(2n+3)\cdot(n+1)!}.$$

Thus, to ensure that $|R_n| < 10^{-4}$, we need to choose n, such that

$$\frac{1}{(2n+3)\cdot(n+1)!} < 10^{-4}, \quad \Longrightarrow \quad n > 6.$$

Thus, the finite series $\sum_{k=0}^{6} \frac{(-1)^k}{(2k+1)\cdot k!}$ gives an approximation that is within 10^{-4} of the exact value of the series.

Thus, we have

$$\int_0^1 e^{-x^2}\ dx \approx s_6 \approx 0.74684.$$

For comparison, $\int_0^1 e^{-x^2}\, dx = 0.7468241324....$ We see that our approximation indeed is within 10^{-4} of the exact value of the series.

Solving Differential Equations

Solve the differential equation $y'(t) = y(t) - 1$ subject to the initial value condition $y(0) = 3$.

solution. We assume the solution has a power series solution of the form

$$y(t) = \sum_{n=0}^{\infty} c_n t^n = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots,$$

where the coefficients $\{c_n\}$ will be determined. Since that $y(0) = 0$, we have $c_0 = 3$. The other coefficients can be determined one by one. In fact, substituting the series solution into the differential equation gives

$$c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + \cdots = (c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots) - 1.$$

Putting $t = 0$ on both sides of the equation gives

$$c_1 = c_0 - 1 \Longrightarrow c_1 = 2.$$

We differentiate both sides of the above equation to have

$$2c_2 + 3 \cdot 2c_3 t + 4 \cdot 3c_4 t^2 + \cdots = (c_1 + 2c_2 t + 3c_3 t^2 + \cdots) - 0.$$

Again, putting $t = 0$ on both sides of the last equation gives

$$2c_2 = c_1 \quad \Longrightarrow \quad c_2 = 1.$$

We repeat the above process by differentiating the equation once more then putting $t = 0$ on the resulted equation. Recursively, we get

$$c_3 = \frac{1}{3}, \quad c_4 = \frac{1}{4 \cdot 3}, \quad c_5 = \frac{1}{5 \cdot 4 \cdot 3}, \quad \cdots$$

Thus, we obtain an "formal" solution below,

$$y(t) = 3 + 2t + t^2 + \frac{t^3}{3} + \frac{t^4}{4 \cdot 3} + \frac{t^5}{5 \cdot 4 \cdot 3} + \cdots$$

The formal solution can be re-written as

$$y(t) = 1 + 2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right).$$

We recognize that the right-hand side is actually the Maclaurin series of the function $1+2e^t$. Finally, we can verify that the formally obtained solution is indeed the true solution of the original problem, since

$$\begin{aligned} y'(t) &= (1+2e^t)' = 2e^t = (1+2e^t) - 1 = y(t) - 1; \\ y(0) &= (1+2e^t)|_{t=0} = 3. \end{aligned}$$

Example (from classviva.org). Let $F(x) = \int_0^x e^{-4t^4}\, dt$. Find the MacLaurin polynomial of degree 5 for $F(x)$ and use this polynomial to estimate the value of $\int_0^{0.7} e^{-4x^4}\, dx$

solution. Note that the MacLaurin Series below,

$$\begin{aligned} e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots, \quad t \in (-\infty, \infty), \\ e^{-4t^4} &= \sum_{n=0}^{\infty} \frac{(-4t^4)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \frac{t^{4n}}{n!} = 1 + \frac{(-4t^4)}{1!} + \frac{(-4t^4)^2}{2!} + \frac{(-4t^4)^3}{3!} + \cdots. \end{aligned}$$

As a result,

$$F(x) = \int_0^x e^{-4t^4}\, dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n \cdot 4^n \frac{t^{4n}}{n!} \right) dx = \sum_{n=0}^{\infty} (-1)^n 4^n \frac{1}{n!} \int_0^x t^{4n}\, dx$$
$$= \sum_{n=0}^{\infty} (-1)^n 4^n \frac{1}{n!} \frac{x^{4n+1}}{4n+1}.$$

Thus, the Maclaurin polynomial is

$$M_n(x) = \sum_{k=0}^{n} (-1)^k 4^k \frac{1}{k!} \frac{x^{4k+1}}{4k+1}.$$

To find the degree less than 5, we should take $n = 1$, we have

$$M_1(x) = x - 4 \cdot \frac{x^5}{5} = x - \frac{4}{5} x^5.$$

Since that

$$\int_0^{0.7} e^{-4x^4}\, dt = F(0.7) \approx M_1(0.7) = \left(x - \frac{4}{5} x^5 \right) |_{x=0.7} = 0.7 - \frac{4}{5} (0.7)^5 = 0.565544.$$

Example (from classviva.org). Let $T_6(x)$: be the Taylor polynomial of degree 6 of the function $f(x) = \ln(1+x)$ at $a = 0$. Suppose you approximate $f(x)$ by $T_6(x)$, find all positive values of x for which this approximation is within 0.001 of the right answer. (Hint: use the alternating series approximation.) $(0 < x \leq ?)$.

solution. Note that the MacLaurin series below,

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots, \quad x \in (-1, 1], \quad R = 1.$$

Thus,

$$f(x) \approx T_6(x) = \sum_{k=1}^{6} (-1)^{k-1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}.$$

Using the bound on the remainder in the alternating series test, we have

$$|R_6| = |f(x) - T_6(x)| \leq b_7 = \frac{x^7}{7}, \quad \text{where } x > 0,$$

to find $x > 0$, such that $|R_6| \leq 10^{-3}, \Longrightarrow \frac{x^7}{7} \leq 10^{-3}, \Longrightarrow 0 < x \leq \sqrt[7]{7 \times 10^{-3}}$.

Example (from classviva.org). Compute the 9-th derivative of

$$f(x) = \tan^{-1} \left(\frac{x^3}{5} \right),$$

at $x = 0$. Find $f^{(9)}(0)$. (Hint: Use the MacLaurin series for $f(x)$).

solution. Note that the Maclaurin series below,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad x \in [-1, 1],$$
$$\tan^{-1} \left(\frac{x^3}{5} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1) \cdot 5^{2n+1}} \quad x \in [-1, 1].$$

By the uniqueness of Taylor expansion, we have

$$\frac{f^{(6n+3)}(0)}{(6n+3)!} = (-1)^n \frac{1}{(2n+1)\cdot 5^{2n+1}}, \quad n \geq 0.$$

Thus,

$$f^{(9)}(0) = (9!)\cdot(-1)\cdot\frac{1}{3\cdot 5^3} = -\frac{9!}{3\cdot 5^3}.$$

Now, that we know how to represent function as power series we have talked about at least a couple of applications of series.

There are in fact many applications of series, unfortunately most of them are beyond the scope of this course. One application of power series (with the occasional use of Taylor Series) is in the field of Ordinary Differential Equations when finding Series Solutions to Differential Equations.

Another application of series arises in the study of Partial Differential Equations. One of the more commonly used methods in that subject makes use of Fourier Series (here, we do not talk about that trigonometric series).

Example. Determine a Taylor Series about $x=0$ for the following integral.

$$\int \frac{\sin x}{x}\, dx.$$

solution. To do this we will first need to find a Taylor Series about $x=0$ for the integrand. We already have a Taylor Series for sine about $x=0$, so we'll just use that as follows,

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}.$$

Thus, from term by term integration.

$$\int \frac{\sin x}{x}\, dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}\, dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}.$$

So, while we can not integrate this function in terms of known functions we can come up with a series representation for the integral.

This idea of deriving a series representation for a function instead of trying to find the function itself is used quite often in several fields. In fact, there are some fields where this is one of the main ideas used and without this idea it would be very difficult to accomplish anything in those fields.

2 Other examples

Example. Find the first three non-zero terms in the Taylor Series for $f(x) = e^x \cos x$ about $x=0$.

solution. Before we start let's acknowledge that the easiest way to do this problem is to simply compute the first 4 derivative evaluate them at $x=0$ plug into the formula and we would be done. However, as we noted prior to this example we want to use this example to illustrate how we multiply series.

We will make use of the fact that we have got Taylor Series for each of these so we can use them in this problem.

$$e^x \cos x = \left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right).$$

Now, let's write down the first few terms of each series and we'll stop at the x^4 in each series,

$$\begin{aligned}
e^x \cos x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + \left(x - \frac{x^3}{2} + \frac{x^5}{24} + \cdots\right) + \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{48} + \cdots\right) \\
&+ \left(\frac{x^3}{6} - \frac{x^5}{12} + \frac{x^7}{144} + \cdots\right) + \left(\frac{x^4}{24} - \frac{x^6}{48} + \frac{x^8}{576} + \cdots\right) \\
&= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{24}\right)x^4 + \cdots \\
&= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots
\end{aligned}$$

We are going to look at another series representation for a function. Before we do this let's first recall the following theorem.

Binomial Theorem If n is any positive integer then,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n,$$

where the notation

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}, \quad k = 1, 2, 3, \cdots, n$$

$$\binom{n}{0} = 1.$$

This is useful for expanding $(a+b)^n$ for large n when straight forward multiplication would not be easy to do. Let's take a quick look at an example.

Example. Use the Binomial Theorem to expand $(2x-3)^4$.

solution. There really isn't much to do other than plugging into the theorem.

$$\begin{aligned}
(2x-3)^4 &= \sum_{i=0}^{4} \binom{4}{k} (2x)^{4-k}(-3)^k \\
&= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-3) + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)(-3)^3 + \binom{4}{4}(-3)^4 \\
&= (2x)^4 + 4(2x)^3(-3) + \frac{4\cdot 3}{2}(2x)^2(-3)^2 + 4\cdot(2x)(-3)^3 + (-3)^4 \\
&= 16x^4 - 96x^3 + 216x^2 - 216x + 81.
\end{aligned}$$

Now, the Binomial Theorem required that n be a positive integer. There is an extension to this however that allows for any number at all.

Binomial Series If p is any number and $|x| < 1$, then

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots,$$

where

$$\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}, \quad n = 1, 2, 3, \cdots$$

$$\binom{p}{0} = 1.$$

So, similar to the binomial theorem except that it is an infinite series and we must have $|x| < 1$ in order to get convergence.

Example. Write down the first four terms in the binomial series for $\sqrt{9-x}$.

solution. Note that $p = \frac{1}{2}$ and

$$\sqrt{9-x} = 3\left(1 - \frac{x}{9}\right)^{\frac{1}{2}} = 3\left(1 + \left(-\frac{x}{9}\right)\right)^{\frac{1}{2}}.$$

The first four terms in the binomial series is then,

$$\begin{aligned}
\sqrt{9-x} &= 3\left(1 + \left(-\frac{x}{9}\right)\right)^{\frac{1}{2}} = 3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(-\frac{x}{9}\right)^n \\
&= 3\left[1 + \frac{1}{2}\cdot\left(-\frac{x}{9}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(-\frac{x}{9}\right)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(-\frac{x}{9}\right)^3 + \cdots\right] \\
&= 3 - \frac{x}{6} - \frac{x^2}{216} - \frac{x^3}{3888} - \cdots
\end{aligned}$$

Example. Expand the function $y = \sqrt{a^2 + x^2}$, $(a > 0)$ as its MacLaurin Series.

solution. Write the function as

$$y = a\left[1 + (x/a)^2\right]^{\frac{1}{2}}.$$

By the binomial expansion

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k, \quad x \in (-1, 1).$$

Thus, for $x \in (-a, a)$,

$$\begin{aligned}
\sqrt{a^2 + x^2} &= a\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\left(\frac{x}{a}\right)^{2k} \\
&= a\left[1 + \frac{\frac{1}{2}}{1!}\cdot\left(\frac{x}{a}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\cdot\left(\frac{x}{a}\right)^4 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\cdot\left(\frac{x}{a}\right)^6 + \cdots\right] \\
&= a + \frac{\frac{1}{2}}{1!}\cdot a^{-1}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\cdot a^{-3}x^4 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\cdot a^{-5}x^6 + \cdots
\end{aligned}$$

Example (from classviva.org). (a) Evaluate the integral

$$\int_0^2 \frac{16}{x^2+4}\,dx$$

Your answer should be in the form $k\pi$, where k is an integer. What is the value of k? (Hint: $(\tan^{-1}(x))' = \frac{1}{1+x^2}$).

(b) Now, let's evaluate the same integral using power series. First, find the power series for the function $f(x) = \frac{16}{x^2+4}$. Then, integrate it from 0 to 2 and call it S. S should be an infinite series $\sum_{n=0}^{\infty} a_n$.

What are the first few terms of S? a_0, a_1, a_2, a_3, a_4?

(c) The answer in part (a) equals the sum of the infinite series in part (b) (why?). Hence, if you divide your infinite series from (b) by k (the answer to (a)), you have found an estimate for the value of π in terms of an infinite series. Approximate the value of π by the first 5 terms.

(d) What is an upper bound for your error of your estimate if you use the first 9 terms? (Use the alternating series estimation.)

solution. For (a), note that

$$\int_0^2 \frac{16}{x^2+4}\,dx = \int_0^2 \frac{4}{1+\left(\frac{x}{2}\right)^2}\,dx = 8\int_0^2 \frac{1}{1+\left(\frac{x}{2}\right)^2}\,d\left(\frac{x}{2}\right) = 8\tan^{-1}\left(\frac{x}{2}\right)|_0^2 = 8\cdot\frac{\pi}{4} = 2\pi.$$

Thus, $k=2$.

For (b), note that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, \quad \text{if } |x|<1,$$

we have

$$f(x) = \frac{16}{4+x^2} = 4\cdot\frac{1}{1+\left(\frac{x}{2}\right)^2} = \sum_{n=0}^{\infty} 4(-1)^n\left(\frac{x}{2}\right)^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{4^{n-1}}.$$

Then,

$$S = \int_0^2\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{4^{n-1}}\right)dx = \sum_{n=0}^{\infty}(-1)^n\frac{1}{4^{n-1}}\int_0^2 x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{1}{4^{n-1}}\frac{2^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{8}{2n+1}.$$

Thus,

$$a_0 = 8, \quad a_1 = -\frac{8}{3}, \quad a_2 = \frac{8}{5}, \quad a_3 = -\frac{8}{7}, \quad a_4 = \frac{8}{9}.$$

For (c), note that

$$\sum_{n=0}^{\infty}(-1)^n\frac{8}{2n+1} = 2\pi,$$

since that the remainder term (by the bound of alternating series)

$$|R_n| = \left|\int_0^2\frac{16}{x^2+4}\,dx - \sum_{k=0}^{n-1}(-1)^k\frac{8}{2k+1}\right| = \left|\int_0^2\frac{16}{x^2+4}\,dx - \sum_{k=1}^{n}(-1)^{k-1}\frac{8}{2(k-1)+1}\right|$$
$$\le b_{n+1} = \frac{8}{2n+1}\to 0, \quad \text{as } n\to\infty.$$

Thus,

$$\pi = \sum_{n=0}^{\infty}(-1)^n\frac{4}{2n+1}.$$

The approximation by the first 5 terms is

$$\pi \approx 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} \approx 3.3397$$

For (d), use the alternating series estimation, we have the upper bound for π

$$|R_9| = \left|\pi - \sum_{k=0}^{8}(-1)^k\frac{4}{2k+1}\right| = \left|\pi - \sum_{k=1}^{9}(-1)^{k-1}\frac{4}{2(k-1)+1}\right| \le b_{10} = \frac{4}{2\cdot 9+1} = \frac{4}{19}.$$

Example. For the following alternating series,

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{(0.3)^2}{2!} + \frac{(0.3)^4}{4!} - \frac{(0.3)^6}{6!} + \frac{(0.3)^8}{8!} + \cdots,$$

how many terms do you have to go for your approximation (your partial sum) to be within 10^{-7} from the convergent value of that series.
solution. Since that

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{f^{(2n)}(0)}{(2n)!} x^{2n},\end{aligned}$$

and use the Lagrange remainder and note that $|f^{(k)}| \le 1$, we have

$$|R_n(x)| = \left| \cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!} \right| \le \frac{|f^{(2n)}(c)|}{(2n)!} |x|^{2n},$$

where $c \in [0, x]$. Taking $x = 0.3$, to find n, we have to take

$$|R_n(0.3)| \le \frac{1}{(2n)!} (0.3)^{2n} \le 10^{-7},$$

in turn, $n = 4$ at least satisfied, such that $1.6272 \times 10^{-9} \le 10^{-7}$.
Counterexample for Taylor series of function. Considering

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that $f(x) = e^{-\frac{1}{x^2}} \to e^0 = 1$ as $x \to \infty$; $f(x) = e^{-\frac{1}{x^2}} \to e^{-\infty} = 0$. The MacLaurin series of f at $x = 0$ is

$$0 + 0 \cdot x + 0 \cdot x^2 + \cdots = 0 \neq f(x), \quad \text{except } x = 0,$$

where $f^{(n)}(0) = 0$. Just look at

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-\frac{1}{h^2}}}{h} = \lim_{h \to 0} \frac{\frac{1}{h}}{e^{\frac{1}{h^2}}} = \lim_{h \to 0} \frac{-\frac{1}{h^2}}{-\frac{2}{h^3} e^{\frac{1}{h^2}}} = \lim_{h \to 0} \frac{1}{2} h e^{-\frac{1}{h^2}} = 0,$$

where the l'Hospital rule has been used. Thus,

$$f'(x) = \begin{cases} \dfrac{2}{x^3} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For the general case of $f^{(n)}(x)$, we could use the definition of derivative, we could get $f^{(n)}(0) = 0$. However, in this example, the function $f(x)$ can not be expanded at $x = 0$ with some Taylor series.