Lecture 23 Three dimensional coordinate systems and vectors

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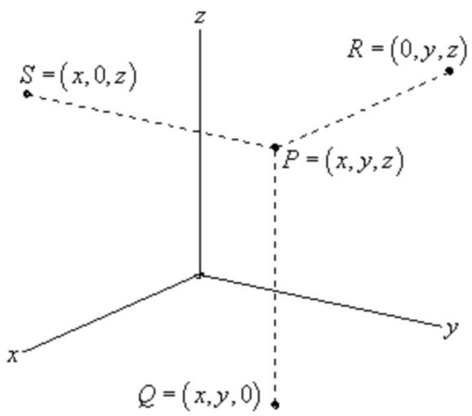
1 3-D coordinate systems

Let us introduce some basic notation out of the way. The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise, the 2-D coordinate system is often denoted by $\mathbb{R}^2$ and 1-D coordinate system is denoted by $\mathbb{R}$. Also, as you might have guessed then a general n-D coordinate system is often denoted by $\mathbb{R}^n$.
Next, let's take a quick look at the basic coordinate system. This is the standard placement of the



axes in this class. It is assumed that only the positive directions are shown by the axes. If we need the negative axes for any reason we will put them in as needed.
Also note the various points on this sketch. The point P is the general point sitting out in 3-D space. If we start at P and drop straight down until we reach a z-coordinate of zero we arrive at the point Q. We say that Q sits in the xy-plane. The xy-plane corresponds to all the points which have a zero z-coordinate. We can also start at P and move in the other two directions as shown to get points in the xz-plane (this is S with a y-coordinate of zero) and the yz-plane (this is R with an x-coordinate of zero). Here, the xy, xz, yz-planes are sometimes called the coordinate planes. Also, the point Q is often referred to as the projection of P in the xy-plane. Likewise, R is the projection of P in the yz-plane and S is the projection of P in the xz-plane.
Many of the formulas that you are used to working with in $\mathbb{R}^2$ have natural extensions in $\mathbb{R}^3$. For instance, the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in $\mathbb{R}^2$ is given by

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

while the distance between any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

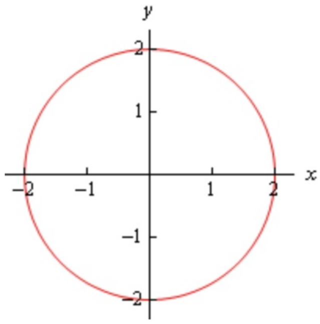
Likewise, the general equation for a circle with center (x_0, y_0) and radius r is given by

$$(x - x_0)^2 + (y - y_0)^2 = r^2,$$

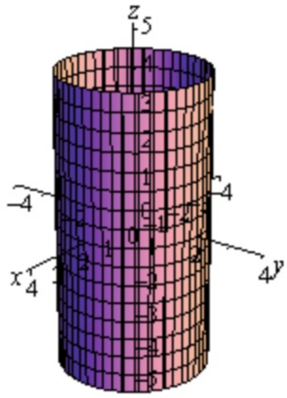
and the general equation for a sphere with center (x_0, y_0, z_0) and radius r is given by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2,$$

With that said we do need to be careful about just translating everything we know about $\mathbb{R}^2$ into $\mathbb{R}^3$ and assuming that it will work the same way.
Example. Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.
solution. In $\mathbb{R}^2$, this is a circle centered at the origin with radius 2. In $\mathbb{R}^3$, since we have not specified z in any way, then we can assume that z can take any value.



(a) $\mathbb{R}^2$

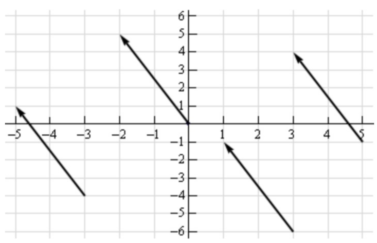


(b) $\mathbb{R}^3$

Exercise. Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

2 Vectors

Motivations. Vectors are used to represent quantities that have both a magnitude and a direction. Good examples of quantities that can be represented by vectors are force and velocity. Both of these have a direction and a magnitude.
Let us take force as an example. A force of say 5 Newtons that is applied in a particular direction can be applied at any point in space. In other words, the point where we apply the force does not change the force itself. Forces are independent of the point of application. To define a force all we need to know is the magnitude of the force and the direction that the force is applied in.
The same idea holds more generally with vectors. Vectors only contains magnitude and direction. They do not contain any information about where the quantity is applied. This is an important idea to always remember in the study of vectors.
Basic Concepts. In a graphical sense vectors are represented by directed line segments. The length of the line segment is the magnitude of the vector and the direction of the line segment is the direction of the vector.
However, because vectors don't contains any information about where the quantity is applied any directed line segment with the same length and direction will represent the same vector.
Consider the sketch below. Each of the directed line segments in the sketch represents the same



vector. In each case the vector starts at a specific point then moves 2 units to the left and 5 units up. The notation that we will use for this vector is,

$$\vec{v} = \langle -2, 5 \rangle,$$

and each of the directed line segments in the sketch are called **representations** of the vector.
Rk. Be careful to distinguish vector notation, $\langle -2, 5 \rangle$, from the notation we use to represent coordinates of points, $(-2, 5)$. The vector denotes a magnitude and a direction of a quantity while the point denotes a location in space. So don't mix the notations up!
A representation of the vector $\vec{v} = \langle a_1, a_2 \rangle$ in two dimensional space is any directed line segment, $\overrightarrow{AB}$, from the point $A = (x, y)$ to the point $B = (x + a_1, y + a_2)$. Likewise a representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ in three dimensional space is any directed line segment $\overrightarrow{AB}$, from the point $A = (x, y, z)$ to the point $B = (x + a_1, y + a_2, z + a_3)$.
Rk. There is one representation of a vector that is special in some way. The representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ that starts at the point $A = (0, 0, 0)$ and ends at the point $B = (a_1, a_2, a_3)$ is called the **position vector** of the point (a_1, a_2, a_3). So, when we talk about position vectors we are specifying the initial and final point of the vector.
Next, we need to discuss briefly how to generate a vector given the initial and final points of the representation. Given the two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ the vector with the representation $\overrightarrow{AB}$ is,

$$\vec{v} = \overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.$$

Note that we have to be very careful with direction here. The vector above is the vector that starts at A and ends at B. However, the vector that starts at B and ends at A, say, with representation $\overrightarrow{BA}$ is,

$$\vec{w} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle,$$

since that $\overrightarrow{AB} = -\overrightarrow{BA}$. These two vectors are different and so we do need to always pay attention to what point is the starting point and what point is the ending point. When determining the vector between two points we always subtract the initial point from the terminal point.
Example. Give the vector for each of the following.

(a) The vector from $(2, -7, 0)$ to $(1, -3, -5)$;

(b) The vector from $(1, -3, -5)$ to $(2, -7, 0)$;

(c) The position vector for $(-90, 4)$.

solution. For (a), remember that to construct this vector we subtract coordinates of the starting point from the ending point.

$$\langle 1 - 2, -3 - (-7), -5 - 0 \rangle = \langle -1, 4, -5 \rangle.$$

For (b), same thing here. We have

$$\langle 2 - 1, -7 - (-3), 0 - (-5) \rangle = \langle 1, -4, 5 \rangle.$$

Notice that the only difference between the first two is the signs are all opposite. This difference is important as it is this difference that tells us that the two vectors point in opposite directions.
For (c), not much to this one other than acknowledging that the position vector of a point is nothing more than a vector with the point's coordinates as its components. Thus, we have

$$\langle -90, 4 \rangle.$$

We now need to start discussing some of the basic concepts that we will run into on occasion.
Definition. The **magnitude**, or **length**, of the vector $\overrightarrow{v} = \langle a_1, a_2, a_3 \rangle$ is given by

$$\|\overrightarrow{\boldsymbol{v}}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Example. Determine the magnitude of each of the following vectors.

(a) $\overrightarrow{a} = \langle 3, -5, 10 \rangle$;

(b) $\overrightarrow{u} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$;

(c) $\overrightarrow{w} = \langle 0, 0 \rangle$;

(d) $\overrightarrow{i} = \langle 1, 0, 0 \rangle$.

solution. There is not too much to these other than plug into the formula.

$$\begin{aligned}
\text{For (a):} \quad & \|\overrightarrow{a}\| = \sqrt{9 + 25 + 100} = \sqrt{134}; \\
\text{For (b):} \quad & \|\overrightarrow{u}\| = \sqrt{\frac{1}{5} + \tfrac{4}{5}} = \sqrt{1} = 1; \\
\text{For (c):} \quad & \|\overrightarrow{w}\| = \sqrt{0 + 0} = 0; \\
\text{For (d):} \quad & \|\overrightarrow{i}\| = \sqrt{1 + 0 + 0} = 1.
\end{aligned}$$

We also have the following fact about the magnitude.

$$\text{If} \quad \|\overrightarrow{a}\| = 0, \quad \text{then} \quad \overrightarrow{a} = \overrightarrow{0}.$$

This should make sense. Because we square all the components the only way we can get zero out of the formula was for the components to be zero in the first place.
Definition. Any vector with magnitude of 1, say $\|\overrightarrow{u}\| = 1$, is called a **unit vector**.
Rk. The vector $\overrightarrow{w} = \langle 0, 0 \rangle$ is called a zero vector since its components are all zero. Zero vectors are often denoted by $\overrightarrow{0}$. Be careful to distinguish 0 (the number) from $\overrightarrow{0}$ (the vector). The number 0 denotes the origin in space, while the vector $\overrightarrow{0}$ denotes a vector that has no magnitude or direction.
Standard Basis Vectors. The vector $\overrightarrow{i} = \langle 1, 0, 0 \rangle$, is called a standard basis vector. In three dimensional space there are three standard basis vectors,

$$\overrightarrow{i} = \langle 1, 0, 0 \rangle, \quad \overrightarrow{j} = \langle 0, 1, 0 \rangle, \quad \overrightarrow{k} = \langle 0, 0, 1 \rangle.$$

In two dimensional space there are two standard basis vectors,

$$\overrightarrow{i} = \langle 1, 0 \rangle, \quad \overrightarrow{j} = \langle 0, 1 \rangle.$$

Note that standard basis vectors are also unit vectors.
Rk. Vectors can exist in general n-dimensional space. The general notation for a n-dimensional vector is,

$$\overrightarrow{v} = \langle a_1, a_2, a_3, \cdots, a_n \rangle.$$

Each of the a_i's are called **components** of the vector.
Vector Arithmetic.
We will start with addition of two vectors. So, given the vectors $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$, the **addition** of the two vectors is given by the following formula.

$$\overrightarrow{a} + \overrightarrow{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

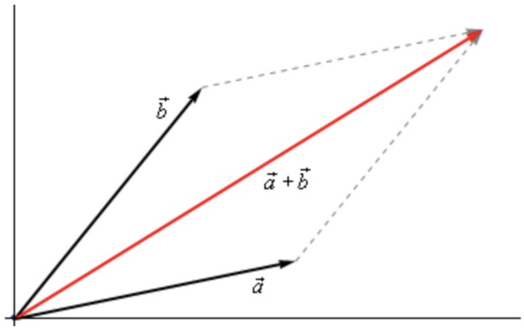


Figure 1: The addition of two vectors.

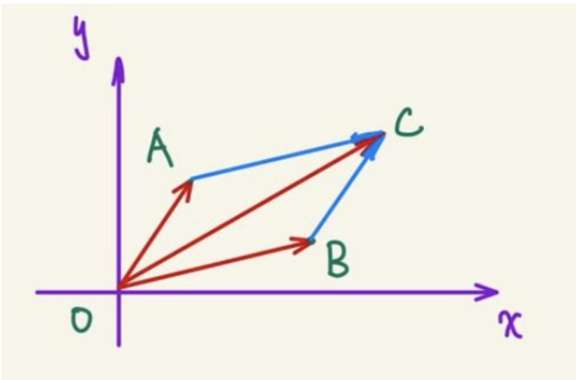


Figure 2: parallelogram law

The following figure gives the geometric interpretation of the addition of two vectors.
This is sometimes called the **parallelogram law** or **triangle law**.
Rk. As is shown in Figure 2, where $OACB$ is a parallelogram, we have

$$\overrightarrow{OA} = \overrightarrow{BC}, \quad \overrightarrow{OB} = \overrightarrow{AC}, \quad \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}, \quad \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OB} = \overrightarrow{BC}, \quad \overrightarrow{OB} = \overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{AC}.$$

and

$$\overrightarrow{OA} \,//\, \overrightarrow{BC}, \quad \overrightarrow{OB} \,//\, \overrightarrow{AC}, \quad \overrightarrow{AO} \,//\, \overrightarrow{BC}, \quad \overrightarrow{BO} \,//\, \overrightarrow{AC},$$

If we use the coordinate $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, we then have

$$\langle a_1, a_2 \rangle = \langle c_1 - b_1, c_2 - b_2 \rangle, \quad \langle b_1, b_2 \rangle = \langle c_1 - a_1, c_2 - a_2 \rangle, \quad \langle c_1, c_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle.$$

Computationally, **subtraction** is very similar. Given the vectors $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$, the difference of the two vectors is given by,

$$\overrightarrow{a} - \overrightarrow{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

Here is the geometric interpretation of the difference of two vectors.
It is a little harder to see this geometric interpretation. To help see this let us instead think of subtraction as the addition of $\overrightarrow{a}$ and $-\overrightarrow{b}$. First, as we will see in a bit $-\overrightarrow{b}$ is the same vector as $\overrightarrow{b}$ with opposite signs on all the components. In other words, $-\overrightarrow{b}$ goes in the opposite direction as $\overrightarrow{b}$. Here is the vector set up for $\overrightarrow{a} + (-\overrightarrow{b})$.
As we can see from this figure we can move the vector representing $\overrightarrow{a} + (-\overrightarrow{b})$ to the position we have got in the first figure showing the difference of the two vectors.

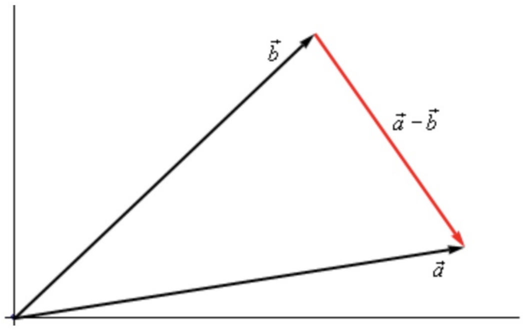
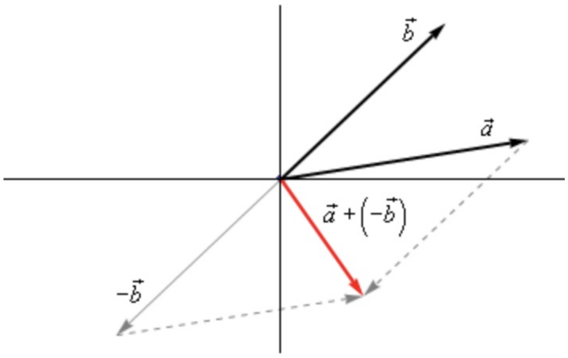


Figure 3: The difference of two vectors.



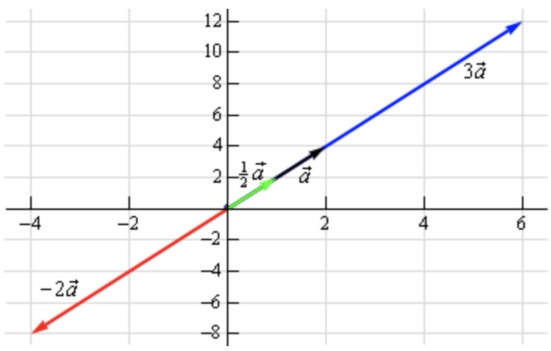
Rk. Note that we can't add or subtract two vectors unless they have the same number of components. If they don't have the same number of components then addition and subtraction can't be done.
scalar multiplication. Given the vector $\overrightarrow{a} = \langle a_1, a_2, a_3\rangle$ and any number c, the scalar multiplication is

$$c\overrightarrow{a} = \langle ca_1, ca_2, ca_3\rangle.$$

So, we multiply all the components by the constant c. To see the geometric interpretation of scalar multiplication let's take a look at an example.
Example. For the vector $\overrightarrow{a} = \langle 2, 4\rangle$ and compute $3\overrightarrow{a}$, $\frac{1}{2}\overrightarrow{a}$ and $-2\overrightarrow{a}$. Graph all four vectors on the same axis system.
solution. Here are the three scalar multiplications.

$$3\overrightarrow{a} = \langle 6, 12\rangle \quad \frac{1}{2}\overrightarrow{a} = \langle 1, 2\rangle \quad -2\overrightarrow{a} = \langle -4, -8\rangle$$

Here is the graph for each of these vectors.



Rk. We can see that if c is positive all scalar multiplication will do is stretch (if $c > 1$) or shrink (if $c < 1$) the original vector, but it won't change the direction. Likewise, if c is negative scalar

multiplication will switch the direction so that the vector will point in exactly the opposite direction and it will again stretch or shrink the magnitude of the vector depending upon the size of c.
There are several nice applications of scalar multiplication that we should now take a look at.

The first is parallel vectors. Two vectors are **parallel** if they have the same direction or are in exactly opposite directions. Now, recall again the geometric interpretation of scalar multiplication. When we performed scalar multiplication we generated new vectors that were parallel to the original vectors (and each other for that matter).

So, let's suppose that $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel vectors. If they are parallel then there must be a number c, such that

$$\overrightarrow{a} = c\,\overrightarrow{b}.$$

So, two vectors are parallel if one is a scalar multiple of the other.
Example. Determine if the sets of vectors are parallel or not.

(a) $\overrightarrow{a} = \langle 2, -4, 1\rangle$, $\overrightarrow{b} = \langle -6, 12, -3\rangle$;

(b) $\overrightarrow{a} = \langle 4, 10\rangle$, $\overrightarrow{b} = \langle 2, -9\rangle$.

solution. For (a), these two vectors are parallel since $\overrightarrow{b} = -3\overrightarrow{a}$; For (b), these two vectors are not parallel. This can be seen by noticing that $4 \cdot \frac{1}{2} = 2$ and yet $10 \cdot \frac{1}{2} = 5 \neq -9$. In other words, we can't make $\overrightarrow{a}$ be a scalar multiple of $\overrightarrow{b}$.
The next application is best seen in an example.
Example. Find a unit vector that points in the same direction as $\overrightarrow{w} = \langle -5, 2, 1\rangle$.
solution. What we are asking for is a new parallel vector (points in the same direction) that happens to be a unit vector. We can do this with a scalar multiplication since all scalar multiplication does is change the length of the original vector (along with possibly flipping the direction to the opposite direction).
Here's what we'll do. First let's determine the magnitude of $\overrightarrow{w}$.

$$\|\overrightarrow{w}\| = \sqrt{25+4+1} = \sqrt{30}.$$

Now, let's form the following new vector,

$$\overrightarrow{u} = \frac{1}{\|\overrightarrow{w}\|}\overrightarrow{w} = \frac{1}{\sqrt{30}}\langle -5, 2, 1\rangle = \left\langle -\frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right\rangle.$$

The claim is that this is a unit vector. That's easy enough to check

$$\|\overrightarrow{u}\| = \sqrt{\frac{25}{30}+\frac{4}{30}+\frac{1}{30}} = \sqrt{\frac{30}{30}} = 1.$$

So, in general, given a vector $\overrightarrow{w}$, $\overrightarrow{u} = \frac{\overrightarrow{w}}{\|\overrightarrow{w}\|}$ will be a unit vector that points in the same direction as $\overrightarrow{w}$.
Standard Basis Vectors Revisited. Previously, we introduced the idea of standard basis vectors without really discussing why they were important. We can now do that. Let's start with the vector

$$\overrightarrow{a} = \langle a_1, a_2, a_3\rangle.$$

We can use the addition of vectors to break this up as follows,

$$\overrightarrow{a} = \langle a_1, a_2, a_3\rangle = \langle a_1, 0, 0\rangle + \langle 0, a_2, 0\rangle + \langle 0, 0, a_3\rangle.$$

Using scalar multiplication we can further rewrite the vector as,

$$\overrightarrow{a} = a_1\langle 1, 0, 0\rangle + a_2\langle 0, 1, 0\rangle + a_3\langle 0, 0, 1\rangle.$$

Finally, notice that these three new vectors are simply the three standard basis vectors for three dimensional space.

$$\langle a_1, a_2, a_3\rangle = a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{k}.$$

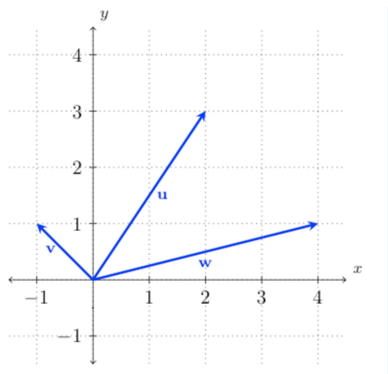
So, we can take any vector and write it in terms of the standard basis vectors. From this point on we will use the two notations interchangeably so make sure that you can deal with both notations.
Example. If $\overrightarrow{a} = \langle 3, -9, 1\rangle$ and $\overrightarrow{w} = -\overrightarrow{i} + 8\overrightarrow{k}$, compute $2\overrightarrow{a} - 3\overrightarrow{w}$.
solution. In order to do the problem we will convert to one notation and then perform the indicated operations.

$$2\overrightarrow{a} - 3\overrightarrow{w} = 2\langle 3, -9, 1\rangle - 3\langle -1, 0, 8\rangle = \langle 6, -18, 2\rangle - \langle -3, 0, 24\rangle = \langle 9, -18, -22\rangle$$

Properties. If $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ are vectors (each with the same number of components) and a and b are two numbers then we have the following properties.

- $\overrightarrow{v} + \overrightarrow{w} = \overrightarrow{w} + \overrightarrow{v}$; $\overrightarrow{v} + \overrightarrow{0} = \overrightarrow{v}$; $a(\overrightarrow{v} + \overrightarrow{w}) = a\overrightarrow{v} + a\overrightarrow{w}$;
- $\overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w}$; $1 \cdot \overrightarrow{u} = \overrightarrow{u}$; $(a+b)\overrightarrow{v} = a\overrightarrow{v} + b\overrightarrow{v}$
- $\|a\overrightarrow{v}\| = |a|\|\overrightarrow{v}\|$;
- $\|\overrightarrow{v} + \overrightarrow{w}\| \leq \|\overrightarrow{v}\| + \|\overrightarrow{w}\|$.

Example (from classviva.org) Find the following expressions using the graph below of vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$.



- $\overrightarrow{u} + \overrightarrow{v} =?$
- $2\overrightarrow{u} + \overrightarrow{w} =?$
- $3\overrightarrow{v} - 6\overrightarrow{w} =?$
- $\|\overrightarrow{w}\| =?$

solution. Since that $\vec{u} = \langle 2, 3\rangle$, $\vec{v} = \langle -1, 1\rangle$, and $\vec{w} = \langle 4, 1\rangle$. Thus

$$\begin{aligned}
\vec{u} + \vec{v} &= \langle 2-1, 3+1\rangle = \langle 1, 4\rangle \\
2\vec{u} + \vec{w} &= \langle 2\cdot 2+4, 2\cdot 3+1\rangle = \langle 8, 7\rangle \\
3\vec{v} - 6\vec{w} &= 3\langle -1, 1\rangle - 6\langle 4, 1\rangle = \langle -3-24, 3-6\rangle = \langle -27, -3\rangle \\
\|\vec{w}\| &= \sqrt{4^2+1^2} = \sqrt{17}.
\end{aligned}$$

Example (from classviva.org) Find $\vec{a}$ that has the same direction as $\vec{b} = \langle -8, 7, 8\rangle$ but has length 5.
solution. Note that the magnitude of $\vec{b}$ below,

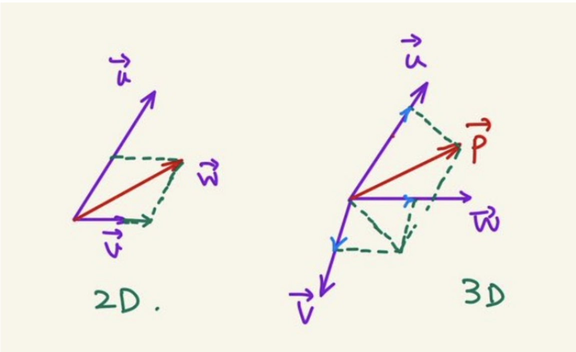
$$\|\vec{b}\| = \sqrt{(-8)^2+7^2+8^2} = \sqrt{177}.$$

Thus, the unit vector has the same direction as $\vec{b}$ is $\frac{1}{\sqrt{177}}\vec{b}$. So the required vector has length 5 is

$$\vec{a} = \frac{5}{\sqrt{177}}\vec{b} = \frac{5}{\sqrt{177}}\langle -8, 7, 8\rangle = \left\langle -\frac{40}{\sqrt{177}}, \frac{35}{\sqrt{177}}, \frac{40}{\sqrt{177}}\right\rangle.$$

Decomposing vectors into various components. The vectors

- in 2D, $\vec{w}$ can be decomposed a component along $\vec{u}$ and a component along $\vec{v}$. ;
- in 3D, $\vec{p}$ can be decomposed a component along $\vec{u}$, a component along $\vec{v}$ a component along $\vec{w}$.



We have

$$\begin{aligned}
&\text{in 2D,} \quad \vec{w} = a\vec{u} + b\vec{v}, \\
&\text{in 3D,} \quad \vec{p} = a\vec{u} + b\vec{v} + c\vec{w},
\end{aligned}$$

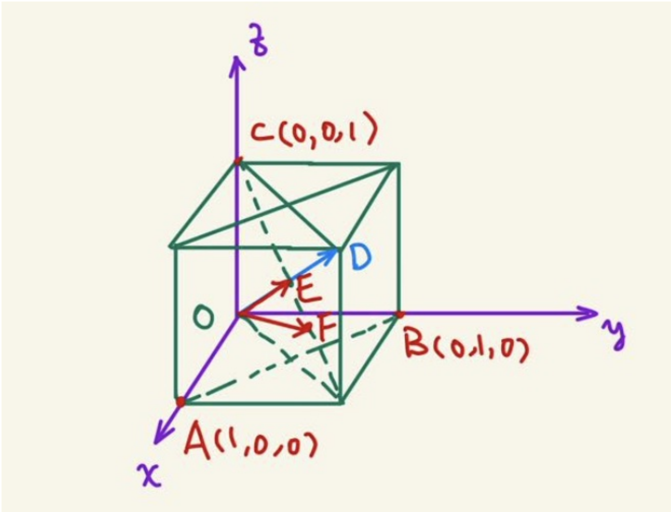
where a, b, c can be determined by the linear system of equations with coordinates. For example in 2D, take $\vec{w} = \langle 3, 5\rangle$, $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 3\rangle$, we have

$$\langle 3, 5\rangle = a\langle 1, 1\rangle + b\langle 1, 3\rangle,$$

say

$$\begin{cases} a+b=3 \\ a+3b=5. \end{cases} \Longrightarrow \begin{cases} a=2, \\ b=1. \end{cases}$$

Thus, $\vec{w} = 2\vec{u} + \vec{v}$.
Example. Consider the unit cube in 3D below. Here $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$,



what is the $\overrightarrow{OE}$ and $\overrightarrow{OF}$, $\overrightarrow{EF}$, where E is the midpoint of OD and $\|\overrightarrow{EF}\| = \frac{1}{2}\|\overrightarrow{CE}\|$.
solution. Since that $D = (1,1,1)$, thus

$$\overrightarrow{OA} = \langle 1,0,0\rangle, \quad \overrightarrow{OB} = \langle 0,1,0\rangle, \quad \overrightarrow{OC} = \langle 0,0,1\rangle, \quad \overrightarrow{OD} = \langle 1,1,1\rangle = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC},$$

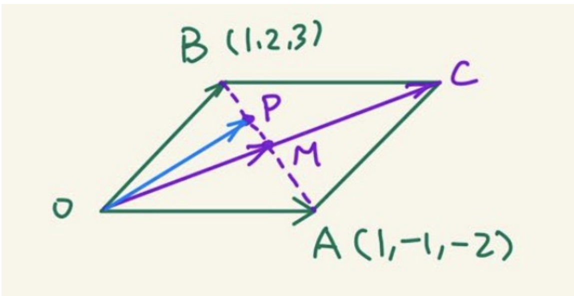
and

$$\overrightarrow{OE} = \frac{1}{2}\overrightarrow{OD} = \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle, \quad \overrightarrow{CE} = \overrightarrow{CO} + \overrightarrow{OE} = \overrightarrow{OE} - \overrightarrow{OC} = \left\langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle,$$

and

$$\overrightarrow{EF} = \frac{1}{2}\overrightarrow{CE} = \left\langle \frac{1}{4}, \frac{1}{4}, -\frac{1}{4} \right\rangle, \quad \overrightarrow{OF} = \overrightarrow{OE} + \overrightarrow{EF} = \left\langle \frac{3}{4}, \frac{3}{4}, \frac{1}{4} \right\rangle.$$

Example. The vectors are shown below. What are the point M and P, such that $\|\overrightarrow{OM}\| = \frac{1}{2}\|\overrightarrow{OC}\|$



and $\overrightarrow{BP} = \frac{1}{3}\|AB\|$.
solution. Since that

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = \langle 1,2,3\rangle + \langle 1,-1,-2\rangle = \langle 2,1,1\rangle,$$

thus

$$\begin{aligned}
\overrightarrow{OM} &= \frac{1}{2}\overrightarrow{OC} = \left\langle 1, \frac{1}{2}, \frac{1}{2} \right\rangle \\
\overrightarrow{OP} &= \overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OB} + \frac{1}{3}\overrightarrow{BA} = \overrightarrow{OB} + \frac{1}{3}\left(\overrightarrow{OA} - \overrightarrow{OB}\right) = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OB} \\
&= \frac{1}{3}\langle 1,-1,-2\rangle + \frac{2}{3}\langle 1,2,3\rangle = \left\langle 1, 1, \frac{4}{3} \right\rangle.
\end{aligned}$$

Thus, we have the coordinates of points $M\left(1, \frac{1}{2}, \frac{1}{2}\right)$ and $P\left(1, 1, \frac{4}{3}\right)$.