Lecture 24 Dot Product

Instructor: Dr. C.J. Xie (macjxie@ust.hk)

1 Dot product

Let us jump right into the definition of the dot product of two vectors. Given the two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, the **dot product** is,

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Sometimes the dot product is called the **scalar product**. The dot product is also an example of an **inner product** and so on occasion you may hear it called an inner product.
Example. Compute the dot product for each of the following.

(a) $\vec{v} = 5\vec{i} - 8\vec{j}, \quad \vec{w} - \vec{i} + 2\vec{j};$

(b) $\vec{a} = \langle 0, 3, -7 \rangle, \quad \vec{b} = \langle 2, 3, 1 \rangle.$

solution. Not much to do with these other than use the formula.

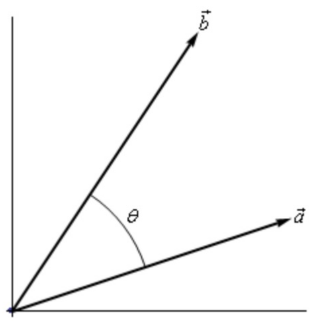
(a) $\vec{v} \cdot \vec{w} = 5 - 16 = -11;$

(b) $\vec{a} \cdot \vec{b} = 0 + 9 - 7 = 2.$

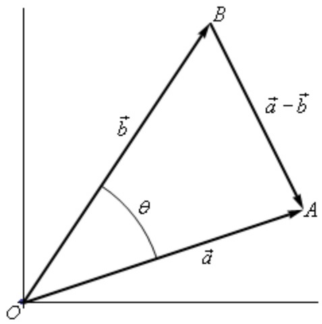
Here are some properties of the dot product.
Properties.

$$\begin{aligned}
&\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}, \qquad (c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w});\\
&\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \qquad \vec{v} \cdot \vec{0} = 0;\\
&\vec{v} \cdot \vec{v} = \|\vec{v}\|^2, \qquad \text{If } \vec{v} \cdot \vec{v} = 0, \quad \text{then } \vec{v} = \vec{0}.
\end{aligned}$$

geometric interpretation of dot product. First suppose that θ is the angle between $\vec{a}$ and $\vec{b}$, such that $0 \le \theta \le \pi$ as shown in the picture below.



We then have the following theorem.
Theorem. $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta.$



Proof. Let's give a modified version of the sketch above.
The three vectors above form the triangle AOB and note that the length of each side is nothing more than the magnitude of the vector forming that side.
The Law of Cosines tells us that,

$$\|\overrightarrow{a}-\overrightarrow{b}\|^2=\|\overrightarrow{a}\|^2+\|\overrightarrow{b}\|^2-2\|\overrightarrow{a}\|\cdot\|\overrightarrow{b}\|\cos\theta.$$

Also using the properties of dot products we can write the left side as,

$$\begin{aligned}\|\overrightarrow{a}-\overrightarrow{b}\|^2&=\left(\overrightarrow{a}-\overrightarrow{b}\right)\cdot\left(\overrightarrow{a}-\overrightarrow{b}\right)\\&=\overrightarrow{a}\cdot\overrightarrow{a}-\overrightarrow{a}\cdot\overrightarrow{b}-\overrightarrow{b}\cdot\overrightarrow{a}+\overrightarrow{b}\cdot\overrightarrow{b}\\&=\|\overrightarrow{a}\|^2-2\overrightarrow{a}\cdot\overrightarrow{b}+\|\overrightarrow{b}\|^2.\end{aligned}$$

Our original equation is then,

$$\begin{aligned}\|\overrightarrow{a}-\overrightarrow{b}\|^2&=\|\overrightarrow{a}\|^2+\|\overrightarrow{b}\|^2-2\|\overrightarrow{a}\|\|\overrightarrow{b}\|\cos\theta\\\|\overrightarrow{a}\|^2-2\overrightarrow{a}\cdot\overrightarrow{b}+\|\overrightarrow{b}\|^2&=\|\overrightarrow{a}\|^2+\|\overrightarrow{b}\|^2-2\|\overrightarrow{a}\|\|\overrightarrow{b}\|\cos\theta\\-2\overrightarrow{a}\cdot\overrightarrow{b}&=-2\|\overrightarrow{a}\|\cdot\|\overrightarrow{b}\|\cos\theta\\\overrightarrow{a}\cdot\overrightarrow{b}&=\|\overrightarrow{a}\|\cdot\|\overrightarrow{b}\|\cos\theta.\end{aligned}$$

□

Rk. The formula from this theorem is often used not to compute a dot product but instead to find the angle between two vectors. Note as well that while the sketch of the two vectors in the proof is for two dimensional vectors the theorem is valid for vectors of any dimension (as long as they have the same dimension of course).
Let's see an example of this.
Example. Determine the angle between $\overrightarrow{a}=\langle 3,-4,-1\rangle$ and $\overrightarrow{b}=\langle 0,5,2\rangle$.
solution. We will need the dot product as well as the magnitudes of each vector.

$$\overrightarrow{a}\cdot\overrightarrow{b}=-22,\quad \|\overrightarrow{a}\|=\sqrt{26},\quad \|\overrightarrow{b}\|=\sqrt{29}.$$

The angle is then,

$$\cos\theta=\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{\|\overrightarrow{a}\|\cdot\|\overrightarrow{b}\|}=\frac{-22}{\sqrt{26}\sqrt{29}}=-0.8011927$$

$$\theta=\cos^{-1}(-0.8011927)=2.5\text{ radians}=143.24\text{ degrees}$$

Example (from classviva.org) What is the angle in radians between the vectors $\overrightarrow{a}=(8,2,8)$ and $\overrightarrow{b}=(5,4,-10)$?

solution. Note that

$$\vec{a}\cdot\vec{b} = 40+8-80 = -32, \quad \|\vec{a}\| = \sqrt{64+4+64} = \sqrt{132}, \quad \|\vec{b}\| = \sqrt{25+16+100} = \sqrt{141}.$$

The angle is then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|} = \frac{-32}{\sqrt{132}\sqrt{141}} = -0.2345597,$$
$$\theta = \cos^{-1}(-0.2345597) = 1.8076 \text{ radians}.$$

The dot product gives us a very nice method for determining if two vectors are perpendicular and it will give another method for determining when two vectors are parallel. Note as well that often we will use the term **orthogonal** in place of perpendicular.
Now, if two vectors are orthogonal then we know that the angle between them is 90° which tells us that if two vectors are orthogonal then,

$$\vec{a}\cdot\vec{b} = 0.$$

Likewise, if two vectors are parallel then the angle between them is either 0° (pointing in the same direction) or 180° (pointing in the opposite direction). This would mean that one of the following would have to be true.

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\| \quad (\theta = 0^\circ), \qquad \text{or} \quad \vec{a}\cdot\vec{b} = -\|\vec{a}\|\|\vec{b}\| \quad (\theta = 180^\circ).$$

Example. Determine if the following vectors are parallel, orthogonal, or neither.

(a) $\vec{a} = \langle 6,-2,-1\rangle$, $\vec{b} = \langle 2,5,2\rangle$;

(b) $\vec{u} = 2\vec{i} - \vec{j}$, $\vec{v} = -\frac{1}{2}\vec{i} + \frac{1}{4}\vec{j}$.

solution. For (a), first get the dot product to see if they are orthogonal.

$$\vec{a}\cdot\vec{b} = 12-10-2 = 0.$$

So, the two vectors are orthogonal. For (b), again, let's get the dot product first.

$$\vec{u}\cdot\vec{v} = -1 - \frac{1}{4} = -\frac{5}{4}.$$

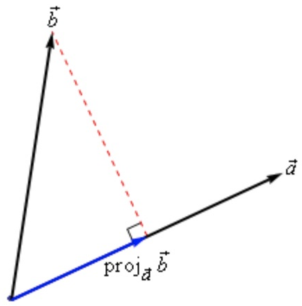
So, they aren't orthogonal. Let's get the magnitudes and see if they are parallel.

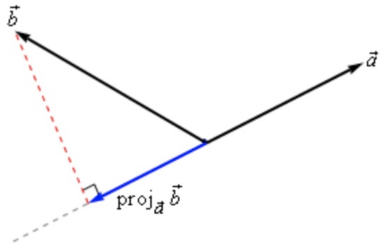
$$\|\vec{u}\| = \sqrt{5}, \quad \|\vec{v}\| = \sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}.$$

Note that

$$\vec{u}\cdot\vec{v} = -\frac{5}{4} = -\sqrt{5}\cdot\frac{\sqrt{5}}{4} = -\|\vec{u}\|\|\vec{v}\|.$$

So, the two vectors are parallel.
There are several nice applications of the dot product as well that we should look at.
Projections. The best way to understand projections is to see a couple of sketches. So, given two vectors $\vec{a}$ and $\vec{b}$, we want to determine the projection of $\vec{b}$ onto $\vec{a}$. The projection is denoted by $\text{proj}_{\vec{a}}\vec{b}$. Here are a couple of sketches illustrating the projection.
So, to get the projection of $\vec{b}$ onto $\vec{a}$ we drop straight down from the end of $\vec{b}$ until we hit (and form a right angle) with the line that is parallel to $\vec{a}$. The projection is then the vector that is





parallel to $\vec{a}$, starts at the same point both of the original vectors started at and ends where the dashed line hits the line parallel to $\vec{a}$.
here is a nice formula for finding the projection of $\vec{b}$ onto $\vec{a}$.

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}.$$

Rk. Note that we also need to be very careful with notation here. The projection of $\vec{a}$ onto $\vec{b}$ is given by

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b}.$$

We can see that this will be a totally different vector. This vector is parallel to $\vec{b}$, while $\text{proj}_{\vec{a}}\,\vec{b}$ is parallel to $\vec{a}$. So, be careful with notation and make sure you are finding the correct projection.
Here's an example.
Example. Determine the projection of $\vec{b} = \langle 2, 1, -1\rangle$ onto $\vec{a} = \langle 1, 0, -2\rangle$.
solution. We need the dot product and the magnitude of $\vec{a}$.

$$\vec{a}\cdot\vec{b} = 4, \quad \|\vec{a}\|^2 = 5.$$

The projection is then,

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a} = \frac{4}{5}\langle 1, 0, -2\rangle = \left\langle \frac{4}{5}, 0, -\frac{8}{5}\right\rangle$$

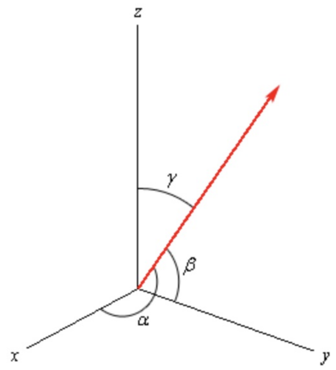
For comparison purposes let's do it the other way around as well.
Example. Determine the projection of $\vec{a} = \langle 1, 0, -2\rangle$ onto $\vec{b} = \langle 2, 1, -1\rangle$.
solution. We need the dot product and the magnitude of $\vec{b}$.

$$\vec{a}\cdot\vec{b} = 4, \quad \|\vec{b}\|^2 = 6.$$

The projection is then,

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b} = \frac{4}{6}\langle 2, 1, -1\rangle = \left\langle \frac{4}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle.$$

As we can see from the previous two examples the two projections are different so be careful.
Direction Cosines. Let's start with a vector, $\vec{a}$ in 3-D space. This vector will form angles with the x-axis, the y-axis and the z-axis. These angles are called **direction angles** and the cosines of these angles are called **direction cosines**.
Here is a sketch of a vector and the direction angles.



The formulas for the direction cosines are,

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{\|\vec{a}\|} = \frac{a_1}{\|\vec{a}\|}, \quad \cos\beta = \frac{\vec{a}\cdot\vec{j}}{\|\vec{a}\|} = \frac{a_2}{\|\vec{a}\|}, \quad \cos\gamma = \frac{\vec{a}\cdot\vec{k}}{\|\vec{a}\|} = \frac{a_3}{\|\vec{a}\|},$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the standard basis vectors.
Let's verify the first dot product above. We will leave the rest to you to verify.

$$\vec{a}\cdot\vec{i} = \langle a_1, a_2, a_3\rangle \cdot \langle 1,0,0\rangle = a_1.$$

Here are a couple of facts about the direction cosines.

- The vector $\vec{u} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$ is a unit vector;
- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$;
- $\vec{a} = \|\vec{a}\|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$.

Let's do a quick example involving direction cosines.
Example. Determine the direction cosines and direction angles for $\vec{a} = \langle 2, 1, -4\rangle$.
solution. We will need the magnitude of the vector.

$$\|\vec{a}\| = \sqrt{4+1+16} = \sqrt{21}.$$

The direction cosines and angles are then,

$$\begin{aligned} \cos\alpha &= \frac{2}{\sqrt{21}}, \quad \alpha = 1.119 \text{ radians} = 64.123 \text{ degrees},\\ \cos\beta &= \frac{1}{\sqrt{21}}, \quad \beta = 1.351 \text{ radians} = 77.396 \text{ degrees},\\ \cos\gamma &= \frac{-4}{\sqrt{21}}, \quad \gamma = 2.632 \text{ radians} = 150.794 \text{ degrees}. \end{aligned}$$

Example (from classviva.org) Find two vectors $\vec{v_1}$ and $\vec{v_2}$ whose sum is $\langle -4, -1, -4\rangle$, where $\vec{v_1}$ is parallel to $\langle 0, 4, 5\rangle$, while $\vec{v_2}$ is perpendicular to $\langle 0, 4, 5\rangle$.
solution. Note that

$$\begin{aligned} \vec{v_1} + \vec{v_2} &= \langle -4, -1, -4\rangle\\ \vec{v_2}\cdot\vec{v_1} &= 0. \end{aligned}$$

We assume that $\vec{v_1} = c\langle 0,4,5\rangle = \langle 0, 4c, 5c\rangle$, so, $\vec{v_2} = \langle -4, -1-4c, -4-5c\rangle$. We have

$$\vec{v_2}\cdot\vec{v_1} = 0 + 4c(-1-4c) + 5c(-4-5c) = 0.$$

Thus, $c = 0$ (omitted), $c = -\frac{24}{41}$. As a result, $\vec{v_1} = \left\langle 0, -\frac{96}{41}, -\frac{120}{41}\right\rangle$ and $\vec{v_2} = \left\langle -4, \frac{55}{41}, -\frac{44}{41}\right\rangle$.