Lecture 3– Integration by parts

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1 Recap last time

substitution method

- for the indefinite integral, let $u = g(x)$, we have

$$\int f(g(x)) \cdot g'(x)\ dx = \int f(g(x))\ dg(x) = \int f(u)\ du$$

- for the definite integral, let $u = g(x)$, we have

$$\int_a^b f(g(x)) \cdot g'(x)\ dx = \int_a^b f(g(x))\ dg(x) = \int_{g(a)}^{g(b)} f(u)\ du.$$

Example. Evaluate $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\ dx$.
solution
Let $u = \tan x$, we have $du = (\tan x)'\ dx = sec^2 x\ dx$, and

$$\begin{cases} x: & 0 \to \frac{\pi}{4} \\ u: & \tan 0 = 0 \to \tan \frac{\pi}{4} = 1. \end{cases}$$

Thus,

$$\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\ dx = \int_0^{\frac{\pi}{4}} \tan x \cdot (\tan x)'\ dx = \int_0^{\frac{\pi}{4}} \tan x\ d\tan x = \int_0^1 u\ du$$
$$= \frac{1}{2}u^2|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Example. Evaluate $\int \tan^3 x \sec x\ dx$.
solution Let $u = \sec x$, we have $du = (\sec x)'\ dx = \sec x \tan x\ dx$ and

$$\int \tan^3 x \sec x\ dx = \int \tan^2 x \cdot (\sec x \tan x)\ dx = \int (\sec^2 x - 1) \cdot (\sec x)'\ dx$$
$$= \int (u^2 - 1)\ du = \frac{1}{3}u^3 - u + C.$$

Let's recall the product rule below (some intuition of **integration by parts**),
product rule Let $u(x)$, $v(x)$ are both differentiable in x, we have

$$(uv)' = u'v + uv'.$$

Say,

$$uv' = (uv)' - u'v. \tag{1}$$

For the indefinite integral $\int uv'dx$, we integrate the both sides of the product rule and have

$$\begin{aligned}\int uv'dx &= \int \left[(uv)' - u'v\right]\ dx = \int (uv)'\ dx - \int u'v\ dx\\ &= uv - \int u'v\ dx,\end{aligned}$$

where the integral $\int u'v\ dx$ is hopefully easier to integrate.

Example. evaluate $\int xe^x\ dx$.

solution

We first need to choose u and v. In this example, we choose $u = x$, and $v = e^x$, and thus have

$$\begin{aligned}\int xe^x\ dx &= \int x(e^x)'\ dx = \int (xe^x)'\ dx - \int 1\cdot e^x\ dx\\ &= xe^x - e^x + C.\end{aligned}$$

Rk. This idea is nothing but the **integration by parts**.

2 Integration by parts

Theorem Let u and v are both differentiable in x. Then

- for the indefinite integral, we have

$$\int uv'\ dx = \int u\ dv = uv - \int v\ du = uv - \int vu'\ dx;$$

- for the definite integral, by the Fundamental Theorem of Calculus (FTC), we have

$$\int_a^b uv'\ dx = (uv)\,|_a^b - \int_a^b vu'\ dx.$$

Example. evaluate $\int x\cos x\ dx$ and $\int_0^\pi x\cos x\ dx$.

solution We take $u = x$ and $v = \sin x$, and thus have

$$\begin{aligned}\int x\cos x\ dx = \int x(\sin x)'\ dx = \int x\ d\sin x &= x\sin x - \int \sin x\ dx\\ &= x\sin x + \cos x + C.\end{aligned}$$

Using the FTC, we have

$$\begin{aligned}\int_0^\pi x\cos x\ dx &= \int_0^\pi x(\sin x)'\ dx = \int_0^\pi x\ d\sin x = x\sin x|_0^\pi - \int_0^\pi \sin x\ dx\\ &= 0 - 0 + \cos x|_0^\pi = \cos(\pi) - \cos 0 = -1 - 1 = -2.\end{aligned}$$

Rk. If we take $v = \cos x$ and $u = \frac{1}{2}x^2$, then

$$\begin{aligned}\int x\cos x\ dx &= \frac{1}{2}\int \cos x\cdot\left(x^2\right)'\ dx = \frac{1}{2}\int \cos x\ d\left(x^2\right)\\ &= \frac{1}{2}x^2\cos x - \frac{1}{2}\int x^2\cdot(\cos x)'\ dx = \frac{1}{2}x^2\cos x + \frac{1}{2}\int x^2\sin x\ dx,\end{aligned}$$

thus the power x^k is higher, that has to use the integration by parts again. This is not an optimal method for $u = \frac{1}{2}x^2$ and $v = \cos x$ since the resulting integration on the right hand side is more complicated than the original one. Obviously, $u = x$ and $v = \sin x$ is a better strategy to evaluate this integral.

Example (from classviva.org). evaluate

$$\int_1^7 \sqrt{t}\ln t\,dt.$$

solution

method 1: Using the integration by parts, we have

$$\begin{aligned}\int_1^7 \sqrt{t}\ln t\,dt &= \int_1^7 \ln t\cdot\left(\frac{2}{3}t^{\frac{3}{2}}\right)'\,dt = \frac{2}{3}\int_1^7 \ln t\,dt^{\frac{3}{2}}\\ &= \frac{2}{3}\ln t\cdot t^{\frac{3}{2}}|_1^7 - \frac{2}{3}\int_1^7 t^{\frac{3}{2}}\cdot(\ln t)'\,dt = \frac{2}{3}\ln t\cdot t^{\frac{3}{2}}|_1^7 - \frac{2}{3}\int_1^7 t^{\frac{1}{2}}\,dt\\ &= \frac{2}{3}\ln t\cdot t^{\frac{3}{2}}|_1^7 - \frac{2}{3}\cdot\frac{2}{3}t^{\frac{3}{2}}|_1^7 = \frac{2}{3}(7^{\frac{3}{2}}\ln 7) - \frac{4}{9}(7^{\frac{3}{2}}-1) \approx 16.24.\end{aligned}$$

method 2: Let $u=\sqrt{t}$ first, we then have $x: 1\to 7$ and $u: 1\to\sqrt{7}$, $t=u^2$, $dt = 2u\,du$ and

$$\begin{aligned}\int_1^7 \sqrt{t}\ln t\,dt &= \int_1^{\sqrt{7}} u\ln u^2\cdot 2u\,du = \int_1^{\sqrt{7}} 2u\ln u\cdot 2u\,du\\ &= 4\int_1^{\sqrt{7}} u^2\ln u\,du = 4\int_1^{\sqrt{7}} \ln u\cdot\left(\frac{1}{3}u^3\right)'\,du = \frac{4}{3}\int_1^{\sqrt{7}}\ln u\,du^3,\end{aligned}$$

using the integration by parts below,

$$\begin{aligned}\int_1^7 \sqrt{t}\ln t\,dt &= \frac{4}{3}\int_1^{\sqrt{7}}\ln u\,du^3 = \frac{4}{3}u^3\ln u|_1^{\sqrt{7}} - \frac{4}{3}\int_1^{\sqrt{7}} u^3(\ln u)'\,du\\ &= \frac{4}{3}u^3\ln u|_1^{\sqrt{7}} - \frac{4}{3}\int_1^{\sqrt{7}} u^2\,du = \frac{4}{3}u^3\ln u|_1^{\sqrt{7}} - \frac{4}{3}\cdot\frac{1}{3}u^3|_1^{\sqrt{7}}\\ &= \frac{4}{3}(\sqrt{7})^3\ln\sqrt{7} - \frac{4}{9}\left((\sqrt{7})^3 - 1\right) \approx 16.24.\end{aligned}$$

Ideas for using integration by parts Two ingradients below,

- choosing the suitable u and v, such that integration is easier to do;
- taking the integration by parts (might be used more than once, usually to be used twice is enough).

Example. evaluate

$$\int e^{2x}\sin x\,dx.$$

solution

$$\int e^{2x}\sin x\,dx = \int e^{2x}\cdot(-\cos x)'\,dx = -\int e^{2x}\cdot d\cos x,$$

take $u = e^{2x}$ and $v = \cos x$, by the integration by parts, we have

$$\begin{aligned}\int e^{2x}\sin x\,dx &= -\int e^{2x}\cdot d\cos x = -\left(e^{2x}\cdot\cos x - \int \cos x\,de^{2x}\right)\\ &= -e^{2x}\cos x + \int\cos x\cdot(e^{2x})'\,dx = -e^{2x}\cos x + 2\int\cos x\cdot e^{2x}\,dx\\ &= -e^{2x}\cos x + 2\int e^{2x}\cdot(\sin x)'\,dx = -e^{2x}\cos x + 2\int e^{2x}\,d\sin x.\end{aligned}$$

We then take $u = e^{2x}$ and $v = \sin x$, and thus have

$$\begin{aligned}\int e^{2x}\sin x\,dx &= -e^{2x}\cos x + 2\int e^{2x}\,d\sin x = -e^{2x}\cos x + 2e^{2x}\sin x - 2\int \sin x\,de^{2x}\\ &= e^{2x}(2\sin x - \cos x) - 2\int \sin x\cdot (e^{2x})'\,dx\\ &= e^{2x}(2\sin x - \cos x) - 4\int e^{2x}\sin x\,dx.\end{aligned}$$

Thus, we have

$$5\int e^{2x}\sin x\,dx = e^{2x}(2\sin x - \cos x) + C,$$

say,

$$\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C.$$

Example (from classviva.org). evaluate

$$\int x^2 e^{2x}\,dx.$$

solution By $u = x^2$, $v = e^{2x}$ for the first integration by parts, and $u = x$, $v = e^{2x}$ for the second one, we have

$$\begin{aligned}\int x^2e^{2x}\,dx &= \frac{1}{2}\int x^2(e^{2x})'\,dx = \frac{1}{2}\int x^2\,de^{2x}\\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}\int e^{2x}\,dx^2 = \frac{1}{2}x^2e^{2x} - \frac{1}{2}\int e^{2x}\cdot 2x\,dx\\ &= \frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \frac{1}{2}\int x(e^{2x})'\,dx\\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}\int x\,de^{2x} = \frac{1}{2}x^2e^{2x} - \frac{1}{2}\left[xe^{2x} - \int e^{2x}\,dx\right]\\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}\left[xe^{2x} - \frac{1}{2}e^{2x}\right] + C = \frac{1}{2}e^{2x}\left(x^2 - x + \frac{1}{2}\right) + C.\end{aligned}$$

Exercise. Evaluate $\int_0^1 (x^2+x)e^{-2x}\,dx$.
Rk. To evaluate some integral like $\int p(x)e^{kx}\,dx$, where $p(x)$ is a polynomial function, we can use the integration by parts for $u = p(x)$ and $v = \frac{1}{k}e^{kx}$.
Example (from classviva.org). evaluate

$$\int x^2\arctan x\,dx.$$

Rk. Sometimes, we also write $\arctan x = \tan^{-1} x$ exactly as the inverse function of $\tan x$. Choose one of it to do the calculation.
solution
Let's take $u = \arctan x$ and $v = x^3$, we have

$$\begin{aligned}\int x^2\arctan x\,dx &= \int \arctan x\cdot\left(\frac{1}{3}x^3\right)'\,dx = \frac{1}{3}\int \arctan x\cdot\,dx^3\\ &= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int x^3\,d\arctan x = \frac{1}{3}x^3\arctan x - \frac{1}{3}\int x^3\cdot(\arctan x)'\,dx\\ &= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int x^3\cdot\frac{1}{x^2+1}\,dx.\end{aligned}$$

Note that

$$\int x^3 \cdot \frac{1}{x^2+1}\,dx = \int \frac{1}{x^2+1}\cdot\left(\frac{1}{4}x^4\right)'\,dx = \frac{1}{4}\int \frac{1}{x^2+1}\,dx^4.$$

Let $u = x^2$, we then have $x^4 = u^2$, $dx^4 = du^2 = 2u\,du$ and

$$\begin{aligned}\int x^3 \cdot \frac{1}{x^2+1}\,dx &= \frac{1}{4}\int \frac{1}{x^2+1}\,dx^4 = \frac{1}{4}\int \frac{1}{u+1}\cdot 2u\,du \\ &= \frac{1}{2}\int \frac{u}{u+1}\,du = \frac{1}{2}\int \frac{u+1-1}{u+1}\,du = \frac{1}{2}\int 1\,du - \frac{1}{2}\int \frac{1}{u+1}\,du \\ &= \frac{1}{2}u - \frac{1}{2}\ln|u+1| + C = \frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + C.\end{aligned}$$

Thus,

$$\begin{aligned}\int x^2 \arctan x\,dx &= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int x^3\cdot\frac{1}{x^2+1}\,dx \\ &= \frac{1}{3}x^3\arctan x - \frac{1}{3}\left(\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1)\right) + C \\ &= \frac{1}{3}x^3\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + C\end{aligned}$$

Example (from classviva.org). Suppose that $f(1) = 8$, $f(4) = -8$, $f'(1) = 3$, $f'(4) = 8$ and f'' is continuous. Evaluate

$$\int_1^4 x f''(x)\,dx.$$

solution Using the integration by parts, we have

$$\int_1^4 x f''(x)\,dx = \int_1^4 x[f'(x)]'\,dx = \int_1^4 x\,df'(x) = xf'(x)|_1^4 - \int_1^4 f'(x)\,dx.$$

By the FTC, we have

$$\begin{aligned}\int_1^4 x f''(x)\,dx &= xf'(x)|_1^4 - f(x)|_1^4 = 4f'(4) - f'(1) - [f(4) - f(1)] \\ &= 4\cdot 8 - 3 - (-8) + 8 = 45.\end{aligned}$$

Example (from classviva.org). Using integration by parts and the formula

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

to evaluate $\int_1^e \ln x\,dx$.

solution (manipulate this) Taking $u = \ln x$ and $v = x$

$$\begin{aligned}\int_1^e \ln x\,dx &= x\ln x|_1^e - \int_1^e x\,d\ln x = x\ln x|_1^e - \int_1^e x\cdot(\ln x)'\,dx \\ &= x\ln x|_1^e - \int_1^e 1\,dx = x\ln x|_1^e - x|_1^e = (x\ln x - x)|_1^e \\ &= e\ln e - e - (\ln 1 - 1) = e - e + 1 = 1.\end{aligned}$$

Rk. If f and f^{-1} are inverse functions to each other (i.e., $f(f^{-1}) = 1$) and f' is continuous, we then have

$$\int_a^b f(x)\,dx = xf(x)|_a^b - \int_a^b x\,df(x) = bf(b) - af(a) - \int_a^b x\,df(x)$$

We take $y = f(x)$, thus by the definition of inverse function $x = f^{-1}(y)$, we have

$$\begin{cases} x: & a \to b \\ y: & f(a) \to f(b) \end{cases}$$

then the fact is

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b x\,df(x) = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy.$$

Note that $\ln x$ and e^y are inverse to each other and

$$\begin{aligned}\int_a^b \ln x\,dx &= b\ln(b) - a\ln(a) - \int_{\ln(a)}^{\ln(b)} e^y\,dy = b\ln(b) - a\ln(a) - e^y\big|_{\ln(a)}^{\ln(b)} \\ &= b\ln(b) - a\ln(a) - \left[e^{\ln(b)} - e^{\ln(a)}\right] = b\ln(b) - a\ln(a) - b + a.\end{aligned}$$

Thus,

$$\int_1^e \ln x\,dx = e\ln e - \ln 1 - e + 1 = e - e + 1 = 1.$$

Integrals with different integration techniques

Example.

Evaluate

$$\int x^p \ln x\,dx,$$

where the constant p is a real number.

solution

- If $p = -1$, we have

$$\int x^{-1}\ln x\,dx = \int \ln x \cdot (\ln x)'\,dx = \int \ln x\,d\ln x.$$

  Using the substitution $u = \ln x$, and the FTC, we have

$$\int x^{-1}\ln x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

- If $p \neq -1$, $p + 1 \neq 0$, let $u = \ln x$, $v = x^{p+1}$, using the integration by parts, we have

$$\begin{aligned}\int x^p \ln x\,dx &= \int \ln x \cdot \left(\frac{x^{p+1}}{p+1}\right)'\,dx = \frac{1}{p+1}\int \ln x\,dx^{p+1} \\ &= \frac{1}{p+1}\left(x^{p+1}\ln x - \int x^{p+1}\,d\ln x\right) = \frac{1}{p+1}\left(x^{p+1}\ln x - \int x^{p+1}\cdot(\ln x)'\,dx\right) \\ &= \frac{1}{p+1}\left(x^{p+1}\ln x - \int x^{p+1}\cdot\frac{1}{x}\,dx\right) = \frac{1}{p+1}\left(x^{p+1}\ln x - \int x^p\,dx\right) \\ &= \frac{1}{p+1}x^{p+1}\left(\ln x - \frac{1}{p+1}\right) + C.\end{aligned}$$

Exercise. Evaluate $\int x^3 \ln x\,dx$.

Example (for the trigonometric integral). Evaluate $\int \sin^4\theta\,d\theta$.

solution This can be evaluated directly by the double angle formula below,

$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

Here, we use the integration by parts,

$$\begin{aligned}\int \sin^4\theta\, d\theta &= \int \sin^3\theta \cdot \sin\theta\, d\theta = -\int \sin^3\theta\, d\cos\theta \\ &= -\cos\theta\sin^3\theta + \int \cos\theta d\sin^3\theta = -\cos\theta\sin^3\theta + \int \cos\theta\cdot 3\sin^2\theta\cos\theta\, d\theta \\ &= -\cos\theta\sin^3\theta + 3\int \sin^2\theta(1-\sin^2\theta)\, d\theta \\ &= -\cos\theta\sin^3\theta + 3\int \sin^2\theta\, d\theta - 3\int \sin^4\theta\, d\theta.\end{aligned}$$

Thus

$$\begin{aligned}\int \sin^4\theta\, d\theta &= -\frac{1}{4}\cos\theta\sin^3\theta + \frac{3}{4}\int \sin^2\theta\, d\theta = -\frac{1}{4}\cos\theta\sin^3\theta + \frac{3}{4}\int \frac{1}{2}(1-\cos 2\theta)\, d\theta \\ &= -\frac{1}{4}\cos\theta\sin^3\theta + \frac{3}{8}\theta - \frac{3}{16}\sin 2\theta + C.\end{aligned}$$

Rk. In general,

$$\int \sin^n\theta\, d\theta = -\frac{\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta\, d\theta,$$

especially, we have $\int \sin^2\theta\, d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\int 1\, d\theta$.