Lecture 4–Trigonometric Integrals

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1 Recap last time

Integration by parts Let u and v are differentiable in x. Then

- for the indefinite integral, we have

$$\int uv'\,dx = \int u\,dv = uv - \int v\,du = uv - \int vu'\,dx,$$

- for the definite integral, by the FTC, we have

$$\int_a^b uv'\,dx = (uv)|_a^b - \int_a^b vu'\,dx$$

Example. evaluate $\int x\sec^2 x\,dx$.
solution Using the integration by parts, take $u = x$, $v = \tan x$, we have

$$\begin{aligned}\int x\sec^2 x\,dx &= \int x\cdot(\tan x)'\,dx = \int x\,d\tan x\\ &= x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x| + C.\end{aligned}$$

Example. evaluate $\int_0^{\frac{\pi}{4}} x\sec x\tan x\,dx$.
solution
Using the integration by parts, take $u = x$, $v = \sec x$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{4}} x\sec x\tan x\,dx &= \int_0^{\frac{\pi}{4}} x(\sec x)'\,dx = \int_0^{\frac{\pi}{4}} x\,d\sec x\\ &= x\sec x|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\sec x\,dx = x\sec x|_0^{\frac{\pi}{4}} - \ln|\sec x + \tan x||_0^{\frac{\pi}{4}}\\ &= \frac{\pi}{4}\frac{1}{\cos\frac{\pi}{4}} - \left[\ln|\frac{1}{\cos\frac{\pi}{4}} + \tan\frac{\pi}{4}|\right] = \frac{\pi}{4}\sqrt{2} - \ln(\sqrt{2}+1)\end{aligned}$$

Trigonometric identities

$$\begin{aligned}\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad \cot^2 x + 1 = \csc^2 x\\ \sin x\csc x = 1, \quad \cos x\sec x = 1, \quad \tan x\cot x = 1,\\ \sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\frac{\sin x}{\cos x} = -\tan x.\end{aligned}$$

Angle / difference sum

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha\\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}$$

Double angle formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha,$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha,$$
$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Thus, we have

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right], \tag{1}$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right], \tag{2}$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right], \tag{3}$$

and

$$\sin 4\alpha = 2\sin 2\alpha\cos 2\alpha = 4\sin\alpha\cos\alpha(2\cos^2\alpha - 1)$$
$$\cos 4\alpha = 2\cos^2(2\alpha) - 1 = 2\left[2\cos^2\alpha - 1\right]^2 - 1.$$

half angle identities

$$\sin^2\alpha = \frac{1}{2}\left[1 - \cos 2\alpha\right]$$
$$\cos^2\alpha = \frac{1}{2}\left[1 + \cos 2\alpha\right]$$
$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha} \quad \text{or} \quad \tan\alpha = \frac{1-\cos 2\alpha}{\sin 2\alpha} = \frac{\sin 2\alpha}{1+\cos 2\alpha}.$$

2 Trigonometric Integrals

The first type:

- Evaluate $\int \sin mx \cdot \cos nx \, dx$, using the fact that (1);
- Evaluate $\int \sin mx \cdot \sin nx \, dx$, using the fact that (2);
- Evaluate $\int \cos mx \cdot \cos nx \, dx$, using the fact that (3).

Example (involving $\sin mx$ and $\cos nx$). Evaluate $\int \sin 2x \cos 3x \, dx$.
solution In view of the fact below,

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right].$$

Thus, we have

$$\int \sin 2x\cos 3x \, dx = \frac{1}{2}\int \sin(2x-3x) + \sin(2x+3x) \, dx = \frac{1}{2}\int \sin(-x) + \sin 5x \, dx$$
$$= \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C.$$

Example (involving $\cos mx$ and $\cos nx$). Evaluate $\int \cos 4x \cos x \, dx$.
solution In view of the fact below,

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right].$$

Thus, we have

$$\int \cos 4x \cos x \, dx = \frac{1}{2}\int \cos(4x-x)+\cos(4x+x)\, dx = \frac{1}{2}\int \cos 3x + \cos 5x \, dx$$
$$= \frac{1}{6}\sin 3x + \frac{1}{10}\sin 5x + C.$$

Example (from classviva.org). Evaluate

$$\int \sin x \sin 2x \sin 3x \, dx.$$

solution

Note that the trigonometric identity

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right],$$

thus we have

$$\sin x \sin 2x = \frac{1}{2}\left[\cos(x-2x)-\cos(x+2x)\right] = \frac{1}{2}\left[\cos x - \cos 3x\right].$$

Also, note that the identity,

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right],$$

thus we have

$$\sin 3x \cos x = \frac{1}{2}\left[\sin 2x + \sin 4x\right]$$
$$\sin 3x \cos 3x = \frac{1}{2}\sin 6x.$$

Then,

$$\int \sin x \sin 2x \sin 3x \, dx = \int \sin 3x \cdot \frac{1}{2}\left[\cos x - \cos 3x\right] \, dx = \frac{1}{2}\int (\sin 3x\cos x - \sin 3x \cos 3x) \, dx$$
$$= \frac{1}{4}\int (\sin 2x + \sin 4x) \, dx - \frac{1}{4}\int \sin 6x \, dx$$
$$= -\frac{1}{8}\cos 2x - \frac{1}{16}\cos 4x + \frac{1}{24}\sin 6x + C.$$

Rk. For the definite integral in a periodic domain $[-\pi,\pi]$, we have

$$\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \int_{-\pi}^{\pi} \sin mx \cdot \cos nx \, dx = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n. \end{cases}$$

Since that (here for $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$)

- If $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m-n)x - \cos(m+n)x\right] \, dx$$
$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right] \Big|_{-\pi}^{\pi} = 0.$$

- If $m = n$, we have

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1-\cos 2mx) \, dx$$
$$= \left[\frac{x}{2} - \frac{\sin 2mx}{2\cdot 2m}\right] \Big|_{-\pi}^{\pi} = \pi.$$

The second type:

1. Evaluate $\int \sin^m x \cdot \cos^n x\, dx$,
   - if m odd, e.g., $m = 3$, we have $\sin^3 x = \sin^2 \sin x = (1 - \cos^2 x)(-\cos x)'$, then this integral only depends on $\cos x$, we can use the substitution $u = \cos x$;
   - by the same way, if n odd, e.g., $n = 3$, we have $\cos^3 x = \cos^2 \cos x = (1 - \sin^2 x)(\sin x)'$, then this integral only depends on $\sin x$, we can use the substitution $u = \sin x$;
   - if m and n all even, e.g., $m = 4$, $n = 2$, by the half angle identities, we have

$$\begin{aligned}\sin^4 x \cos^2 x = \sin^2 x(\sin x \cos x)^2 &= \frac{1}{2}(1 - \cos 2x) \cdot \left(\frac{1}{2}\sin 2x\right)^2 \\ &= \frac{1}{8}(1 - \cos 2x) \cdot \frac{1}{2}(1 - \cos 4x).\end{aligned}$$

   By the substitution $u = \cos 2x$ and the integral involving $\int \cos 2x \cos 4x\, dx$ before.

2. Evaluate $\int \tan^m x \cdot \sec^n x\, dx$ (similar as before),
   - if n even, e.g., $n = 2$, let $u = \tan x$, $du = \sec^2 x\, dx$; transfer the trigonometric integral to be an integral of a polynomial in u;
   - if m odd, e.g., $m = 3$, let $u = \sec x$, $du = \sec x \tan x\, dx$, transfer the trigonometric integral to be an integral of a polynomial in u.

Example (involving $\sin^m x$ and $\cos^n x$). Evaluate

$$\int \sin^2 x \cos x\, dx, \quad \int \sin^3 x \cos^2 x\, dx, \quad \int \sin^2 x \cos^2 x\, dx.$$

solution

- Let $u = \sin x$, we have

$$\begin{aligned}\int \sin^2 x \cos x\, dx = \int \sin^2 x \cdot (\sin x)'\, dx &= \int \sin^2 x\, d\sin x \\ = \int u^2\, du &= \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C.\end{aligned}$$

- Let $u = \cos x$, we have

$$\begin{aligned}\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \cos^2 x \cdot \sin x\, dx = -\int \sin^2 x \cos^2 x \cdot (\cos x)'\, dx \\ &= -\int (1 - \cos^2 x)\cos^2 x\, d\cos x = -\int (1 - u^2)u^2\, du \\ &= \int u^4 - u^2\, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.\end{aligned}$$

- Since by the double-angle formula, we have

$$\begin{aligned}\sin^2 x \cos^2 x &= \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) = \frac{1}{4}(1 - \cos^2 2x) = \frac{1}{4}\left[1 - \frac{1}{2}(1 + \cos 4x)\right] \\ &= \frac{1}{8}(1 - \cos 4x),\end{aligned}$$

  or

$$\sin x \cos x = \frac{1}{2}\sin 2x, \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x),$$

Thus,

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int \sin^2 2x \, dx = \frac{1}{4}\int \frac{1}{2}(1-\cos 4x) \, dx$$
$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$$

Example (from classviva.org). Evaluate

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^{14} x \, dx.$$

solution Let $u = \cos x$, we then have $x : 0 \to \frac{\pi}{2}$, $u : \cos 0 = 1 \to \cos\frac{\pi}{2} = 0$ and

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^{14} x \, dx = \int_0^{\frac{\pi}{2}} \sin^4 x \cos^{14} x \cdot \sin x \, dx = -\int_0^{\frac{\pi}{2}} \sin^4 x \cos^{14} x \cdot (\cos x)' \, dx$$
$$= -\int_0^{\frac{\pi}{2}} (1-\cos^2 x)^2 \cos^{14} x \, d\cos x = -\int_1^0 (1-u^2)^2 u^{14} \, du$$
$$= \int_0^1 (1-2u^2+u^4)u^{14} \, du = \int_0^1 \left(u^{14} - 2u^{16} + u^{18}\right) \, du$$
$$= \left[\frac{1}{15}u^{15} - 2\cdot\frac{1}{17}u^{17} + \frac{1}{19}u^{19}\right]|_0^1 = \frac{1}{15} - \frac{2}{17} + \frac{1}{19} = \frac{8}{4845}.$$

Exercise. Evaluate $\int \sin^4\theta \cos^3\theta \, d\theta$ (hint: let $u = \sin\theta$).

Example (from classviva.org). Evaluate

$$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^4 x \, dx.$$

solution Let $u = \tan x$, we then have $x : 0 \to \frac{\pi}{3}$, $u : \tan 0 = 0 \to \tan\frac{\pi}{3} = \sqrt{3}$, and

$$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^4 x \, dx = \int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x \cdot \sec^2 x \, dx = \int_0^{\frac{\pi}{3}} \tan^5 x (1+\tan^2 x) \cdot (\tan x)' \, dx$$
$$= \int_0^{\frac{\pi}{3}} \tan^5 x(1+\tan^2 x) \, d\tan x = \int_0^{\sqrt{3}} u^5(1+u^2) \, du$$
$$= \int_0^{\sqrt{3}} \left(u^5 + u^7\right) \, du = \left[\frac{1}{6}u^6 + \frac{1}{8}u^8\right]|_0^{\sqrt{3}} = \frac{1}{6}(\sqrt{3})^6 + \frac{1}{8}(\sqrt{3})^8$$
$$= \frac{9}{2} + \frac{81}{8} = \frac{117}{8}.$$

Exercise. Evaluate $\int \tan^3 x \sec^5 x \, dx$ (hint: let $u = \sec x$).

The third type:

1. for the even power,
   - evaluate $\int \cos^{2m}\theta \, d\theta$, $m \geq 1$ being the integer, e.g., $m = 2$, $\int \cos^4\theta \, d\theta$: it can be handled by the "double angle formula" and some substitution rule $u = \sin k\theta$;
   - evaluate $\int \sin^{2n}\theta \, d\theta$, $n \geq 1$ being the integer, e.g., $m = 2$, $\int \sin^4\theta \, d\theta$: it also can be handled by "double angle formula" and some substitution rule $u = \cos k\theta$;
   - evaluate $\int \tan^{2m}\theta \, d\theta$, $m \geq 1$ being the integer, e.g., $m = 2$, $\int \tan^4\theta \, d\theta$: it can be handled by the identity $\tan^2\theta = \sec^2\theta - 1$ and some substitution rule $u = \tan\theta$;
   - evaluate $\int \sec^{2n}\theta \, d\theta$, $n \geq 1$ being the integer, e.g., $m = 2$, $\int \sec^4\theta \, d\theta$: it can be handled by the identity $\sec^2\theta = 1 + \tan^2\theta$ and some substitution rule $u = \tan\theta$;

2. for the odd power,

- evaluate $\int \cos^{2m+1}\theta\, d\theta$, $m \geq 1$ being the integer, e.g., $m = 2$, let $u = \sin\theta$,

$$\begin{aligned}\int \cos^5\theta\, d\theta &= \int \cos^4\theta\, d\sin\theta = \int \left(1-\sin^2\theta\right)^2\, d\sin\theta = \int (1-u^2)^2\, du \\ &= \int \left(1-2u^2+u^4\right)\, du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C.\end{aligned}$$

- evaluate $\int \sin^{2n+1}\theta\, d\theta$, $n \geq 1$ being the integer, e.g., $n = 1$, let $u = \cos\theta$,

$$\begin{aligned}\int \sin^3\theta\, d\theta &= \int \sin^2\theta\cdot\sin\theta\, d\theta = \int \sin^2\theta\cdot(-\cos\theta)'\, d\theta \\ &= -\int (1-\cos^2\theta)\, d\cos\theta = \int (u^2-1)\, du = \frac{1}{3}u^3 - u + C \\ &= \frac{1}{3}\cos^3\theta - \cos\theta + C.\end{aligned}$$

- evaluate $\int \tan^{2m+1}\theta\, d\theta$, $m \geq 1$ being the integer, e.g., $m = 2$, using $\tan^2\theta = \sec^2\theta - 1$, $(\tan\theta)' = \sec^2\theta$,

$$\begin{aligned}\int \tan^5\theta\, d\theta &= \int \tan^3\theta\cdot\tan^2\theta\, d\theta = \int \tan^3\theta\cdot(\sec^2\theta - 1)\, d\theta \\ &= \int \tan^3\theta\cdot\sec^2\theta\, d\theta - \int \tan^3\theta\, d\theta = \int \tan^3\theta\, d\tan\theta - \int \tan^3\theta\, d\theta \\ &= \frac{1}{4}\tan^4\theta\, d\theta - \int \tan^3\theta\, d\theta,\end{aligned}$$

note that

$$\begin{aligned}\int \tan^3\theta\, d\theta &= \int \tan\theta\cdot\tan^2\theta\, d\theta = \int \tan\theta\cdot(\sec^2\theta - 1)\, d\theta \\ &= \int \tan\theta\sec^2\theta\, d\theta - \int \tan\theta\, d\theta = \int \tan\theta\, d\tan\theta - \int \tan\theta\, d\theta \\ &= \frac{1}{2}\tan^2\theta - \int \tan\theta\, d\theta,\end{aligned}$$

also, note

$$\int \tan\theta\, d\theta = \ln|\sec\theta| + C.$$

Thus, we have

$$\begin{aligned}\int \tan^3\theta\, d\theta &= \frac{1}{2}\tan^2\theta - \ln|\sec\theta| + C \\ \int \tan^5\theta\, d\theta &= \frac{1}{4}\tan^4\theta - \frac{1}{2}\tan^2\theta + \ln|\sec\theta| + C, \\ \int \tan^7\theta\, d\theta &= \int \tan^5\theta(\sec^2\theta - 1)\, d\theta = \frac{1}{6}\tan^6\theta - \int \tan^5\theta\, d\theta \\ &= \frac{1}{6}\tan^6\theta - \frac{1}{4}\tan^4\theta + \frac{1}{2}\tan^2\theta - \ln|\sec\theta| + C.\end{aligned}$$

- evaluate $\int \sec^{2n+1}\theta\, d\theta$, $n \geq 1$ being the integer, e.g., $n = 2$, using the integration by parts, and $(\tan\theta)' = \sec^2\theta$, $(\sec\theta)' = \tan\theta\sec\theta$ and $\tan^2\theta = \sec^2\theta - 1$,

$$\begin{aligned}\int \sec^5\theta\, d\theta &= \int \sec^3\theta \cdot \sec^2\theta\, d\theta = \int \sec^3\theta\, d\tan\theta \\ &= \tan\theta\sec^3\theta - \int \tan\theta\, d(\sec^3\theta) = \tan\theta\sec^3\theta - \int \tan\theta \cdot 3\sec^2\theta \cdot \tan\theta\sec\theta\, d\theta \\ &= \tan\theta\sec^3\theta - 3\int \tan^2\theta \cdot \sec^3\theta\, d\theta = \tan\theta\sec^3\theta - 3\int (\sec^2\theta - 1)\cdot\sec^3\theta\, d\theta \\ &= \tan\theta\sec^3\theta - 3\int \sec^5\theta\, d\theta + 3\int \sec^3\theta\, d\theta.\end{aligned}$$

Thus, we have

$$4\int \sec^5\theta\, d\theta = \tan\theta\sec^3\theta + 3\int \sec^3\theta\, d\theta.$$

Note that (by the same way of power reduction)

$$\begin{aligned}\int \sec^3\theta\, d\theta &= \int \sec\theta \cdot \sec^2\theta\, d\theta = \int \sec\theta\, d\tan\theta \\ &= \tan\theta\sec\theta - \int \tan\theta\, d\sec\theta = \tan\theta\sec\theta - \int \tan\theta \cdot (\tan\theta\sec\theta)\, d\theta \\ &= \tan\theta\sec\theta - \int \tan^2\theta\sec\theta\, d\theta = \tan\theta\sec\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta \\ &= \tan\theta\sec\theta - \int \sec^3\theta + \int \sec\theta\, d\theta.\end{aligned}$$

Thus, we have

$$2\int \sec^3\theta\, d\theta = \tan\theta\sec\theta + \int \sec\theta\, d\theta.$$

Also, note

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Totally, we have

$$\begin{aligned}\int \sec^3\theta\, d\theta &= \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C, \\ \int \sec^5\theta\, d\theta &= \frac{1}{4}\tan\theta\sec^3\theta + \frac{3}{4}\left[\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right] + C \\ &= \frac{1}{4}\tan\theta\sec^3\theta + \frac{3}{8}\tan\theta\sec\theta + \frac{3}{8}\ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

Example. Evaluate $\int \cos^4\theta\, d\theta$.
solution By the double angle formula

$$\begin{aligned}\cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta), \\ \cos^2 2\theta &= \frac{1}{2}(1 + \cos 4\theta),\end{aligned}$$

we have

$$\begin{aligned}\int \cos^4\theta\, d\theta &= \int (\cos^2\theta)^2\, d\theta = \frac{1}{4}\int (\cos 2\theta + 1)^2 = \frac{1}{4}\int \left(1 + 2\cos 2\theta + \cos^2 2\theta\right)\, d\theta \\ &= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)\, d\theta = \frac{1}{4}\left(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right) + C \\ &= \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C.\end{aligned}$$

Exercise. Evaluate $\int \sin^6\theta\, d\theta$.
Example. Evaluate $\int \tan^4\theta\, d\theta$.
solution Let $u = \tan\theta$, we thus have $du = \sec^2\theta\, d\theta$ and

$$\int \tan^4\theta\, d\theta = \int \tan^2\theta \tan^2\theta\, d\theta = \int \tan^2\theta(\sec^2\theta - 1)\, d\theta = \int \tan^2\theta \sec^2\theta\, d\theta - \int \tan^2\theta\, d\theta$$
$$= \int u^2 du - \int (\sec^2\theta - 1)\, d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta + C.$$

Exercise. Evaluate $\int \tan^6\theta\, d\theta$.
Example (from classviva.org). Evaluate

$$\int_0^{\frac{\pi}{2}} \sec^4\left(\frac{t}{2}\right)\, dt.$$

solution Let first $u = \frac{t}{2}$, we then have $t : 0 \to \frac{\pi}{2}$, $u : 0 \to \frac{\pi}{4}$, $dt = 2\, du$ and

$$\int_0^{\frac{\pi}{2}} \sec^4\left(\frac{t}{2}\right)\, dt = 2\int_0^{\frac{\pi}{4}} \sec^4 u\, du2 = \int_0^{\frac{\pi}{4}} \sec^2 u \cdot (\tan u)'\, du$$
$$= \int_0^{\frac{\pi}{4}} (1 + \tan^2 u)\, d\tan u.$$

Then use $v = \tan u$, we thus have $u : 0 \to \frac{\pi}{4}$, $v : \tan 0 = 0 \to \tan\frac{\pi}{4} = 1$ and

$$\int_0^{\frac{\pi}{2}} \sec^4\left(\frac{t}{2}\right)\, dt = \int_0^{\frac{\pi}{4}} (1 + \tan^2 u)\, d\tan u = \int_0^1 (1 + v^2)\, dv$$
$$= \left(v + \frac{1}{3}v^3\right)|_0^1 = 1 + \frac{1}{3} = \frac{4}{3}.$$

Example (evaluate the integral with trigonometric substitutions). Note that the trigonometric substitutions below,

- let $x = a\sin\theta$, then $a^2 - x^2 = a^2\cos^2\theta$;
- let $x = a\tan\theta$, then $a^2 + x^2 = a^2\sec^2\theta$;
- let $x = a\sec\theta$, then $x^2 - a^2 = a^2\tan^2\theta$.

Evaluate $\int x^2\sqrt{9 - x^2}\, dx$
solution Let $\frac{x}{3} = \sin\theta$, we then have $\theta = \sin^{-1}(\frac{x}{3})$, $\cos\theta = \frac{\sqrt{9-x^2}}{3}$, $dx = 3\cos\theta\, d\theta$,

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta = 2\sin\theta\cos\theta \cdot (1 - 2\sin^2\theta)$$
$$= \frac{2x\sqrt{9 - x^2}}{9}\left(1 - \frac{2x^2}{9}\right)$$

and

$$\int x^2\sqrt{9 - x^2}\, dx = \int x^2\sqrt{9\left(1 - \left(\frac{x}{3}\right)^2\right)}\, dx = 3\int x^2\sqrt{1 - \left(\frac{x}{3}\right)^2}\, dx$$
$$= 3\int 9\sin^2\theta\cos\theta \cdot 3\cos\theta\, d\theta = 81\int \sin^2\theta\cos^2\theta\, d\theta$$
$$= 81\int (\frac{1}{2}\sin 2\theta)^2\, d\theta = \frac{81}{4}\int \sin^2 2\theta\, d\theta = \frac{81}{4}\int \frac{1}{2}(1 - \cos 4\theta)\, d\theta$$
$$= \frac{81}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right) + C = \frac{81}{8}\sin^{-1}\left(\frac{x}{3}\right) - \frac{81}{32}\frac{2x\sqrt{9 - x^2}}{9}\left(1 - \frac{2x^2}{9}\right) + C.$$

Exercise. Evaluate

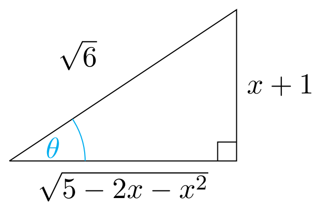
$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^3\sqrt{9x^2-1}},$$

(hint: using the substitution $3x = \sec\theta$ and $\sec^2\theta - 1 = \tan^2\theta$).

Example (variations of completing the square). Evaluating

$$\int \frac{x^2}{\sqrt{5-2x-x^2}}\,dx.$$

solution Let $x+1 = \sqrt{6}\sin\theta$, we then have $\theta = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$, $dx = \sqrt{6}\cos\theta$, and



$$\cos\theta = \frac{\sqrt{5-2x-x^2}}{\sqrt{6}},$$

and

$$\begin{aligned}
\int \frac{x^2}{\sqrt{5-2x-x^2}}\,dx &= \int \frac{x^2}{\sqrt{6-(2x+x^2+1)}}\,dx = \int \frac{x^2}{\sqrt{6-(x+1)^2}}\,dx \\
&= \int \frac{(\sqrt{6}\sin\theta - 1)^2}{\sqrt{6\cos^2\theta}} \cdot \sqrt{6}\cos\theta\,d\theta = \int \left(6\sin^2\theta - 2\sqrt{6}\sin\theta + 1\right)\,d\theta \\
&= \int \left(6\cdot\frac{1}{2}(1-\cos 2\theta) - 2\sqrt{6}\sin\theta + 1\right)\,d\theta = 4\theta - \frac{3}{2}\sin 2\theta + 2\sqrt{6}\cos\theta + C \\
&= 4\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) - \frac{(x+1)\sqrt{5-2x-x^2}}{2} + 2\sqrt{5-2x-x^2} + C.
\end{aligned}$$

Rk. For the integral involving ax^2+bx+c, $a \neq 0$, we have

$$ax^2+bx+c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

- if $a > 0$, $c - \frac{b^2}{4a} > 0$, i.e., $b^2 - 4ac < 0$, (or $a < 0$, $c - \frac{b^2}{4a} < 0$, i.e., $b^2 - 4ac > 0$), let $\sqrt{\frac{a}{c-\frac{b^2}{4a}}}\left(x + \frac{b}{2a}\right) = \tan\theta$ and using $1 + \tan^2\theta = \sec^2\theta$;
- if $a > 0$, $c - \frac{b^2}{4a} < 0$, i.e., $b^2 - 4ac > 0$, let $\sqrt{\frac{a}{\frac{b^2}{4a}-c}}\left(x + \frac{b}{2a}\right) = \sec\theta$ and using $\sec^2\theta - 1 = \tan^2\theta$;
- if $a < 0$, $c - \frac{b^2}{4a} > 0$, i.e., $b^2 - 4ac < 0$, let $\sqrt{\frac{-a}{c-\frac{b^2}{4a}}}\left(x + \frac{b}{2a}\right) = \sin\theta$ and using $1 - \sin^2\theta = \cos^2\theta$.