Lecture 5–The method of partial fractions

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1 Recap last time

The first type trigonometric integrals

Evaluate $\int \sin mx \cdot \cos nx \, dx$ or $\int \sin mx \cdot \sin nx \, dx$ or $\int \cos mx \cdot \cos nx \, dx$ by using the fact below,

$$\begin{aligned}\sin\alpha\cos\beta &= \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]\\ \sin\alpha\sin\beta &= \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]\\ \cos\alpha\cos\beta &= \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]\end{aligned}$$

The second type trigonometric integrals

Evaluate $\int \sin^m x \cdot \cos^n x \, dx$, where m, n are positive integers.

- If $m = n = 2$,

$$\begin{aligned}\int \sin^2 x \cdot \cos^2 x \, dx &= \frac{1}{4}\int (2\sin x\cos x)^2 \, dx = \frac{1}{4}\int \sin^2 2x \, dx = \frac{1}{4}\int \frac{1}{2}\left[1-\cos 4x\right] \, dx\\ &= \frac{1}{8}\int (1-\cos 4x) \, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C,\end{aligned}$$

- if $m = 3, n = 2$, let $u = \cos x$,

$$\begin{aligned}\int \sin^3 x \cdot \cos^2 x \, dx = \int \sin^2 x\cos^2 x\sin x \, dx &= -\int (1-\cos^2 x)\cos^2 x \, d\cos x\\ &= -\int (1-u^2)u^2 \, du = \int (u^4-u^2) \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C,\end{aligned}$$

- if $m = 2, n = 3$, let $u = \sin x$,

$$\begin{aligned}\int \sin^2 x \cdot \cos^3 x \, dx = \int \sin^2 x\cos^2 x\cos x \, dx &= \int \sin^2 x(1-\sin^2 x) \, d\sin x\\ &= \int u^2(1-u^2) \, du = \int (u^2-u^4) \, du = \frac{1}{3}u^2 - \frac{1}{5}u^5 + C\\ &= \frac{1}{3}\sin^2 x - \frac{1}{5}\sin^5 x + C,\end{aligned}$$

- if $m = 3, n = 3$, let $u = \sin x$,

$$\begin{aligned}\int \sin^3 x \cdot \cos^3 x \, dx = \int \sin^3 x\cos^2 x \, d\sin x &= \int \sin^3 x \cdot (1-\sin^2 x) \, d\sin x\\ &= \int u^3(1-u^2) \, du = \int (u^3-u^5) \, du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C\\ &= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.\end{aligned}$$

Evaluate $\int \tan^m x \cdot \sec^n x \, dx$, where m, n are positive integers.

- if $m = 2, n = 4$, let $u = \tan x$,

$$\begin{aligned}\int \tan^2 x \sec^4 x \, dx &= \int \tan^2 x \sec^2 x \cdot \sec^2 x \, dx = \int \tan^2 x \sec^2 x \, d\tan x \\ &= \int \tan^2 x \cdot (\tan^2 x + 1) \, d\tan x = \int u^2(u^2+1) \, du = \int (u^4 + u^2) \, du \\ &= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C,\end{aligned}$$

- if $m = 3, n = 2$, let $u = \tan x$,

$$\int \tan^3 x \sec^2 x \, dx = \int \tan^3 x \, d\tan x = \int u^3 \, du = \frac{1}{4}u^4 + C = \frac{1}{4}\tan^4 x + C,$$

- if $m = 2, n = 3$,

$$\begin{aligned}\int \tan^2 x \sec^3 x \, dx &= \int \tan x \sec^2 x \cdot \tan x \sec x \, dx = \int \tan x(1 + \tan^2 x) \, d\sec x \\ &= \int \tan x \, d\sec x + \int \tan^3 x \, d\sec x \\ &= \tan x \sec x - \int \sec x \, d\tan x + \tan^3 x \sec x - \int \sec x \, d\tan^3 x \\ &= \tan x \sec x - \int \sec^3 x \, dx + \tan^3 x \sec x - 3\int \tan^2 x \sec^3 x \, dx.\end{aligned}$$

We have

$$\int \tan^2 x \sec^3 x \, dx = \frac{1}{4}\tan x \sec x + \frac{1}{4}\tan^3 x \sec x - \frac{1}{4}\int \sec^3 x \, dx$$

From below, we know that

$$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

In turn,

$$\int \tan^2 x \sec^3 x \, dx = \frac{1}{8}\tan x \sec x + \frac{1}{4}\tan^3 x \sec x - \frac{1}{8}\ln|\sec x + \tan x| + C.$$

- if $m = 3, n = 3$, let $u = \sec x$,

$$\begin{aligned}\int \tan^3 x \sec^3 x \, dx &= \int \tan^2 x \sec^2 x \cdot (\tan x \sec x) \, dx = \int (\sec^2 x - 1) \cdot \sec^2 x \, d\sec x \\ &= \int (u^2 - 1)u^2 \, du = \int (u^4 - u^2) \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.\end{aligned}$$

The third type trigonometric integrals

Evaluate $\int \cos^m x \, dx$, or $\int \sin^n x \, dx$ or $\int \tan^m x \, dx$ or $\int \sec^n x \, dx$

- if $m = 4$,

$$\begin{aligned}\int \cos^4 x \, dx &= \int (\cos^2 x)^2 \, dx = \int \left(\frac{1}{2}[1 + \cos 2x]\right)^2 \, dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2}[1 + \cos 4x] \, dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C,\end{aligned}$$

- if $m = 3$, let $u = \sin x$,

$$\int \cos^3 x\, dx = \int \cos^2 x \cdot \cos x\, dx = \int (1 - \sin^2 x)\, d\sin x = \int (1 - u^2)\, du$$
$$= u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C.$$

- if $m = 4$, $u = \tan x$,

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx$$
$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx = \int \tan^2 x\, d\tan x - \int (\sec^2 x - 1)\, dx$$
$$= \frac{1}{3}\tan^3 x - \tan x + x + C.$$

- if $m = 3$, note that $(\ln|\sec x|)' = \tan x$

$$\int \tan^3 x\, dx = \int \tan x \cdot \tan^2 x\, dx = \int \tan x \cdot (\sec^2 x - 1)\, dx = \int \tan x \sec^2 x\, dx - \int \tan x\, dx$$
$$= \int \tan x\, d\tan x - \int \tan x\, dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C.$$

- if $m = 4$, $u = \tan x$,

$$\int \sec^4 x\, dx = \int \sec^2 x \cdot \sec^2 x\, dx = \int (1 + \tan^2 x)\, d\tan x = \int (1 + u^2)\, du$$
$$= u + \frac{1}{3}u^3 + C = \tan x + \tan^3 x + C.$$

- if $m = 3$, note that $(\ln|\sec x + \tan x|)' = \sec x$,

$$\int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx = \int \sec x\, d\tan x$$
$$= \sec x \tan x - \int \tan x\, d\sec x = \sec x \tan x - \int \tan^2 x \sec x\, dx$$
$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx.$$

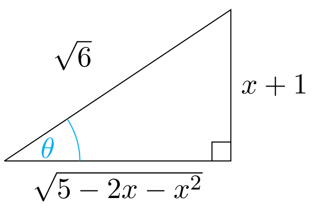
We have

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Example for trigonometric substitutions (variations of completing the square) Evaluate

$$\int \frac{x^2}{\sqrt{5 - 2x - x^2}}\, dx.$$

solution Let $x + 1 = \sqrt{6}\sin\theta$, we then have $\theta = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$, $dx = \sqrt{6}\cos\theta$, and



$$\cos\theta = \frac{\sqrt{5-2x-x^2}}{\sqrt{6}},$$

and

$$\begin{aligned}
\int \frac{x^2}{\sqrt{5-2x-x^2}}\,dx &= \int \frac{x^2}{\sqrt{6-(2x+x^2+1)}}\,dx = \int \frac{x^2}{\sqrt{6-(x+1)^2}}\,dx \\
&= \int \frac{(\sqrt{6}\sin\theta-1)^2}{\sqrt{6\cos^2\theta}}\cdot\sqrt{6}\cos\theta\,d\theta = \int \left(6\sin^2\theta - 2\sqrt{6}\sin\theta+1\right)\,d\theta \\
&= \int \left(6\cdot\frac{1}{2}(1-\cos 2\theta) - 2\sqrt{6}\sin\theta+1\right)\,d\theta = 4\theta - \frac{3}{2}\sin 2\theta + 2\sqrt{6}\cos\theta + C \\
&= 4\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) - \frac{(x+1)\sqrt{5-2x-x^2}}{2} + 2\sqrt{5-2x-x^2} + C.
\end{aligned}$$

2 The method of partial fractions

Example (the intuition of partial fractions for rational functions). Evaluate

$$\int \left(\frac{1}{x-2}+\frac{1}{x+1}\right)\,dx.$$

solution Let $u = x-2$, $v = x+1$, we have $du = dx$, $dv = dx$ and

$$\begin{aligned}
\int \left(\frac{1}{x-2}+\frac{1}{x+1}\right)\,dx &= \int \frac{1}{x-2}\,dx + \int \frac{1}{x+1}\,dx = \int \frac{1}{u}\,du + \int \frac{1}{v}\,dv \\
&= \ln|u| + \ln|v| + C = \ln|x-2| + \ln|x+1| + C.
\end{aligned}$$

Rk. Note that

$$\frac{1}{x-2}+\frac{1}{x+1} = \frac{x+1+x-2}{(x-2)(x+1)} = \frac{2x-1}{x^2-x-2}.$$

Thus, we can evaluate the integral of a rational function below,

$$\int \frac{2x-1}{x^2-x-2}\,dx = \int \left(\frac{1}{x-2}+\frac{1}{x+1}\right)\,dx = \ln|x-2| + \ln|x+1| + C.$$

Example (procedure of decomposition). Evaluate

$$\int \frac{2x-1}{x^2-x-2}\,dx.$$

solution

step 1. Factorize the denominator (the degree of each factor $\leq$ that of original polynomial): $x^2-x-2 = (x-2)(x+1)$.

step2. Write the partial fraction decomposition with coefficients to be determined:

$$\frac{2x-1}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}.$$

step3. Solve A and B:

$$\frac{2x-1}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1)+B(x-2)}{x^2-x-2}.$$

Thus, we have $2x-1 = A(x+1) + B(x-2)$,

✓ take $x = 2$, we have $2 \cdot 2 - 1 = A(2+1)$, then $A = 1$;

✓ take $x = -1$, we have $2 \cdot (-1) - 1 = B \cdot (-3)$, the $B = 1$.

So, we have the decomposition below,

$$\frac{2x-1}{x^2-x-2} = \frac{1}{x-2} + \frac{1}{x+1}.$$

Thus,

$$\int \frac{2x-1}{x^2-x-2}\, dx = \int \frac{1}{x-2}\, dx + \frac{1}{x+1}\, dx = \ln|x-2| + \ln|x+1| + C.$$

Rk.

- Ideas of partial fractions: Decomposition+substitution;
- In this example, The denominator has been decomposed as **two linear factor**. Let's look at more general cases later.
- For this method, we should make sure **the degree of nominator is less than that of denominator**. If we use the linear factor in the nominator here, we have

$$\frac{2x-1}{x^2-x-2} = \frac{Ax+B}{x-2} + \frac{Cx+D}{x+1},$$

in turn, we have $2x - 1 = (Ax+B)(x+1) + (Cx+D)(x-2)$. Thus,

✓ take $x = 2$, we have $3 = 3(2A+B)$;

✓ take $x = -1$, we have $-3 = -3 \cdot (-C+D)$;

✓ take $x = 0$, we have $-1 = B - 2D$;

or using

$$\begin{aligned} 2x-1 &= (Ax+B)(x+1) + (Cx+D)(x-2) \\ &= Ax^2 + (A+B)x + B + Cx^2 + (D-2C)x - 2D \\ &= (A+C)x^2 + (A+B+D-2C)x + B - 2D, \end{aligned}$$

thus

✓ $A + C = 0$;

✓ $A + B + D - 2C = 2$;

✓ $B - 2D = -1$.

We have known that

$$\begin{cases} A + C = 0, \\ A + B + D - 2C = 2, \\ B - 2D = -1, \end{cases} \iff \begin{cases} 2A + B = 1, \\ D - C = 1, \\ B - 2D = -1. \end{cases}$$

(Three equation with four unknowns). Thus,

$$\begin{aligned} A &= 1 - D, \\ B &= 2D - 1, \\ C &= D - 1. \end{aligned}$$

If we take $D=0$, we have $A=1$, $B=-1$, $C=-1$, and then

$$\frac{2x-1}{x^2-x-2}=\frac{x-1}{x-2}-\frac{x}{x+1}=\frac{x-2+1}{x-2}-\frac{x+1-1}{x+1}$$
$$=1+\frac{1}{x-2}-1+\frac{1}{x+1}=\frac{1}{x-2}+\frac{1}{x+1},$$

or take $D=1$, we have $A=0$, $B=1$, $C=0$, and then

$$\frac{2x-1}{x^2-x-2}=\frac{1}{x-2}+\frac{1}{x+1},$$

or take $D=-1$, we have $A=2$, $B=-3$, $C=-2$, and then

$$\frac{2x-1}{x^2-x-2}=\frac{2x-3}{x-2}+\frac{-2x-1}{x+1}=\frac{2(x-2)+1}{x-2}-\frac{2(x+1)-1}{x+1}$$
$$=2+\frac{1}{x-2}-2+\frac{1}{x+1}.$$

That means different ways bring the same result! Basically, we could choose the nominator parameterized unknowns whose degree is less than that of the denominator.

Rk. For an improper rational function $\frac{p(x)}{q(x)}$, where p and q are polynomials with $\deg(p)\geq\deg(q)$, we can reduce it to be

$$\frac{p(x)}{q(x)}=s(x)+\frac{r(x)}{q(x)},$$

where $s(x)$ and $r(x)$ are polynomials and $\frac{r(x)}{q(x)}$ is a proper rational function with $\deg(r)<\deg(q)$. Essentially, we only need to handle the proper rational function.
Let's use the method partial fractions for four types.

The first type

The denominator $q(x)$ has only distinct real roots.
Example. Evaluate

$$\int \frac{2x^2+1}{x^3-6x^2+11x-6}\,dx$$

solution Note that the real roots of the denominator $q(x)=x^3-6x^2+11x-6$ are $x=1,2,3$, so we have the decomposition below,

$$x^3-6x^2+11x-6=(x-1)(x-2)(x-3).$$

If we take

$$\frac{2x^2+1}{x^3-6x^2+11x-6}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}=\frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)},$$

we have

$$2x^2+1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2).$$

In turn,

- ✓ take $x=1$, we have $3=A\cdot(-1)\cdot(-2)$, which gives $A=\frac{3}{2}$;
- ✓ take $x=2$, we have $9=B\cdot 1\cdot(-1)$, which gives $B=-9$;

✓ take $x = 3$, we have $19 = C \cdot 2 \cdot 1$, which gives $C = \frac{19}{2}$.

Thus,

$$\frac{2x^2+1}{x^3-6x^2+11x-6} = \frac{\frac{3}{2}}{x-1} + \frac{-9}{x-2} + \frac{\frac{19}{2}}{x-3},$$

and then

$$\begin{aligned}\int \frac{2x^2+1}{x^3-6x^2+11x-6}\,dx &= \frac{3}{2}\int \frac{1}{x-1}\,dx - 9\int \frac{1}{x-2}\,dx + \frac{19}{2}\int \frac{1}{x-3}\,dx \\ &= \frac{3}{2}\ln|x-1| - 9\ln|x-2| + \frac{19}{2}\ln|x-3| + C.\end{aligned}$$

Rk What about using

$$\begin{aligned}\frac{2x^2+1}{x^3-6x^2+11x-6} &= \frac{2x^2+1}{(x-1)(x-2)(x-3)} \\ &= \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} + \frac{E}{(x-1)(x-2)} + \frac{F}{(x-2)(x-3)} \\ &= \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)+E(x-3)+F(x-1)}{(x-1)(x-2)(x-3)},\end{aligned}$$

Say,

$$\begin{aligned}2x^2+1 &= A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)+E(x-3)+F(x-1) \\ &= A(x^2-5x+6)+B(x^2-4x+3)+C(x^2-3x+2)+E(x-3)+F(x-1) \\ &= (A+B+C)x^2+(-5A-4B-3C+E+F)x+6A+3B+2C-3E-F.\end{aligned}$$

The linear algebraic systems of equation

$$\begin{cases} A+B+C=2 \\ -5A-4B-3C+E+F=0 \\ 6A+3B+2C-3E-F=1, \end{cases} \Longrightarrow \begin{cases} A = \frac{3}{2}+E \\ B=-9-E+F \\ C=\frac{19}{2}-F \end{cases}$$

If we take $E = F = 0$, we have $A = \frac{3}{2}$, $B = -9$, $C = \frac{19}{2}$ which is the original one. If we take $E \neq 0$ and $F \neq 0$, we have to integrate both factor below,

$$\int \frac{E}{(x-1)(x-2)} + \frac{F}{(x-2)(x-3)}\,dx,$$

should be decomposed again.

Rk. For the decomposition of the rational functions: the basic ideas below,

- only use the irreducible factor, (e.g., $x-1$, or x^2-x+1 since $b^2-4ac = 1-4 < 0$, but not $(x-1)(x-2)$);
- use less parameters to be determined for each reducible factor rather than use more parameters;
- each factor should be a proper rational function (i.e., the degree of nominator < the degree of denominator, e.g., $\frac{1}{x-1}$ or $\frac{x-1}{x^2+x+1}$).

The second type

The denominator $q(x)$ has the repeated real roots.

Example. Evaluate

$$\int \frac{3x^2+2x-1}{(x-1)^3}\,dx.$$

solution

Instead of using the linear factor like $\frac{A}{x-1}$, one needs to use the form

$$\frac{3x^2+2x-1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2+B(x-1)+C}{(x-1)^3}.$$

We then have

$$\begin{aligned} 3x^2+2x-1 &= A(x-1)^2+B(x-1)+C = Ax^2-2Ax+A+Bx-B+C \\ &= Ax^2+(B-2A)x+A-B+C. \end{aligned}$$

In turn,

$$\begin{cases} A=3 \\ B-2A=2 \\ A-B+C=-1. \end{cases} \Longrightarrow \begin{cases} A=3 \\ B=8 \\ C=4. \end{cases}$$

Thus, we have

$$\frac{3x^2+2x-1}{(x-1)^3} = \frac{3}{x-1} + \frac{8}{(x-1)^2} + \frac{4}{(x-1)^3}$$

and

$$\begin{aligned} \int \frac{3x^2+2x-1}{(x-1)^3}\,dx &= 3\int \frac{1}{x-1}\,dx + 8\int \frac{1}{(x-1)^2}\,dx + 4\int \frac{1}{(x-1)^3}\,dx \\ &= 3\ln|x-1| - 8\cdot\frac{1}{x-1} + 4\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{(x-1)^2} + C \\ &= 3\ln|x-1| - \frac{8}{x-1} - \frac{2}{(x-1)^2} + C. \end{aligned}$$

Exercise. Evaluate

$$\int \frac{x^3+x-1}{(x-1)(x-2)(x+4)^2}\,dx.$$

(hint: using the decomposition

$$\begin{aligned} \frac{x^3+x-1}{(x-1)(x-2)(x+4)^2} &= \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+4} + \frac{D}{(x+4)^2} \\ &= \frac{A}{x-1} + \frac{B}{x-2} + \frac{Cx+4C+D}{(x+4)^2} \end{aligned}$$

or

$$\begin{aligned} \frac{x^3+x-1}{(x-1)(x-2)(x+4)^2} &= \frac{A}{x-1} + \frac{B}{x-2} + \frac{Cx+D}{(x+4)^2} \\ &= \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+4} + \frac{D-4C}{(x+4)^2} \end{aligned}$$

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Rk. Why not using

$$\frac{x^3+x-1}{(x-1)(x-2)(x+4)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x+4)^2}$$
$$=\frac{A(x-2)(x+4)^2+B(x-1)(x+4)^2+C(x-1)(x-2)}{(x-1)(x-2)(x+4)^2}$$

since that

$$\begin{cases} A+B=1 \\ 16A+7B+C=0 \\ 8B-3C=1 \\ 2C-32A-16B=-1 \end{cases}$$

here four equations but three unknowns, thus the algebraic system has no solutions.
How about using

$$\frac{x^3+x-1}{(x-1)(x-2)(x+4)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{Cx^2+Dx+E}{(x+4)^2}$$
$$=\frac{A(x-2)(x+4)^2+B(x-1)(x+4)^2+(Cx^2+Dx+E)(x-1)(x-2)}{(x+4)^2},$$

say,

$$x^3+x-1=Cx^4+(A+B+D-3C)x^3+(6A+7B+2C-3D+E)x^2$$
$$+(8B+2D-3E)x-32A-16B+2E.$$

So,

$$\begin{cases} C=0 \\ A+B+D-3C=1 \\ 6A+7B+2C-3D+E=0 \\ 8B+2D-3E=1 \\ -32A-16B+2E=-1, \end{cases} \Longrightarrow \begin{cases} A=-\frac{1}{25} \\ B=\frac{1}{4} \\ C=0 \\ D=\frac{79}{100} \\ E=\frac{43}{25} \end{cases}$$

where five equations and five unknowns to be determined. This decomposition works. However, we usually choose the polynomial of nominator with degree $\le$ that of denominator. This is much more simpler! Say, the linear factor is enough when the denominator is quadratic.

The third type

The denominator $q(x)$ contains irreducible quadratic factors and none of them is repeated.
Example. Evaluate

$$\int \frac{2x+3}{x^2+x+1}\,dx.$$

solution Since that $x^2+x+1:=ax^2+bx+c$, where $b^2-4ac=1-4=-3<0$ (no real roots), the denominator is irreducible. We can take

$$\frac{2x+3}{x^2+x+1}=\frac{Ax+B}{x^2+x+1},\ \text{i.e.,}\ A=2,\ B=3.$$

Actually, we have to integrate the original one. We have

$$\begin{aligned}\int \frac{2x+3}{x^2+x+1}\,dx &= \int \frac{(x^2+x+1)'+2}{x^2+x+1}\,dx = \int \frac{d(x^2+x+1)}{x^2+x+1} + 2\int \frac{1}{x^2+x+1}\,dx \\ &= \frac{1}{x^2+x+1} + 2\int \frac{1}{x^2+x+1}\,dx.\end{aligned}$$

Note that

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right] = \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1\right].$$

Let $\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}} = \tan\theta$, using $\tan^2\theta + 1 = \sec^2\theta$, we have $dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and

$$\int \frac{1}{x^2+x+1}\,dx = \frac{4}{3}\int \frac{1}{\sec^2\theta}\cdot\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta = \frac{2}{\sqrt{3}}\theta + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right) + C.$$

The fourth type

The denominator $q(x)$ contains a repeated irreducible quadratic factor.
Example. Evaluate

$$\int \frac{2x^3+1}{(x^2+x+1)^3}\,dx.$$

solution Since that x^2+x+1 is the irreducible quadratic factor, instead of the single factor $\frac{Ax+B}{x^2+x+1}$, one needs to use the form below,

$$\begin{aligned}\frac{2x^3+1}{(x^2+x+1)^3} &= \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{Ex+F}{(x^2+x+1)^3} \\ &= \frac{(Ax+B)(x^2+x+1)^2+(Cx+D)(x^2+x+1)+Ex+F}{(x^2+x+1)^3}.\end{aligned}$$

Thus, we have

$$\begin{aligned}2x^3+1 &= (Ax+B)(x^2+x+1)^2 + (Cx+D)(x^2+x+1) + Ex + F \\ &= Ax^5 + (2A+B)x^4 + (3A+2B+C)x^3 \\ &\quad + (2A+3B+C+D)x^2 + (A+2B+C+D+E)x + B+D+F,\end{aligned}$$

in turn,

$$\begin{cases} A=0 \\ 2A+B=0 \\ 3A+2B+C=2 \\ 2A+3B+C+D=0 \\ A+2B+C+D+E=0 \\ B+D+F=1, \end{cases} \iff \begin{cases} A=0 \\ B=0 \\ C=2 \\ D=-2 \\ E=0 \\ F=3. \end{cases}$$

or by taking the specified $x=-3,-2,-1,0,1,2$, we have

$$\begin{cases} -53 = (-3A+B)\cdot 49 + (-3C+D)\cdot 7 - 3E + F \\ -15 = (-2A+B)\cdot 9 + (-2C+D)\cdot 3 - 2E + F \\ -1 = (-A+B)\cdot 1 + (-C+D)\cdot 1 - E + F \\ 1 = B+D+F \\ 3 = (A+B)\cdot 9 + (C+D)\cdot 3 + E + F \\ 17 = (2A+B)\cdot 49 + (2C+D)\cdot 7 + 2E + F \end{cases} \iff \begin{cases} A=0 \\ B=0 \\ C=2 \\ D=-2 \\ E=0 \\ F=3. \end{cases}$$

As a result, we have

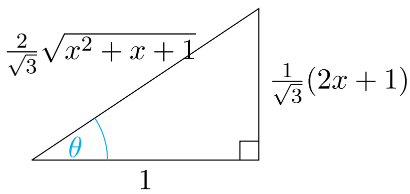
$$\frac{2x^3+1}{(x^2+x+1)^3} = \frac{2x-2}{(x^2+x+1)^2} + \frac{3}{(x^2+x+1)^3}$$

and

$$\begin{aligned}\int \frac{2x^3+1}{(x^2+x+1)^3}\,dx &= \int \frac{2x-2}{(x^2+x+1)^2}\,dx + \int \frac{3}{(x^2+x+1)^3}\,dx \\ &= \int \frac{(x^2+x+1)'-3}{(x^2+x+1)^2}\,dx + \int \frac{3}{(x^2+x+1)^3}\,dx \\ &= \int \frac{d(x^2+x+1)}{(x^2+x+1)^2} - 3\int \frac{1}{(x^2+x+1)^2}\,dx + 3\int \frac{1}{(x^2+x+1)^3}\,dx.\end{aligned}$$

Using the trigonometric substitution $\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}} = \tan\theta$ and the identity $\tan^2\theta + 1 = \sec^2\theta$, we have $dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$, in the right triangle $\sin\theta = \frac{2x+1}{2\sqrt{x^2+x+1}}$, $\cos\theta = \frac{\sqrt{3}}{2\sqrt{x^2+x+1}}$ and

$$\begin{aligned}x^2+x+1 &= \frac{3}{4}\sec^2\theta \\ (x^2+x+1)^2 &= \frac{9}{16}\sec^4\theta \\ (x^2+x+1)^3 &= \frac{27}{64}\sec^6\theta,\end{aligned}$$



and

$$\begin{aligned}\int \frac{1}{(x^2+x+1)^2}\,dx &= \frac{16}{9}\int \frac{1}{\sec^4\theta}\cdot\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta = \frac{8\sqrt{3}}{9}\int \cos^2\theta\,d\theta = \frac{8\sqrt{3}}{9}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta \\ &= \frac{4\sqrt{3}}{9}\int (1+\cos 2\theta)\,d\theta = \frac{4\sqrt{3}}{9}\theta + \frac{2\sqrt{3}}{9}\sin 2\theta + C \\ &= \frac{4\sqrt{3}}{9}\tan^{-1}\left[\frac{1}{\sqrt{3}}(2x+1)\right] + \frac{2x+1}{3(x^2+x+1)} + C \\ \int \frac{1}{(x^2+x+1)^3}\,dx &= \frac{64}{27}\int \frac{1}{\sec^6\theta}\cdot\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta = \frac{32\sqrt{3}}{27}\int \cos^4\theta\,d\theta = \frac{32\sqrt{3}}{27}\int \left[\frac{1}{2}(1+\cos 2\theta)\right]^2\,d\theta \\ &= \frac{8\sqrt{3}}{27}\int (1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \frac{8\sqrt{3}}{27}\theta + \frac{8\sqrt{3}}{27}\sin 2\theta + \frac{8\sqrt{3}}{27}\int \frac{1}{2}(1+\cos 4\theta)\,d\theta \\ &= \frac{8\sqrt{3}}{27}\theta + \frac{8\sqrt{3}}{27}\sin 2\theta + \frac{4\sqrt{3}}{27}\theta + \frac{\sqrt{3}}{27}\sin 4\theta + C \\ &= \frac{4\sqrt{3}}{9}\tan^{-1}\left[\frac{1}{\sqrt{3}}(2x+1)\right] + \frac{4(2x+1)}{9(x^2+x+1)} + \frac{(2x+1)(1-2x^2-2x)}{18(x^2+x+1)^2} + C.\end{aligned}$$

We then have

$$\int \frac{2x^3+1}{(x^2+x+1)^3}\,dx = \frac{2(x-1)}{3(x^2+x+1)} + \frac{(2x+1)(1-2x^2-2x)}{6(x^2+x+1)^2} + C.$$

More example sets

Example. Evaluate

$$\int \frac{x^6}{x^2+1}\,dx.$$

solution Rewriting the improper rational function to be proper below,

$$\frac{x^6}{x^2+1} = \frac{x^4(x^2+1) - x^2(x^2+1) + x^2 + 1 - 1}{x^2+1} = x^4 - x^2 + 1 - \frac{1}{x^2+1}.$$

Note that x^2+1 is irreducible, we have

$$\begin{aligned}\int \frac{x^6}{x^2+1}\,dx &= \int (x^4 - x^2 + 1)\,dx - \int \frac{1}{x^2+1}\,dx \\ &= \frac{1}{5}x^5 - \frac{1}{3}x^3 + x - \tan^{-1} x + C.\end{aligned}$$

Example. Evaluate

$$\int \frac{x+4}{(x+1)(x^2+1)}\,dx.$$

solution
We use

$$\frac{x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)}.$$

Say,

$$\begin{cases} A+B=0 \\ B+C=1 \\ A+C=4, \end{cases}$$

which gives $A = \frac{3}{2}$, $B = -\frac{3}{2}$, $C = \frac{5}{2}$. We then have

$$\frac{x+4}{(x+1)(x^2+1)} = \frac{3}{2}\cdot\frac{1}{x+1} + \frac{-\frac{3}{2}x + \frac{5}{2}}{x^2+1},$$

and

$$\begin{aligned}\int \frac{x+4}{(x+1)(x^2+1)}\,dx &= \frac{3}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{3x-5}{x^2+1}\,dx = \frac{3}{2}\ln|x+1| - \frac{3}{4}\int \frac{(x^2+1)' - \frac{10}{3}}{x^2+1}\,dx \\ &= \frac{3}{2}\ln|x+1| - \frac{3}{4}\ln(x^2+1) + \frac{5}{2}\int \frac{1}{x^2+1}\,dx.\end{aligned}$$

Note that let $x = \tan\theta$ and using the identity $1 + \tan^2\theta = \sec^2\theta$, we have $dx = \sec^2\theta\,d\theta$ and

$$\int \frac{1}{x^2+1}\,dx = \int \frac{1}{\sec^2\theta}\cdot\sec^2\theta\,d\theta = \theta = \tan^{-1} x + C,$$

or we could use the FTC and note that $(\tan^{-1} x)' = \frac{1}{x^2+1}$. In turn,

$$\int \frac{x+4}{(x+1)(x^2+1)}\,dx = \frac{3}{2}\ln|x+1| - \frac{3}{4}\ln(x^2+1) + \frac{5}{2}\tan^{-1} x + C.$$

Rk. Why not taking

$$\frac{x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x^2+1},$$

since this gives $x+4 = Ax^2 + Bx + A + B$, $\Longrightarrow A = 0, B = 1$, but $A + B = 1 \neq 4$, this A and B do not exist.

Exercise. Evaluate

$$\int \frac{x^2+x+2}{(x-1)(x^2+4)^2}\,dx.$$

(hint: using the decomposition below,

$$\frac{x^2+x+2}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}.$$

)

Rk. Using the decomposition for

$$\frac{x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1},$$

why not using

$$\frac{x+4}{(x+1)(x^2+1)} = \frac{Ax+B}{x+1} + \frac{C}{x^2+1} = \frac{(Ax+B)(x^2+1)+C(x+1)}{(x+1)(x^2+1)},$$

say, $x+4 = Ax^3 + Bx^2 + (A+C)x + B + C$, since that $A = 0$, $B = 0$, $A + C = 1$, $B + C = 4$, this is impossible!

Example. Evaluate

$$\int \frac{x+1}{(x^2+1)(x^2+4)}\,dx.$$

solution

We use

$$\frac{x+1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} = \frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+4)},$$

which gives

$$\begin{aligned} x+1 &= (Ax+B)(x^2+4)+(Cx+D)(x^2+1) \\ &= (A+C)x^3 + (B+D)x^2 + (4A+C)x + 4B + D. \end{aligned}$$

So we have

$$\begin{cases} A+C=0 \\ B+D=0 \\ 4A+C=1 \\ 4B+D=1, \end{cases} \Longrightarrow \begin{cases} A=\dfrac{1}{3} \\ B=\dfrac{1}{3} \\ C=-\dfrac{1}{3} \\ D=-\dfrac{1}{3}. \end{cases}$$

In turn, we have

$$\frac{x+1}{(x^2+1)(x^2+4)} = \frac{\frac{1}{3}x+\frac{1}{3}}{x^2+1} + \frac{-\frac{1}{3}x-\frac{1}{3}}{x^2+4},$$

and

$$\int \frac{x+1}{(x^2+1)(x^2+4)}\,dx = \frac{1}{3}\int \frac{x+1}{x^2+1}\,dx - \frac{1}{3}\int \frac{x+1}{x^2+4}\,dx$$
$$= \frac{1}{6}\int \frac{(x^2+1)'+2}{x^2+1}\,dx - \frac{1}{6}\int \frac{(x^2+4)'+2}{x^2+4}\,dx$$
$$= \frac{1}{6}\ln(x^2+1) - \frac{1}{6}\ln(x^2+4) + \frac{1}{3}\int \frac{1}{x^2+1}\,dx - \frac{1}{3}\int \frac{1}{x^2+4}\,dx$$

Note that

- let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$,

$$\int \frac{1}{x^2+1}\,dx = \int \frac{1}{\sec^2\theta}\cdot\sec^2\theta\,d\theta = \theta + C = \tan^{-1}x + C,$$

- let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$,

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\sec^2\theta}\cdot 2\sec^2\theta\,d\theta = \frac{1}{2}\theta + C = \frac{1}{2}\tan^{-1}\frac{x}{2} + C.$$

In turn,

$$\int \frac{x+1}{(x^2+1)(x^2+4)}\,dx = \frac{1}{6}\ln(x^2+1) - \frac{1}{6}\ln(x^2+4) + \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C.$$

Rk. If we take

$$\frac{x+1}{(x^2+1)(x^2+4)} = \frac{A}{x^2+1} + \frac{B}{x^2+4} = \frac{A(x^2+4)+B(x^2+1)}{(x^2+1)(x^2+4)},$$

since that $x+1 \neq (A+B)x^2 + 4A + B$, no x term. This treatment is invalid.

Example. Evaluate

$$\int \frac{2x^2+1}{(x+1)(x-1)^3}\,dx.$$

solution Note that $q(x) = (x+1)(x-1)^3$ has single root $x = -1$ and the repeat root $x = 1$, we have

$$\frac{2x^2+1}{(x+1)(x-1)^3} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$
$$= \frac{A(x-1)^3 + B(x+1)(x-1)^2 + C(x+1)(x-1) + D(x+1)}{(x-1)^3}.$$

In turn,

$$2x^2+1 = A(x-1)^3 + B(x+1)(x-1)^2 + C(x+1)(x-1) + D(x+1),$$

✓ take $x = 1$, we have $3 = 2D$, then $D = \frac{3}{2}$;

✓ take $x = -1$, we have $3 = -8A$, then $A = -\frac{3}{8}$;

✓ take $x = 0$, we have $1 = -A + B - C + D$;

✓ take $x = 2$, we have $9 = A + 3B + 3C + 3D$;

So we have $A = -\frac{3}{8}$, $B = \frac{3}{8}$, $C = \frac{5}{4}$, $D = \frac{3}{2}$. Thus,

$$\begin{aligned}\int \frac{2x^2+1}{(x+1)(x-1)^3}\,dx &= -\frac{3}{8}\int \frac{1}{x+1}\,dx + \frac{3}{8}\int \frac{1}{x-1} + \frac{5}{4}\int \frac{1}{(x-1)^2} + \frac{3}{2}\int \frac{1}{(x-1)^3}\,dx \\ &= -\frac{3}{8}\ln|x+1| + \frac{3}{8}\ln|x-1| - \frac{5}{4}\frac{1}{x-1} - \frac{3}{4}\frac{1}{(x-1)^2} + C.\end{aligned}$$

Example. Evaluate

$$\int \frac{x}{x^3+1}\,dx.$$

solution

Note that $x^3+1 = (x+1)(x^2-x+1)$, where $x+1$ and x^2-x+1 ($b^2-4ac = 1-4 = -3 < 0$) are both irreducible polynomials. We have

$$\frac{x}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}.$$

We then have

$$x = A(x^2-x+1)+(Bx+C)(x+1) = (A+B)x^2 + (B+C-A)x + A + C.$$

In turn,

$$\begin{cases} A+B=0 \\ B+C-A=1 \\ A+C=0, \end{cases} \Longrightarrow \begin{cases} A = -\frac{1}{3} \\ B = \frac{1}{3} \\ C = \frac{1}{3} \end{cases}$$

We then obtain

$$\begin{aligned}\int \frac{x}{x^3+1}\,dx &= -\frac{1}{3}\int \frac{1}{x+1}\,dx + \frac{1}{3}\int \frac{x+1}{x^2-x+1}\,dx = -\frac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{(x^2-x+1)'+3}{x^2-x+1}\,dx \\ &= -\frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{x^2-x+1}\,dx.\end{aligned}$$

Note that

$$x^2-x+1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\frac{4}{3}\left(x-\frac{1}{2}\right)^2+1\right] = \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2+1\right].$$

Let $\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}} = \tan\theta$, using $\tan^2\theta + 1 = \sec^2\theta$, we have $dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and

$$\int \frac{1}{x^2-x+1}\,dx = \frac{4}{3}\int \frac{1}{\sec^2\theta}\cdot\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta = \frac{2\sqrt{3}}{3}\theta + C = \frac{2\sqrt{3}}{3}\tan^{-1}\left[\frac{1}{\sqrt{3}}(2x-1)\right] + C.$$

Thus,

$$\int \frac{x}{x^3+1}\,dx = -\frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| + \frac{\sqrt{3}}{3}\tan^{-1}\left[\frac{1}{\sqrt{3}}(2x-1)\right] + C.$$

Example. Evaluate

$$\int \frac{x^2+1}{x^4+1}\,dx.$$

solution Note that $x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-(\sqrt{2}x)^2 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$.
Using the method of partial fractions, we have

$$\frac{x^2+1}{x^4+1} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}$$

After simple calculation, we have $A=0,\ B=\frac{1}{2},\ C=0,\ D=\frac{1}{2}$.
Of course, one can use

$$\frac{x^2+1}{x^4+1} = \frac{A}{x^2-\sqrt{2}x+1} + \frac{B}{x^2+\sqrt{2}x+1}.$$

We have $A=B=\frac{1}{2}$. Using the substitution $x-\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan\theta$ of completing the square, we obtain

$$\int \frac{x^2+1}{x^4+1}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x-1) + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x+1) + C.$$

Example. Evaluate

$$\int \frac{x+1}{\sqrt{x}-1}\,dx.$$

solution Let $u=\sqrt{x}$, i.e., $x=u^2$we then have $dx = 2u\,du$ and

$$\begin{aligned}\int \frac{x+1}{\sqrt{x}-1}\,dx &= \int \frac{u^2+1}{u-1}\cdot 2u\,du = 2\int \frac{u^3+u}{u-1}\,du = 2\int \frac{u^2(u-1)+u(u-1)+2(u-1)+2}{u-1}\,du\\ &= 2\int (u^2+u+2)\,du + 4\int \frac{1}{u-1}\,du = \frac{2}{3}u^3+u^2+4u+4\ln|u-1|+C\\ &= \frac{2}{3}(\sqrt{x})^3 + x + 4\sqrt{x} + 4\ln|\sqrt{x}-1| + C.\end{aligned}$$

Example. Evaluate

$$\int \frac{1}{(e^x-1)(2e^x+1)}\,dx.$$

solution Let $u=e^x$, i.e., $x=\ln u$ we then have $dx = \frac{1}{u}\,du$ and

$$\int \frac{1}{(e^x-1)(2e^x+1)}\,dx = \int \frac{1}{u(u-1)(2u+1)}\,du.$$

Using

$$\frac{1}{u(u-1)(2u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{2u+1} = \frac{A(u-1)(2u+1)+Bu(2u+1)+Cu(u-1)}{u(u-1)(2u+1)},$$

we have

$$\begin{cases} 2A+2B+C=0\\ B-C-A=0\\ A=-1, \end{cases} \Longrightarrow \begin{cases} A=-1\\ B=\dfrac{1}{3}\\ C=\dfrac{4}{3}. \end{cases}$$

We have

$$\begin{aligned}\int \frac{1}{(e^x-1)(2e^x+1)}\,dx &= -\int \frac{1}{u}\,du + \frac{1}{3}\int \frac{1}{u-1}\,du + \frac{2}{3}\int \frac{1}{2u+1}\,d(2u+1)\\ &= -\ln|u| + \frac{1}{3}\ln|u-1| + \frac{2}{3}\ln|2u+1| + C\\ &= -x + \frac{1}{3}\ln|e^x-1| + \frac{2}{3}\ln|2e^x+1| + C.\end{aligned}$$

Example. Evaluate

$$\int \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta.$$

Solution Let $x=\tan\theta$, we have $\theta=\tan^{-1}x$, $d\theta=\frac{1}{x^2+1}\,dx$ and

$$\int \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta=\int\frac{1}{\tan\theta+1}\,d\theta=\int\frac{1}{x+1}\cdot\frac{1}{x^2+1}\,dx=\int\frac{1}{(x+1)(x^2+1)}\,dx.$$

Using

$$\frac{1}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}=\frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}.$$

In turn,

$$\begin{cases}A+B=0\\B+C=0\\A+C=1,\end{cases}\Longrightarrow\begin{cases}A=\dfrac{1}{2}\\B=-\dfrac{1}{2}\\C=\dfrac{1}{2}.\end{cases}$$

Thus,

$$\begin{aligned}\int \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta&=\int\frac{1}{(x+1)(x^2+1)}\,dx=\frac{1}{2}\int\frac{1}{x+1}\,dx-\frac{1}{2}\int\frac{x-1}{x^2+1}\,dx\\&=\frac{1}{2}\ln|x+1|-\frac{1}{4}\int\frac{(x^2+1)'-2}{x^2+1}\,dx\\&=\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\tan^{-1}x+C\\&=\frac{1}{2}\ln|\tan\theta+1|-\frac{1}{4}\ln(\tan^2\theta+1)+\frac{1}{2}\theta+C.\end{aligned}$$

Rk. The techniques of decomposition

- Fundamental Theorem of Algebra (FTA): The polynomials with degree of n by $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, $a_n\neq 0$ has n roots (whatever the real roots or the complex-valued roots). Solving roots of polynomial $\Longleftrightarrow$ factorizing the polynomials;
- for the quadratic factor like ax^2+bx+c. The real roots from the formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

  e.g, $x^2-3x+2=(x-1)(x-2)$;
- special product below,

$$\begin{aligned}&x^2-a^2=(x-a)(x+a),\quad \text{e.g., } x^2-9=(x-3)(x+3)\\&x^3-a^3=(x-a)(x^2+ax+a^2),\quad \text{e.g., } x^3-8=(x-2)(x^2+2x+4)\\&x^3+a^3=(x+a)(x^2-ax+a^2),\quad \text{e.g., } x^3+64=(x+4)(x^2-4x+16)\\&x^4-a^4=(x-a)(x+a)(x^2+a^2),\quad \text{e.g., } x^4-16=(x-2)(x+2)(x^2+4)\\&x^4+a^4=x^4+2a^2x^2+a^4-2a^2x^2=(x+a^2)^2-2a^2x^2=(x+a^2+\sqrt{2}ax)(x+a^2-\sqrt{2}ax);\end{aligned}$$

- power factorization below,

$$\begin{aligned}
&(x+a)^2 = x^2 + 2ax + a^2, \quad \text{e.g., } (x+3)^2 = x^2 + 6x + 9 \\
&(x-a)^2 = x^2 - 2ax + a^2, \quad \text{e.g., } (x^2-5)^2 = x^4 - 10x^2 + 25 \\
&(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3, \quad \text{e.g., } (x+2)^3 = x^3 + 6x^2 + 12x + 8 \\
&(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3, \quad \text{e.g., } (x-1)^3 = x^3 - 3x^2 + 3x - 1 \\
&(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4, \quad \text{e.g., } (x+2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16 \\
&(x-a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4, \quad \text{e.g., } (x-4)^4 = x^4 - 16x^3 + 96x^2 - 256x + 256;
\end{aligned}$$

- group factorization below,

$$\begin{aligned}
acx^3 + adx^2 + bcx + bd &= ax^2(cx+d) + b(cx+d) \\
&= (ax^2+b)(cx+d),
\end{aligned}$$

e.g., $3x^3-2x^2-6x+4 = x^2(3x-2)-2(3x-2) = (x^2-2)(3x-2) = (x+\sqrt{2})(x-\sqrt{2})(3x-2)$;

- Finding the real roots of the polynomials with degree $n \geq 3$ may be nontrivial. The ideas for handling is to find a real root first, then to use the division of polynomials.

- Using the Rational Zero Theorem (RZT): If a polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has the integer coefficients a_i, then each rational root is $x = \frac{p}{q}$, p is the factor of a_0 and q is the factor of a_n.

  Example. Factorize $2x^3 + 3x^2 - 8x + 3$.

  solution

  ✓ the factor of $a_0 = 3$: ± 1, ± 3;

  ✓ the factor of $a_n = 2$: ± 1, ± 2;

  according to RZT, the possible rational roots are

$$\mathcal{R} = \left\{1,\ -1,\ 3,\ -3,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{3}{2},\ -\frac{3}{2}\right\}.$$

  We verified that $x = 1$ is a root of the polynomial. Let

$$2x^3 + 3x^2 - 8x + 3 = (x-1)(ax^2 + bx + c),$$

  we have $a = 2$, $b = 5$, $c = -3$ and $2x^3 + 3x^2 - 8x + 3 = (x-1)(2x^2 + 5x - 3) = (x-1)(2x-1)(x+3)$. Thus the roots are $x = 1$, $x = \frac{1}{2}$, $x = -3$, from $\mathcal{R}$.

  Exercise Factorize $2x^3 - 3x^2 + 5x - 2$. (hint: $= (2x-1)(x^2 - x + 2)$).