Lecture 7–Improper integrals

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1 Improper integrals

Proper integral

$$\int_a^b f(x)\,dx, \quad \text{where } [a,b] \quad \text{is a finite domain;}$$

where the integral exists, e.g., $\int_1^2 \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)|_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}$.

Improper integral

- the domain of integration is unbounded;

$$\int_a^b f(x)\,dx, \quad \text{where } [a,b] \quad \text{is an infinite domain;}$$

  e.g.,

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{T\to+\infty}\int_1^T \frac{1}{x^2}\,dx = \lim_{T\to+\infty}\left(-\frac{1}{x}\right)|_1^T = \lim_{T\to+\infty} = \lim_{T\to+\infty}\left(1-\frac{1}{T}\right) = 1,$$

  where the limit of the proper integral $\int_1^T \frac{1}{x^2}\,dx$ exists, or

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{T\to+\infty}\int_1^T \frac{1}{x}\,dx = \lim_{T\to+\infty}(\ln T - 1) = +\infty,$$

  where the limit of the proper integral $\int_1^T \frac{1}{x}\,dx$ does not exist.

- the integrand is unbounded;

$$\int_a^b f(x)\,dx, \text{ is an unbounded integrand;}$$

  e.g., $\int_0^1 \ln x\,dx = x\ln x|_0^1 - \int_0^1 x\cdot\frac{1}{x}\,dx$. Since $\ln x$ is unbounded near $x=0$. Thus $\int_0^1 \ln x\,dx$ is unbounded as well.

Improper integral over unbounded interval suppose the function f is defined over $[a,+\infty)$, such that f is integrable over finite domain $[a,b]$, $a<b$, then the improper integral is defined by

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx$$
$$\text{or } \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

Rk.

- If the limit exists, then the improper integral is **convergent**;

- If $\int_a^{+\infty} f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are both convergent, where a is any fixed real number, then

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx$$

Example. Evaluate $\int_{-\infty}^0 xe^x\,dx$.
solution By definition,

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{a\to-\infty}\int_a^0 xe^x\,dx.$$

Note that using the integration by parts, we have

$$\int_a^0 xe^x\,dx = \int_a^0 x\,de^x = xe^x|_a^0 - \int_a^0 e^x\,dx = (x-1)e^x|_a^0 = -1-(a-1)e^a,$$

and

$$\lim_{a\to-\infty}\int_a^0 xe^x\,dx = \lim_{a\to-\infty}[-1-ae^a+e^a] = -1-\lim_{a\to-\infty} ae^a = -1-\lim_{a\to-\infty}\frac{a}{e^{-a}}$$
$$\overset{l'Hospital}{\underset{(\frac{\infty}{\infty})}{=}} -1-\lim_{a\to-\infty}\frac{1}{-e^{-a}} = -1,$$

so,

$$\int_{-\infty}^0 xe^x\,dx = -1.$$

Thus, the improper integral is **convergent** to -1.
Example. Evaluate $\int_{-\infty}^0 \cos x\,dx$.
solution By definition,

$$\int_{-\infty}^0 \cos x\,dx = \lim_{a\to-\infty}\int_a^0 \cos x\,dx.$$

Note that by the FTC, we have

$$\int_a^0 \cos x\,dx = \sin x|_a^0 = -\sin a.$$

Thus,

$$\int_{-\infty}^0 \cos x\,dx = -\lim_{a\to-\infty}\sin a.$$

Since that the subsequences have different limits below,

- take $a_n^1 = \frac{3}{2}\pi - 2n\pi$, $n=1,2,\cdots$, $\lim_{n\to\infty}\sin\left(\frac{3}{2}\pi-2n\pi\right) = \lim_{n\to\infty}\sin\left(\frac{3}{2}\pi\right) = -1$,
- take $a_n^2 = \frac{1}{2}\pi - 2n\pi$, $n=1,2,\cdots$, $\lim_{n\to\infty}\sin\left(\frac{1}{2}\pi-2n\pi\right) = \lim_{n\to\infty}\sin\left(\frac{1}{2}\pi\right) = 1$,

thus, $\lim_{a\to-\infty}\sin a$ does not exist. So, the given improper integral is **divergent**.
Example (p-improper integral). Evaluate $\int_1^{+\infty}\frac{1}{x^p}\,dx$, where $p>0$ and $p\neq 1$.
solution By definition,

$$\int_1^{+\infty}\frac{1}{x^p}\,dx = \lim_{b\to+\infty}\int_1^b \frac{1}{x^p}\,dx.$$

Note that

$$\int_1^b \frac{1}{x^p}\,dx = \frac{1}{1-p}x^{-p+1}|_1^b = \frac{1}{1-p}(b^{1-p}-1),$$

- if $p < 1$, say, $1 - p > 0$, we have

$$\lim_{b \to +\infty} b^{1-p} = +\infty,$$

- if $p > 1$, say, $1 - p < 0$, we have

$$\lim_{b \to +\infty} b^{1-p} = 0,$$

in turn,

$$\int_1^{+\infty} \frac{1}{x^p}\, dx = \lim_{b \to +\infty} \int_1^b \frac{1}{x^p}\, dx = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1, \\ +\infty, & \text{if } p < 1. \end{cases}$$

Example (from classviva.org). Evaluate the integral

$$\int_0^{+\infty} \frac{-x \arctan x}{(1+x^2)^2}\, dx.$$

solution. By definition,

$$\int_0^{+\infty} \frac{-x \arctan x}{(1+x^2)^2}\, dx = \lim_{b \to +\infty} \int_0^b \frac{-x \arctan x}{(1+x^2)^2}\, dx.$$

Note that taking $x = \tan\theta$, we have $dx = \sec^2\theta\, d\theta$, $x : 0 \to b$, $\theta : 0 \to \arctan b$ and

$$\begin{aligned} \int_0^b \frac{-x \arctan x}{(1+x^2)^2}\, dx &= \int_0^{\arctan b} \frac{-\tan\theta \cdot \theta}{\sec^4\theta} \cdot \sec^2\theta\, d\theta = -\int_0^{\arctan b} \theta \sin\theta \cos\theta\, d\theta \\ &= -\frac{1}{2}\int_0^{\arctan b} \sin 2\theta \cdot \theta\, d\theta = \frac{1}{4}\int_0^{\arctan b} \theta\, d\cos 2\theta \\ &= \frac{1}{4}\theta \cos 2\theta\Big|_0^{\arctan b} - \frac{1}{4}\int_0^{\arctan b} \cos 2\theta\, d\theta \\ &= \frac{1}{4}\arctan b \cdot \cos(2\arctan b) - \frac{1}{8}\sin(2\arctan b). \end{aligned}$$

Note that $\lim_{b \to +\infty} \arctan b = \frac{\pi}{2}$. Thus,

$$\int_0^{+\infty} \frac{-x \arctan x}{(1+x^2)^2}\, dx = \lim_{b \to +\infty} \int_0^b \frac{-x \arctan x}{(1+x^2)^2}\, dx = \frac{1}{4} \cdot \frac{\pi}{2} \cos\pi - \frac{1}{8}\sin\pi = -\frac{\pi}{8},$$

which indicates the improper integral is **convergent**.

Improper integral with unbounded integrand Suppose the function f is defined over $[a, b)$, such that f is integrable over $[a, c]$, where $a < c < b$ and suppose f is **unbounded** in $[c, b)$, then the improper integral is defined by

$$\int_a^b f(x)\, dx = \lim_{c \to b} \int_a^c f(x)\, dx.$$

Rk.

- If the limit exists, then the improper integral is **convergent**;
- If $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ are both convergent, where $a < c < b$ is any fixed real number, and f is unbounded near a and b, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Example. Evaluate $\int_0^1 \ln x \, dx$.
solution By definition,

$$\int_0^1 \ln x \, dx = \lim_{c \to 0} \int_c^1 \ln x \, dx.$$

Note that using the integration by parts, we have

$$\int_c^1 \ln x \, dx = x \ln x|_c^1 - \int_c^1 x \cdot \frac{1}{x} \, dx = -c \ln c - (1-c) = -c \ln c - 1 + c.$$

Thus,

$$\begin{aligned}\int_0^1 \ln x \, dx &= \lim_{c \to 0} \int_c^1 \ln x \, dx = \lim_{c \to 0} (-c \ln c - 1 + c) = -1 - \lim_{c \to 0} c \ln c \\ &\overset{l'Hospital}{\underset{(\frac{\infty}{\infty})}{=}} -1 - \lim_{c \to 0} \frac{\ln c}{\frac{1}{c}} = -1 - \lim_{c \to 0} \frac{\frac{1}{c}}{-\frac{1}{c^2}} = -1 + \lim_{c \to 0} c = -1.\end{aligned}$$

So $\int_0^1 \ln x \, dx = -1$, we say the given improper integral is convergent to -1.
Example. Evaluate $\int_0^1 x^p \, dx$, where $p < 0$ and $p \neq -1$.
solution.
By definition,

$$\int_0^1 x^p \, dx = \lim_{c \to 0} \int_c^1 x^p \, dx.$$

Note that $(c > 0)$,

$$\int_c^1 x^p \, dx = \frac{1}{p+1} x^{p+1}|_c^1 = \frac{1}{p+1} \left(1 - c^{p+1}\right),$$

- if $-1 < p < 0$, say $p + 1 > 0$, we have

$$\lim_{c \to 0} c^{p+1} = 0,$$

- if $p < -1$, say $p + 1 < 0$, we have

$$\lim_{c \to 0} c^{p+1} = +\infty.$$

In turn,

$$\int_0^1 x^p \, dx = \lim_{c \to 0} \int_c^1 x^p \, dx = \begin{cases} \dfrac{1}{p+1}, & \text{if} \quad -1 < p < 0, \\ +\infty, & \text{if} \quad p < -1. \end{cases}$$

which indicates when $-1 < p < 0$, the given improper integral is **convergent**; when $p < -1$, it is **divergent**.

Rk. How about the assumption that p is a real number. (hint: $p \geq 0$ the integral is proper, so exists. if $p = -1$, the integral would be divergent. Thus when $p > -1$, the given improper integral is convergent; when $p \leq -1$, it is divergent.)
Exercise (hint: if $p > 1$, then $\int_0^a \frac{1}{x^p}$ converges; if $p \leq 1$, then $\int_0^a \frac{1}{x^p} \, dx$ diverges). Evaluate $\int_0^a \frac{1}{x^p} \, dx$, where $a > 0$, p is a real number.
Example. Evaluate $\int_{-1}^1 \frac{1}{x} \, dx$.
solution. Since that $0 \in [-1, 1]$, we have

$$\int_{-1}^1 \frac{1}{x} \, dx = \int_{-1}^0 \frac{1}{x} \, dx + \int_0^1 \frac{1}{x} \, dx.$$

By the definition of improper integral, we have

$$\int_{-1}^{0} \frac{1}{x}\,dx = \lim_{b\to 0}\int_{-1}^{b} \frac{1}{x}\,dx = \lim_{b\to 0} \ln x|_{-1}^{b}$$
$$\int_{0}^{1} \frac{1}{x}\,dx = \lim_{a\to 0}\int_{a}^{1} \frac{1}{x}\,dx = \lim_{a\to 0} \ln x|_{a}^{1}.$$

Since that the $\ln x$ is unbounded near $x = 0$. Thus, the two parts $\int_{-1}^{0} \frac{1}{x}\,dx$ and $\int_{0}^{1} \frac{1}{x}\,dx$ are both divergent. Thus, the original one $\int_{-1}^{1} \frac{1}{x}\,dx$ is **divergent**.

Comparison test for improper integrals

Theorem for the improper integrals over unbounded interval. Let $f(x)$ and $g(x)$ be continuous over the unbounded domain $[a, +\infty)$, and $0 \le f(x) \le g(x)$. Then

- $\int_{a}^{+\infty} g(x)\,dx$ converges $\Longrightarrow \int_{a}^{+\infty} f(x)\,dx$ converges;
- $\int_{a}^{+\infty} f(x)\,dx$ diverges $\Longrightarrow \int_{a}^{+\infty} g(x)\,dx$ diverges.

Theorem for the improper integrals with unbounded integrand. Let $f(x)$ and $g(x)$ be continuous on $[a, b)$, and suppose $f(x)$ and $g(x)$ are both **unbounded** in $[c, b)$, where $a < c < b$, and $0 \le f(x) \le g(x)$. Then

- $\int_{a}^{b} g(x)\,dx$ converges $\Longrightarrow \int_{a}^{b} f(x)\,dx$ converges;
- $\int_{a}^{b} f(x)\,dx$ diverges $\Longrightarrow \int_{a}^{b} g(x)\,dx$ diverges.

Example. Consider the improper integral $\int_{1}^{+\infty} \frac{1+\sin x}{x^2}\,dx$ is convergent or divergent.
solution. Note that

$$\frac{1+\sin x}{x^2} \le \frac{2}{x^2},$$

and $\int_{1}^{+\infty} \frac{2}{x^2}\,dx$ converges. Thus, $\int_{1}^{+\infty} \frac{1+\sin x}{x^2}\,dx$ converges.
Exercises. Determine if the following integrals are convergent or divergent.

$$\int_{3}^{+\infty} \frac{1}{x+e^x}\,dx$$
$$\int_{3}^{+\infty} \frac{1}{x-e^{-x}}\,dx$$
$$\int_{1}^{+\infty} \frac{1+3\sin^4 2x}{\sqrt{x}}\,dx$$
$$\int_{2}^{+\infty} \frac{1+\cos^2 x}{\sqrt{x}\,(2-\sin^4 x)}\,dx.$$

solution. Note that

$$\frac{1}{x+e^x} < \frac{1}{e^x} = e^{-x}.$$

Since that $\int_{3}^{+\infty} e^{-x}\,dx = e^{-3}$, thus it is convergent, so do the original one. Note that

$$\frac{1}{x-e^{-x}} > \frac{1}{x}.$$

Since that $\int_3^{+\infty} \frac{1}{x}\,dx$ diverges, so do the original one. Other cases note that

$$\frac{1+3\sin^4 2x}{\sqrt{x}} > \frac{1}{\sqrt{x}}$$
$$\frac{1+\cos^2 x}{\sqrt{x}\,(2-\sin^4 x)} > \frac{1}{\sqrt{x}\,(2-\sin^4 x)} > \frac{1}{2\sqrt{x}}.$$

Example (from classviva.org). Let $f(x)$ be a continuous function over $(2,\infty)$. Assume that

$$f(4) = 6,$$
$$|f(x)| \leq x^4 + 2,$$
$$\int_4^{\infty} f(x)e^{-\frac{x}{5}}\,dx = 4.$$

Evaluate

$$\int_4^{\infty} f'(x)e^{-\frac{x}{5}}\,dx.$$

solution.

$$\int_4^{\infty} f'(x)e^{-\frac{x}{5}}\,dx = \int_4^{\infty} e^{-\frac{x}{5}}\,df(x) = f(x)e^{-\frac{x}{5}}|_4^{\infty} + \frac{1}{5}\int_4^{\infty} f(x)e^{-\frac{x}{5}}\,dx.$$

Note that

$$\left|f(x)e^{-\frac{x}{5}}\right| \leq (x^4+2)e^{-\frac{x}{5}},$$

which suggests

$$\lim_{x\to\infty}(x^4+2)e^{-\frac{x}{5}} = \lim_{x\to\infty}\frac{x^4+2}{e^{\frac{x}{5}}} \overset{l'Hospital}{\underset{(\frac{\infty}{\infty})}{=}} 0.$$

In turn,

$$\lim_{x\to\infty} f(x)e^{-\frac{x}{5}} = 0.$$

Thus,

$$\int_4^{\infty} f'(x)e^{-\frac{x}{5}}\,dx = 0 - f(4)e^{-\frac{4}{5}} + \frac{4}{5} = -6e^{-\frac{4}{5}} + \frac{4}{5}.$$