Lecture 8 (Applications of integration)–Areas between curves

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1 Areas between curves

If we let $f(x) > 0$ and $g(x) < 0$ on $[a, b]$, see Figure 1, and denote the areas between the graphs and x-axis by A_f and A_g. Thus,

$$A_f = \int_a^b f(x)\, dx, \quad A_g = -\int_a^b g(x)\, dx.$$

For example,

$$\int_1^2 f(x)\, dx = \int_1^2 x\, dx = \frac{1}{2}x^2|_1^2 = \frac{1}{2}(4-1) = \frac{3}{2} = A_f,$$
$$\int_1^2 g(x)\, dx = \int_1^2 -x\, dx = -\frac{1}{2}x^2|_1^2 = -\frac{1}{2}(4-1) = -\frac{3}{2} = -A_g.$$

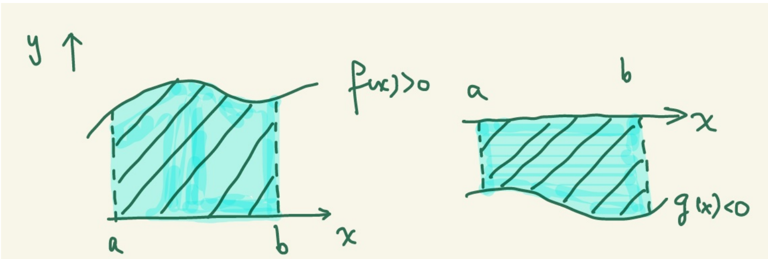


Figure 1: The areas between the graphs and x-axis over $[a, b]$, where $f(x) > 0$ and $g(x) < 0$.

Area of a region bounded by two curves

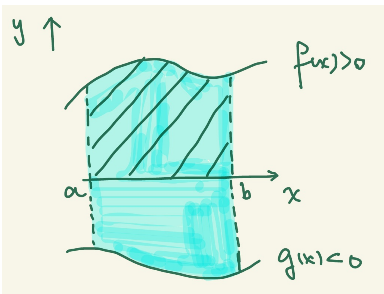


Figure 2: Area of a region bounded by two curves.

Note that in Figure 2, the area over $[a, b]$ is bounded by two curves $f(x)$ and $g(x)$ which is given by

$$A = A_f + A_g = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx.$$

Theorem. Let f and $g(x)$ be two continuous functions with $f(x) \geq g(x)$ over $[a, b]$. The area between the graphs of f and g over $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]\ dx.$$

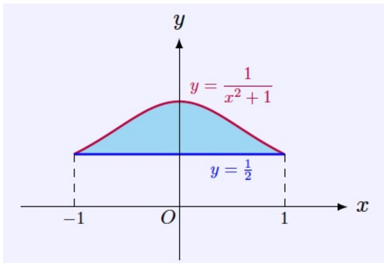
Example. Find the area of the closed region bounded by the graphs of the functions $f(x) = \frac{1}{x^2+1}$ and $g(x) = \frac{1}{2}$.
solution. To find the intersections of two curves, we take

$$\frac{1}{x^2+1} = \frac{1}{2}, \quad \Longrightarrow x = \pm 1.$$

Note that

- when $-1 \leq x \leq 1$, $x^2 + 1 \leq 2$, we have $f(x) \geq g(x)$;
- when $x < -1$ or $x > 1$, $x^2 + 1 > 2$, we have $f(x) \leq g(x)$.

The closed domain is presented below. Thus, the area of the region bounded by two curves is



$$A = \int_{-1}^1 [f(x) - g(x)]\ dx = \int_{-1}^1 \left[\frac{1}{x^2+1} - \frac{1}{2}\right]\ dx = \arctan x|_{-1}^1 - \frac{1}{2} \cdot 2$$
$$= \frac{\pi}{4} - (-\frac{\pi}{4}) - 1 = \frac{\pi}{2} - 1.$$

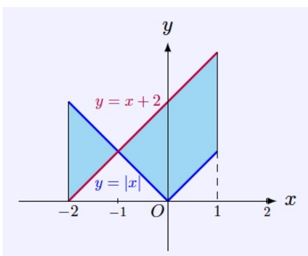
Example. Find the area of the closed region bounded by the graphs of the functions $f(x) = x + 2$ and $g(x) = |x|$ over the interval $[-2, 1]$.
solution. To find the intersection of $f(x)$ and $g(x)$, we take

$$x + 2 = |x|, \quad \Longrightarrow x = -1.$$

Note that

- when $x \leq -1$, $x + 2 \leq |x|$, we have $f(x) \leq g(x)$;
- when $x \geq -1$, $x + 2 \geq |x|$, we have $f(x) \geq g(x)$.

The closed region is presented below. Thus, the area of the region bounded by f and g is



$$\begin{aligned} A &= \int_{-2}^{-1} [g(x) - f(x)]\ dx + \int_{-1}^{1} [f(x) - g(x)]\ dx \\ &= \int_{-2}^{-1} [|x| - (x+2)]\ dx + \int_{-1}^{1} [x+2-|x|]\ dx \\ &= \int_{-2}^{-1} [-x-(x+2)]\ dx + \int_{-1}^{0} [x+2-(-x)]\ dx + \int_{0}^{1} [x+2-x]\ dx \\ &= (-x^2-2x)|_{-2}^{-1} + (x^2+2x)|_{-1}^{0} + 2x|_0^1 \\ &= (-1+2) - (-4+4) + 0 - (1-2) + 2 - 0 = 4. \end{aligned}$$

Rk. Note that

$$A = \int_{-2}^{-1} [g(x) - f(x)]\ dx + \int_{-1}^{1} [f(x) - g(x)]\ dx = \int_{-2}^{1} |f(x) - g(x)|\ dx,$$

since that $f(x) \le g(x)$ when $x \le -1$ and $f(x) \ge g(x)$ when $x \ge -1$.
Theorem. Generally, the area between the curve $f(x)$ and $g(x)$ over $[a, b]$ is given by

$$\int_a^b |f(x) - g(x)|\ dx.$$

Example. Find the area between $f(x) = \cos x$ and $g(x) = \sin x$ over $[0, \frac{\pi}{2}]$.
solution. The area is

$$\int_0^{\frac{\pi}{2}} |f(x) - g(x)|\ dx = \int_0^{\frac{\pi}{2}} |\cos x - \sin x|\ dx.$$

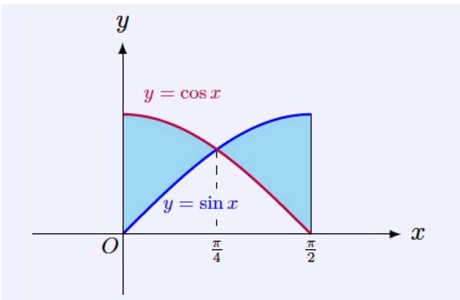
To find the intersections, we take

$$\cos x = \sin x, \quad \Longrightarrow \cos^2 x = \sin^2 x = 1 - \cos^2 x \quad \Longrightarrow \cos^2 x = \frac{1}{2},$$

due to $x \in [0, \frac{\pi}{2}]$, we have $x = \frac{\pi}{4}$. Note that

- when $0 \le x \le \frac{\pi}{4}$, $\sin x \le \frac{\sqrt{2}}{2} \le \cos x$, we have $f(x) \ge g(x)$;
- when $\frac{\pi}{4} \le x \le \frac{\pi}{2}$, $\sin x \ge \frac{\sqrt{2}}{2} \ge \cos x$, we have $f(x) \le g(x)$.

The area between two curves is presented below. Thus, the area is



$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} |f(x) - g(x)|\ dx = \int_0^{\frac{\pi}{4}} [f(x) - g(x)]\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [g(x) - f(x)]\ dx \\ &= \int_0^{\frac{\pi}{4}} [\cos x - \sin x]\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - \cos x]\ dx \\ &= (\sin x + \cos x)|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 + (-1) - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) = 2\sqrt{2} - 2. \end{aligned}$$

Example (from classviva.org). Find the area of the region that is enclosed between $y = 7x^2 - x^3 + x$ and $y = x^2 + 9x$.
solution.
To find the intersection of two curves, we take

$$\begin{cases} y = f(x) = 7x^2 - x^3 + x \\ y = g(x) = x^2 + 9x, \end{cases} \implies \begin{cases} x = 0, 2, 4 \\ y = 0, 22, 52. \end{cases}$$

Note that $f(x) - g(x) = 7x^2 - x^3 + x - (x^2 + 9x) = -x(x-2)(x-4)$ and

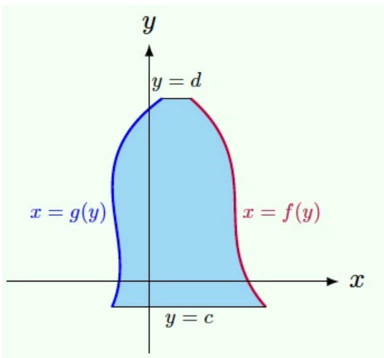
- when $x \leq 0$, thus, $-x(x-2)(x-4) \geq 0$, we have $f(x) \geq g(x)$;
- when $0 \leq x \leq 2$, thus, $-x(x-2)(x-4) \leq 0$, we have $f(x) \leq g(x)$;
- when $2 \leq x \leq 4$, thus, $-x(x-2)(x-4) \geq 0$, we have $f(x) \geq g(x)$;
- when $x \geq 4$, thus, $-x(x-2)(x-4) \leq 0$, we have $f(x) \leq g(x)$.

Thus, the area is

$$\begin{aligned} A &= \int_0^4 |f(x) - g(x)|\, dx = \int_0^2 [g(x) - f(x)]\, dx + \int_2^4 [f(x) - g(x)]\, dx \\ &= \int_0^2 [x^2 + 9x - (7x^2 - x^3 + x)]\, dx + \int_2^4 [7x^2 - x^3 + x - (x^2 + 9x)]\, dx \\ &= \int_0^2 \left[-6x^2 + x^3 + 8x\right] dx + \int_2^4 [6x^2 - x^3 - 8x]\, dx \\ &= (-2x^3 + \frac{1}{4}x^4 + 4x^2)|_0^2 + (2x^3 - \frac{1}{4}x^4 - 4x^2)|_2^4 = 8. \end{aligned}$$

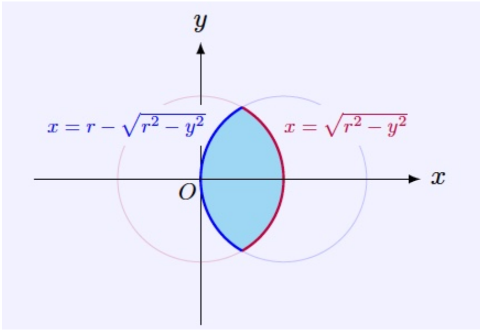
Theorem (area bounded by two curves along the y-axis). Let A be the area of the region that lies between two curves $x = f(y)$ and $x = g(y)$ over $y \in [c, d]$, where f and g are continuous functions and suppose $f(y) \geq g(y)$ (see the figure below), then

$$A = \int_c^d [f(y) - g(y)]\, dy.$$



Example. Find the common area of two disks $x^2 + y^2 \leq r^2$ and $(x - r)^2 + y^2 \leq r^2$.
solution. To find the intersections between two curves, we take

$$\begin{cases} x^2 + y^2 = r^2 \\ (x - r)^2 + y^2 = r^2, \end{cases} \implies \begin{cases} x = \dfrac{r}{2} \\ y = \pm\dfrac{\sqrt{3}}{2}r. \end{cases}$$



The area between two curves is presented above. We take

$$x = f(y) = \sqrt{r^2 - y^2}$$
$$x = g(y) = r - \sqrt{r^2 - y^2},$$

where $f(y) \geq g(y)$. Thus, the common area is given by

$$A = \int_{-\frac{\sqrt{3}}{2}r}^{\frac{\sqrt{3}}{2}r} [f(y) - g(y)]\ dy = \int_{-\frac{\sqrt{3}}{2}r}^{\frac{\sqrt{3}}{2}r} \left[\sqrt{r^2 - y^2} - (r - \sqrt{r^2 - y^2})\right]\ dy$$
$$= \int_{-\frac{\sqrt{3}}{2}r}^{\frac{\sqrt{3}}{2}r} \left[2\sqrt{r^2 - y^2} - r\right]\ dy.$$

Note that let $y = r\sin\theta$, we have $dy = r\cos\theta\ d\theta$, $y : -\frac{\sqrt{3}}{2}r \to \frac{\sqrt{3}}{2}r$, $\theta : -\frac{\pi}{3} \to \frac{\pi}{3}$ and

$$\int_{-\frac{\sqrt{3}}{2}r}^{\frac{\sqrt{3}}{2}r} \left[2\sqrt{r^2 - y^2} - r\right]\ dy = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [2r\cos\theta - r] \cdot r\cos\theta\ d\theta = r^2 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[2\cos^2\theta - \cos\theta\right]\ d\theta$$
$$= r^2 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [\cos 2\theta + 1 - \cos\theta]\ d\theta = r^2 \left(\frac{1}{2}\sin 2\theta + \theta - \sin\theta\right) |_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$
$$= \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right) r^2.$$

Theorem (area under a parametric curve) If the curve is parameterized by $x = f(t)$ and $y = g(t)$ $(y > 0)$, where $\alpha \leq t \leq \beta$. Then the area under the curve is given by

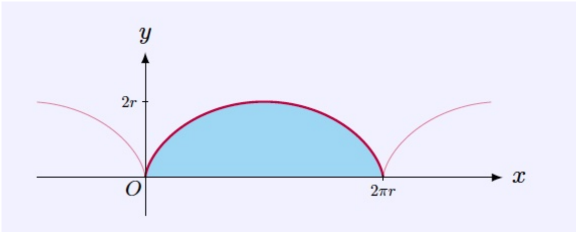
$$A = \int_{f(\alpha)}^{f(\beta)} y\ dx = \int_{\alpha}^{\beta} g(t)\ df(t) = \int_{\alpha}^{\beta} g(t)f'(t)\ dt.$$

Example. Find the area under one arch of the cycloid

$$\begin{cases} x = r(t - \sin t) \\ y = r(1 - \cos t), \end{cases}$$

where $t \in [0, 2\pi]$.
solution. Note that

- when $t = 0$, we have $x = 0$, $y = 0$;
- when $t = 2\pi$, we have $x = 2\pi r$, $y = 0$.
- when $0 \leq t \leq 2\pi$, we have $1 - \cos t \geq 0$, $\Longrightarrow y \geq 0$.



The area under the curve over $x \in [0, 2\pi r]$ is presented above. Thus, we have

$$A = \int_0^{2\pi r} y\, dx = \int_0^{2\pi} r(1-\cos t)\cdot[r(t-\sin t)]'\, dt = r^2\int_0^{2\pi}(1-\cos t)^2\, dt$$
$$= r^2\int_0^{2\pi}\left(1-2\cos t+\cos^2 t\right)\, dt = r^2\,(t-2\sin t)\,|_0^{2\pi} + r^2\int_0^{2\pi}\frac{1}{2}\left[\cos 2t+1\right]\, dt$$
$$= r^2(2\pi-0)+\frac{1}{2}r^2(\frac{1}{2}\sin 2t+t)|_0^{2\pi} = 2\pi r^2+\pi r^2 = 3\pi r^2.$$