Lecture 9 (Applications of integration)–Volumes by slicing

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1 Recap last time

Comparison Theorem of improper integral

- If $0 \le f(x) \le g(x)$, then $\int_a^\infty g(x)\,dx < \infty$ is convergent $\Longrightarrow \int_a^\infty f(x)\,dx < \infty$ is convergent;
- If $0 \le f(x) \le g(x)$, then $\int_a^\infty f(x)\,dx > \infty$ is divergent $\Longrightarrow \int_a^\infty g(x)\,dx > \infty$ is divergent;
- If $0 \le |f(x)| \le |g(x)|$, then $\int_a^\infty |g(x)|\,dx < \infty$ is convergent $\Longrightarrow \int_a^\infty |f(x)|\,dx < \infty$ is convergent; $\Longrightarrow \int_a^\infty f(x)\,dx < \infty$ is convergent;
- If $0 \le |f(x)| \le |g(x)|$, then $\int_a^\infty |f(x)|\,dx > \infty$ is divergent $\Longrightarrow \int_a^\infty |g(x)|\,dx > \infty$ is divergent.

Rk. Property of improper integrals below,

- $\int_a^\infty f(x)\,dx$ is convergent $\Longleftrightarrow \int_b^\infty$ is convergent for some $b \ge a$;
- $\int_a^\infty f(x) + g(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$ whenever the improper integrals at right hand side are convergent.
- $\int_a^\infty cf(x)\,dx = c\int_a^\infty f(x)\,dx$ for any constant c;
- $\int_a^\infty |f(x)|\,dx < \infty$ is convergent $\Longrightarrow \int_a^\infty f(x)\,dx < \infty$ is convergent.
- $\int_a^\infty f(x)\,dx > \infty$ is divergent $\Longrightarrow \int_a^\infty |f(x)|\,dx > \infty$ is divergent.

The above property holds for other improper integrals.
Example. $\int_1^\infty e^{-x^2}\,dx$, convergent of divergent?
solution. Since that $e^{-x^2} \le e^{-x}$ for $x \ge 1$ and

$$\int_1^\infty e^{-x}\,dx = (-e^{-x})|_1^\infty = \lim_{x\to\infty} -e^{-x} + e^{-1} = e^{-1} < \infty.$$

Thus the original improper integral is convergent.
Example. $\int_1^\infty \frac{1}{x^3+\sqrt[3]{x}}\,dx$, convergent of divergent?
solution. Since that

$$\frac{1}{x^3+\sqrt[3]{x}} \le \frac{1}{x^3} \quad \text{for all } x \ge 1$$

and $\int_1^\infty \frac{1}{x^3}\,dx$ is convergent, so does the original improper integral.
Example. $\int_1^\infty \frac{\sqrt{x}}{2x+\sqrt{x}+3}\,dx$, convergent of divergent?
solution. Since that $2x + \sqrt{x} + 3 = 2x\left(1 + \frac{1}{2\sqrt{x}} + \frac{3}{2x}\right)$ and

$$1 + \frac{1}{2\sqrt{x}} + \frac{3}{2x} < 1 + \frac{1}{2\sqrt{x}} + \frac{3}{2\sqrt{x}} = 1 + \frac{2}{\sqrt{x}} \le 3, \quad \text{for } x \ge 1.$$

Thus,

$$\frac{\sqrt{x}}{2x+\sqrt{x}+3} \geq \frac{\sqrt{x}}{2x \cdot 3} = \frac{1}{6x^{\frac{1}{2}}}.$$

Since $\int_1^\infty \frac{1}{x^{\frac{1}{2}}}\, dx = \infty$ is divergent, thus the original one is divergent.

Example. $\int_1^\infty \frac{\sqrt{x}}{x^2+2x+1}\, dx$, convergent of divergent?

solution. Since that

$$\frac{\sqrt{x}}{x^2+2x+1} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{x^{\frac{3}{2}}},$$

and $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\, dx < \infty$ is convergent, thus the original one is convergent.

Example. $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\, dx$, convergent of divergent?

solution. Since that

$$\frac{1+\sin^2 x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}},$$

we have

$$\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\, dx \geq \int_1^\infty \frac{1}{\sqrt{x}}\, dx > \infty.$$

The original one is divergent.

Example. $\int_1^\infty \frac{\ln x}{x^2}\, dx$, convergent of divergent?

solution. Since that $\ln x < x$ for all $x \geq 1$, thus

$$\frac{2\ln\sqrt{x}}{x^2} \leq \frac{2\sqrt{x}}{x^2} = \frac{2}{x^{\frac{3}{2}}}.$$

Note that $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\, dx$ is convergent, so does the original one.

Example. $\int_1^\infty \frac{\ln x}{\sin^2 x + x^2}\, dx$, convergent of divergent?

solution.

Since that

$$\frac{\ln x}{\sin^2 x + x^2} < \frac{\ln x}{x^2}.$$

From above example, we have the convergent original improper integral.

Example (other types improper integral). $\int_3^7 \frac{1}{\sqrt{(x-3)(x^2+x+1)}}\, dx$, convergent of divergent?

solution. Since that

$$\frac{1}{\sqrt{(x-3)(x^2+x+1)}} < \frac{1}{\sqrt{x-3}}, \qquad \text{for all } 3 \leq x \leq 7,$$

and

$$\int_3^7 \frac{1}{\sqrt{x-3}}\, d(x-3) = \int_0^4 u^{-\frac{1}{2}}\, du = 2u^{\frac{1}{2}}|_0^4 = 4.$$

Thus, the original one is convergent.

Example. $\int_1^\infty \frac{x^2+2x+3}{2x^5-x^2}\, dx$, convergent of divergent?

solution. For x is larger enough, we have

$$\frac{x^2+2x+3}{2x^5-x^2} \approx \frac{x^2}{2x^5} = \frac{1}{2x^3}$$

and $\int_1^\infty \frac{1}{x^3}\,dx$ is convergent. Since that

$$\frac{x^2+2x+3}{2x^5-x^2} \le \frac{x^2+2x^2+3x^2}{2x^5\left(1-\frac{1}{2x^3}\right)}, \qquad \text{for all } x \ge 1,$$

and

$$\frac{1}{2x^3} \le \frac{1}{2} \Longrightarrow 1-\frac{1}{2x^3} \ge \frac{1}{2} \Longrightarrow \frac{1}{1-\frac{1}{2x^3}} \le 2.$$

We have

$$\frac{x^2+2x+3}{2x^5-x^2} \le \frac{6x^2}{2x^5}\cdot 2 = \frac{6}{x^3}.$$

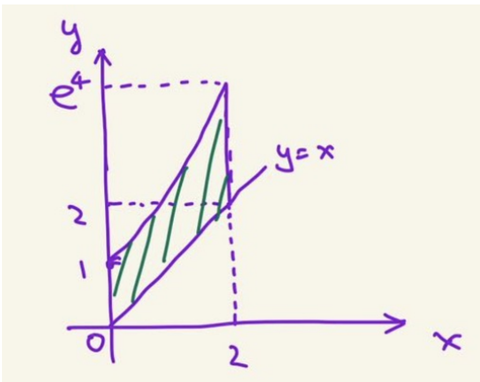
Since $\int_1^\infty \frac{1}{x^3}\,dx$ is convergent, thus the original one is convergent as well.
Example. $\int_1^\infty \frac{\sin x}{x^2}\,dx$, convergent of divergent?
solution. Since that

$$\left|\frac{\sin x}{x^2}\right| \le \frac{1}{x^2},$$

and $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent. Thus, $\int_1^\infty \left|\frac{\sin x}{x^2}\right|\,dx$ is convergent $\Longrightarrow \int_1^\infty \frac{\sin x}{x^2}\,dx$ is convergent.

Area between curves

Example. Find the area enclosed by $y = e^{2x}$, $y = x$, $x = 2$ and the y-axis.
solution. The enclosed sketch is shown below. We have



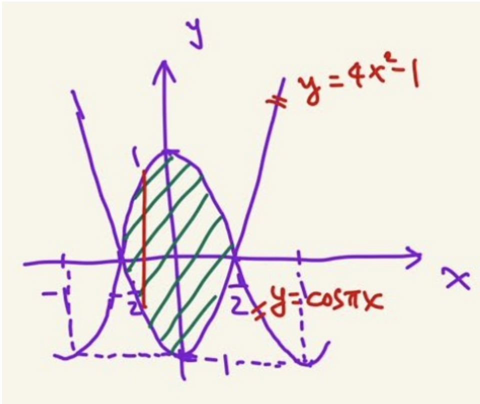
$$A = \int_0^2 \left[e^{2x}-x\right]\,dx = \left(\frac{e^{2x}}{2}-\frac{x^2}{2}\right)\Big|_0^2 = \frac{e^4}{2}-\frac{5}{2}.$$

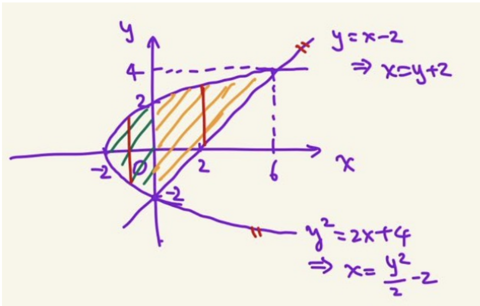
Example. Find the area enclosed by $y = 4x^2-1$ and $y = \cos\pi x$.
solution. The enclosed sketch is shown below. We have

$$A = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\cos\pi x-(4x^2-1)\right]\,dx = \left(\frac{\sin\pi x}{\pi}-\frac{4}{3}x^3+x\right)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \left(\frac{1}{\pi}-\frac{1}{6}+\frac{1}{2}\right)-\left(-\frac{1}{\pi}-\frac{1}{6}-\frac{1}{2}\right)$$
$$= \frac{2}{\pi}+\frac{2}{3}.$$

Example. Find the area enclosed by $y^2 = 2x+4$ and $y = x-2$.
solution. To find the intersections, we have to take

$$\begin{cases} y^2 = 2x+4 \\ y = x-2 \end{cases} \Longrightarrow \begin{cases} x = 6, & y = 4 \\ x = 0, & y = -2. \end{cases}$$





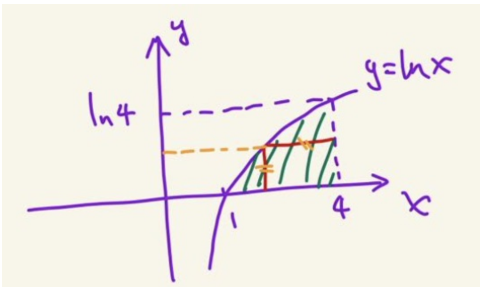
The enclosed sketch is shown below. Taking $u = 2x + 4$, we have

$$\begin{aligned} A &= \int_{-2}^{0} 2\sqrt{2x+4}\, dx + \int_{0}^{6} \left[\sqrt{2x+4} - (x-2)\right]\, dx \\ &= \int_{-2}^{0} \sqrt{2x+4}\, d(2x+4) + \frac{1}{2}\int_{0}^{6} \sqrt{2x+4}\, d(2x+4) + (2x - \frac{1}{2}x^2)|_0^6 \\ &= \frac{2}{3}u^{\frac{3}{2}}|_0^4 + \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}|_4^{16} + (2x - \frac{1}{2}x^2)|_0^6 \\ &= 18. \end{aligned}$$

Rk. It can be easier as below.

$$A = \int_{-2}^{4} \left[(y+2) - (\frac{y^2}{2} - 2)\right]\, dy = \left(\frac{y^2}{2} + 4y - \frac{1}{6}y^3\right)|_{-2}^4 = 18.$$

Example. Find the area enclosed by $y = \ln x$ and $x = 4$ and $y = 0$.
solution. The enclosed sketch is shown below. We have



$$A = \int_{1}^{4} \ln x\, dx = x\ln x|_1^4 - \int_{1}^{4} x\cdot\frac{1}{x}\, dx = 4\ln 4 - 3,$$

or

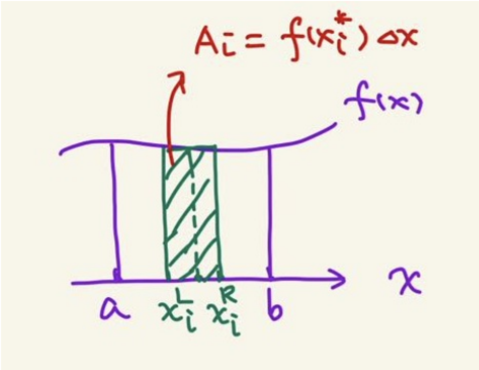
$$A = \int_0^{\ln 4} [4 - (e^y)]\ dy = 4\ln 4 - 3.$$

2 Volumes by slicing

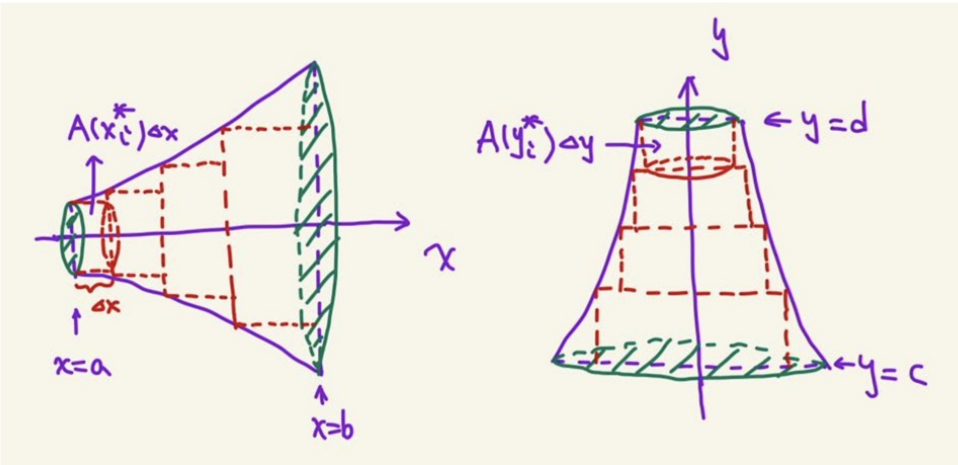
area If $f(x) > 0$ is continuous over $[a, b]$, the area between the function f and the x-axis over $[a, b]$ is given by

$$A = \int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

see below for the definition of the definite integral of $f(x)$ over $[a, b]$.



volume (finding **volume by using slices**) Let S be a solid



- that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is denoted by $A(x)$, where A is continuous function, then the **volume** of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\ dx;$$

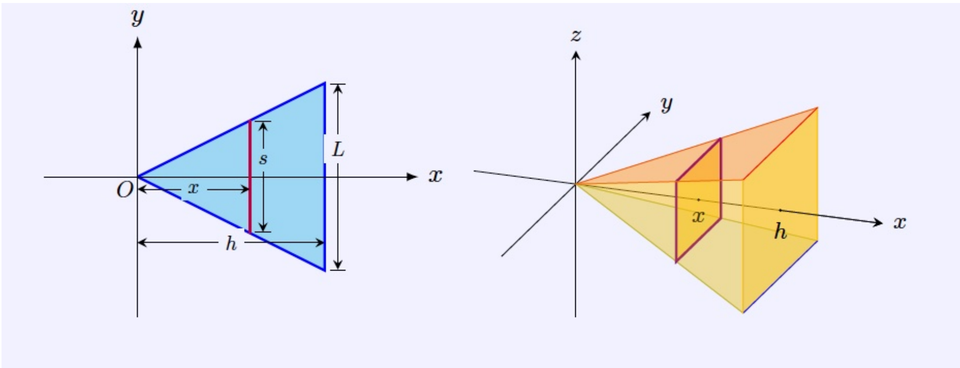
- that lies between $y = c$ and $y = d$ and if the cross-sectional area of S through y perpendicular to the y-axis, is denoted by $A(y)$, where $A(y)$ is a continuous function, then the **volume** of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(y_i^*)\Delta y = \int_a^b A(y)\ dy.$$

Rk. Using the **volume of slices**, the idea is basically given below,

- build the coordinates, including the **origin**, the symmetric axis (x-axis or y-axis);
- formulate the expression of the cross-sectional area $A(x)$;
- compute $V = \int_a^b A(x)\, dx$ or $V = \int_a^b A(y)\, dy$.

Example (volume of a pyramid) Find the volume of a pyramid whose base is a square with side L and whose height is h.
solution. Build the coordinates below, where left graph is the base, the cross-section are all squares



Note that

$$\frac{x}{s} = \frac{h}{L},$$

we have $s = \frac{xL}{h}$. Thus, the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2} x^2.$$

The volume of the pyramid over $[0, h]$ is

$$V = \int_0^h A(x)\, dx = \int_0^h \frac{L^2}{h^2} x^2\, dx = \frac{L^2}{h^2} \left(\frac{1}{3} x^3 \right) |_0^h = \frac{1}{3} L^2 h = \frac{1}{3} Sh,$$

where S is the bottom area.
Rk. There are many ways to get the cross-sectional area $A(x)$ or $A(y)$. Whether we will use $A(x)$ or $A(y)$ will depend upon the method and the axis of rotation used for each problem.
Example (from classviva.org). As is shown in fig. 1, the base of a certain solid is the area bounded

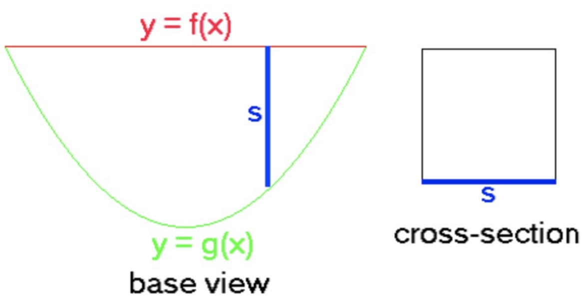


Figure 1: Base and cross-section view.

above by the graph of $y = f(x) = 16$ and and below by the graph of $y = g(x) = 25x^2$. Cross-sections perpendicular to the x-axis are squares. Using the formula $V = \int_a^b A(x)\, dx$ to find the volume.

<u>solution</u>. To find the intersections, we have to take

$$16 = 25x^2, \quad \Longrightarrow x = \pm\frac{4}{5}.$$

Thus, the volume is

$$V = \int_{-\frac{4}{5}}^{\frac{4}{5}} (16 - 25x^2)^2\, dx = 2\int_0^{\frac{4}{5}} (16 - 25x^2)^2\, dx = \frac{8192}{75} \cdot 2 = \frac{16384}{75}.$$

<u>Method of rings</u> To find the specific $A(x)$ or $A(y)$, for example, see Figure 2. We take $y = x^2$ over $[1, 2]$, and rotate the function about x-axis or y-axis, respectively. We get the cross-section area as

$$A = \pi(\text{radius})^2.$$

If we have two function $f(x)$ and $g(x)$ with $0 \le f(x) \le g(x)$ over $[a, b]$, see Figure 3. If we rotate

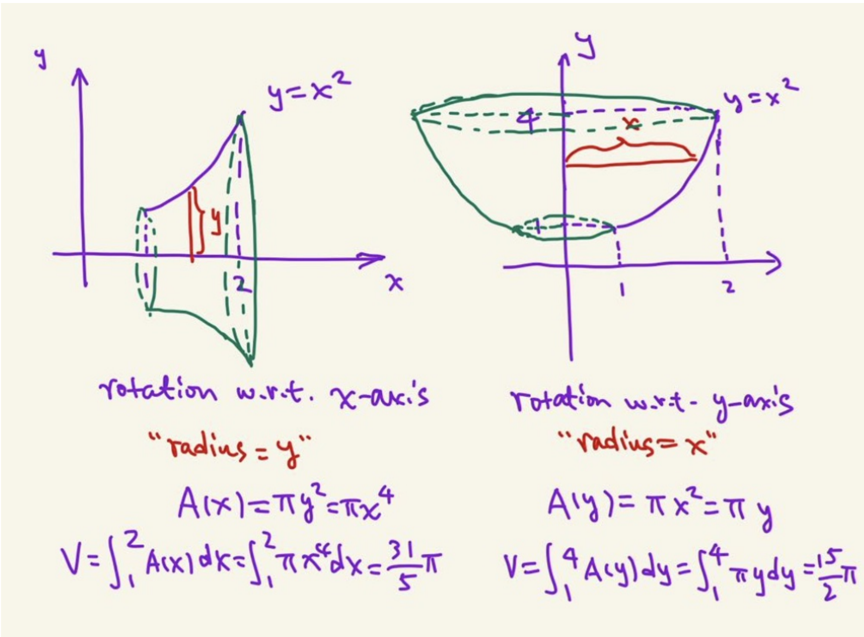


Figure 2: Method of rings.

both functions about x-axis. We get the cross-section area as

$$A = \pi(r_g^2 - r_f^2) = \pi(g(x)^2 - f(x)^2).$$

If there are two functions $x = f(y)$ and $x = g(y)$ and $0 \le f(y) \le g(y)$ over $[c, d]$, similarly, the cross-section area is given by

$$A = \pi(g(y)^2 - (f(y)^2)).$$

<u>Rk</u>. If we rotate about a horizontal axis (the x-axis), then the cross-sectional area will be a function of x. Likewise, if we rotate about a vertical axis (y-axis), then the cross-sectional area will be a function of y.

<u>Example</u>. Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$ and the x-axis about x-axis.

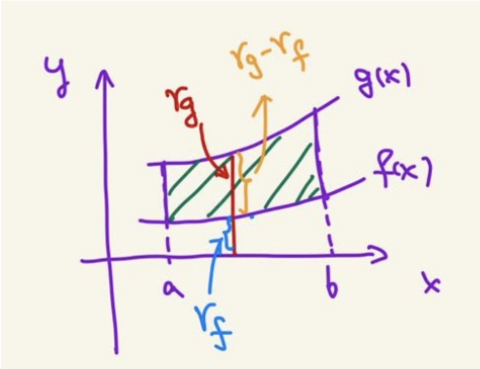
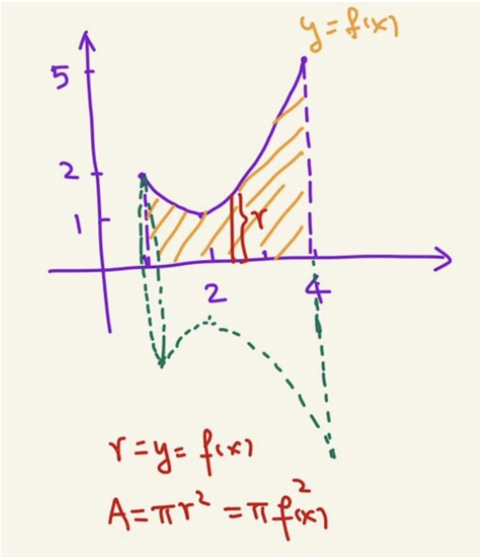


Figure 3: The volume by rotating the closed region about x-axis between the $f(x)$ and $g(x)$ over $[a, b]$



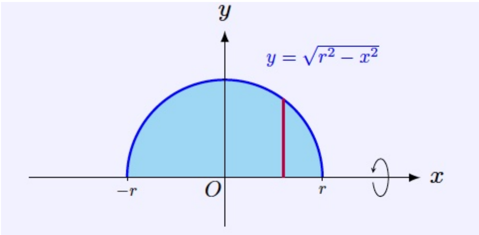
solution. Note that $y = x^2 - 4x + 5 = (x-2)^2 + 1$, and when $x = 1$, $y = 2$; when $x = 4$, $y = 5$. The graph is shown in below.
We have the cross-sectional area

$$A(x) = \pi y^2 = \pi(x^2 - 4x + 5)^2 = \pi \left[x^4 - 8x^3 + 26x^2 - 40x + 25\right].$$

Thus, the volume is given by

$$V = \int_1^4 A(x)\,dx = \int_1^4 \pi \left[x^4 - 8x^3 + 26x^2 - 40x + 25\right]\,dx$$
$$= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Big|_1^4 = \frac{78}{5}\pi.$$

Example (volume of a ball) Find the volume of the ball of radius r.
solution Note that a ball of radius r can be generated by rotating the upper semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis presented below.



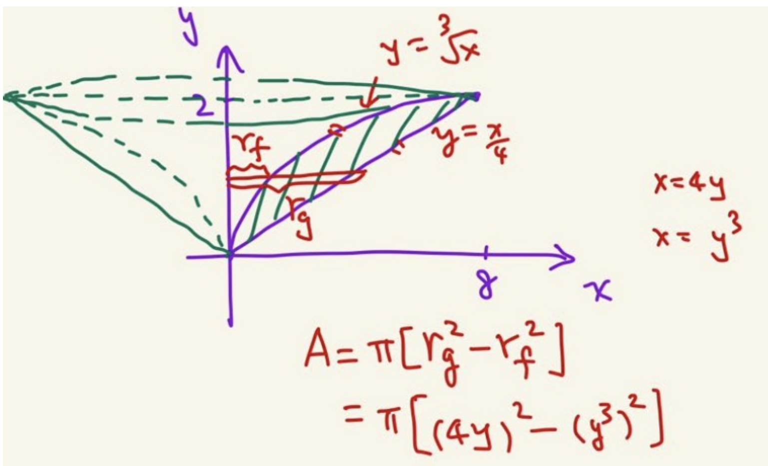
For $x \in [-r, r]$, the cross-section of a ball is a circular disk, with area $\pi y^2 = \pi(r^2 - x^2)$. Thus, the volume of the ball is

$$V = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = 2\int_0^r \pi(r^2 - x^2)\,dx = 2\pi\left(r^2 x - \frac{1}{3}x^3\right)\Big|_0^r = \frac{4}{3}\pi r^3.$$

Example. Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ hat lies in the first quadrant about the y-axis.
solution. To find the intersections, we have to take

$$\sqrt[3]{x} = \frac{x}{4},$$

we have $x = 0$ and $x = 0$ in the first quadrant. The graph is shown below. We have



$$V = \int_0^2 A(y)\,dy = \int_0^2 \pi\left[(4y)^2 - (y^3)^2\right]\,dy = \pi\left[\frac{16}{3}y^3 - \frac{1}{7}y^7\right]\Big|_0^2 = \frac{512}{21}\pi.$$

Example.
Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.
solution. To find the intersections, we have to take

$$x^2 - 2x = x, \Longrightarrow x = 0, x = 3.$$

The graph is shown in fig. 4. We have

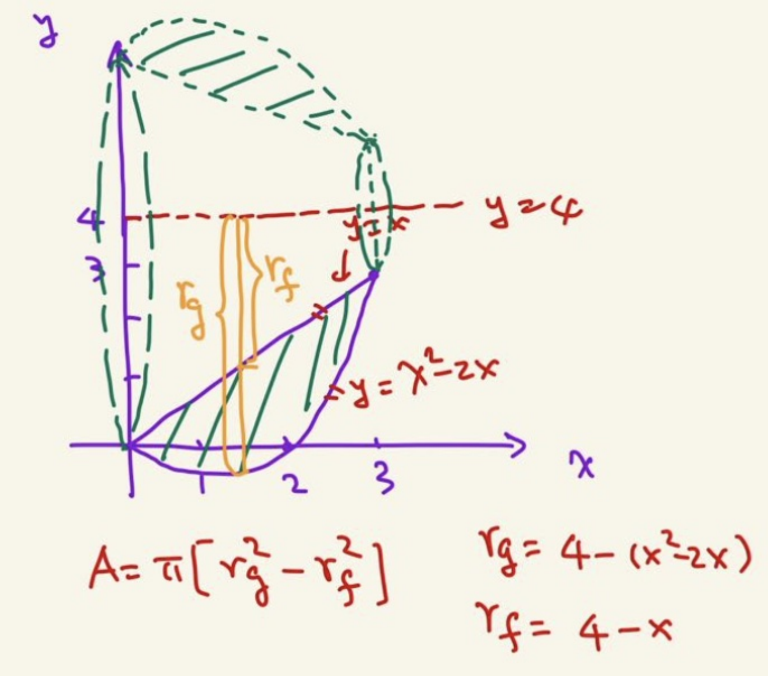


Figure 4: The closed region between $y = x^2 - 2x$,and $y = x$.

$$V = \int_0^3 A(x)\, dx = \int_0^3 \pi \left[(4 - (x^2 - 2x))^2 - (4 - x)^2\right] dx$$

$$= \pi \int_0^2 \left[x^4 - 4x^3 - 5x^2 + 24x\right] dx = \pi \left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2\right)\Big|_0^3 = \frac{153}{5}\pi.$$

Exercise (hint: answer: $V = \frac{96\pi}{5}$). Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$.
Rk. If the rotation is about the line $y = a$ of the closed region, we have the cross-section area $A(x)$. If the rotation is about the line $x = b$ of the closed region, we have the cross-section area $A(y)$.