Solution Tutorial of Midterm Exam

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Part I: MCQs

Q1: Find the area under the graph of the function $y = x\sqrt{9-x^2}$ over the interval $[0,3]$.
solution. Note that $x \in [0,3]$, we have $y \geq 0$. The area is

$$\begin{aligned} A &= \int_0^3 x\sqrt{9-x^2}\,dx = \frac{1}{2}\int_0^3 \sqrt{9-x^2}\,dx^2 = \frac{1}{2}\int_0^9 \sqrt{9-u}\,du \\ &= -\frac{1}{2}\cdot\frac{2}{3}(9-u)^{\frac{3}{2}}\,|_0^9 = 0 + \frac{1}{3}\cdot 9^{\frac{3}{2}} = 9 \end{aligned}$$

where $u = x^2$, $x : 0 \to 3$, $u : 0 \to 9$ has been used.
Rk. Variation: Find the area under the function $y = 3x\sqrt{4-x^2}$ over the interval $[0,2]$. The answer would be

$$\begin{aligned} A &= \int_0^2 3x\sqrt{4-x^2}\,dx = \frac{3}{2}\int_0^2 \sqrt{4-x^2}\,dx^2 = \frac{3}{2}\int_0^4 \sqrt{4-u}\,du \\ &= -\frac{3}{2}\cdot\frac{2}{3}(4-u)^{\frac{3}{2}}|_0^4 = 0 + 4^{\frac{3}{2}} = 8, \end{aligned}$$

where $u = x^2$, $x : 0 \to 2$, $u : 0 \to 4$ has been used.
Q2: Evaluate the integral

$$\int_0^2 4\cos(\pi x)\cos(2\pi x)\cos(3\pi x)\,dx.$$

solution. Note that

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right].$$

We have

$$\begin{aligned} \int_0^2 4\cos(\pi x)\cos(2\pi x)\cos(3\pi x)\,dx &= \int_0^2 2\left[\cos(\pi x) + \cos(3\pi x)\right]\cos(3\pi x)\,dx \\ &= 2\int_0^2 \left[\cos(\pi x)\cos(3\pi x) + \cos(3\pi x)\cos(3\pi x)\right]\,dx \\ &= \int_0^2 \left[\cos(2\pi x) + \cos(4\pi x) + 1 + \cos(6\pi x)\right]\,dx \\ &= \left[\frac{1}{2\pi}\sin(2\pi x) + \frac{1}{4\pi}\sin(4\pi x) + x + \frac{1}{6\pi}\sin(6\pi x)\right]|_0^2 = 2. \end{aligned}$$

Rk. Variation: Evaluate the integral

$$\int_0^2 2\cos(\pi x)\cos(3\pi x)\cos(4\pi x)\,dx.$$

The answer would be

$$\begin{aligned}\int_0^2 2\cos(\pi x)\cos(3\pi x)\cos(4\pi x)\,dx &= \int_0^2 [\cos(2\pi x)+\cos(4\pi x)]\cos(4\pi x)\,dx\\ &= \int_0^2 [\cos(2\pi x)\cos(4\pi x)+\cos(4\pi x)\cos(4\pi x)]\,dx\\ &= \int_0^2 \frac{1}{2}[\cos(2\pi x)+\cos(6\pi x)+1+\cos(8\pi x)]\,dx\\ &= \frac{1}{2}\left[\frac{1}{2\pi}\sin(2\pi x)+\frac{1}{6\pi}\sin(6\pi x)+x+\frac{1}{8\pi}\sin(8\pi x)\right]|_0^2 = 1.\end{aligned}$$

Q3: Evaluate the integral $\int_0^\pi (\pi - x)\sin x\cos^2 x\,dx$.
solution. Method 1 below,

$$\begin{aligned}\int_0^\pi (\pi-x)\sin x\cos^2 x\,dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi}{2}-u\right)\sin\left(\frac{\pi}{2}+u\right)\cos^2\left(\frac{\pi}{2}+u\right)\,du\\ &= \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos u\sin^2 u\,du - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\cos u\sin^2 u\,du\\ &= \frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}} \cos u\sin^2 u\,du = \pi\int_0^{\frac{\pi}{2}} \sin^2 u\,d\sin u = \pi\cdot\frac{1}{3}\sin^3 u|_0^{\frac{\pi}{2}} = \frac{\pi}{3}\end{aligned}$$

where the substitution $u = x - \frac{\pi}{2}$ and $\sin(\frac{\pi}{2}+\alpha) = \cos(\alpha)$, $\cos(\frac{\pi}{2}+\alpha) = -\sin(\alpha)$,

$$\int_{-a}^a f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd,}\\ 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even,}\end{cases}$$

have been used.
Method 2 below,

$$\begin{aligned}\int_0^\pi (\pi-x)\sin x\cos^2 x\,dx &= \int_0^\pi -(\pi-x)\,d\left(\frac{\cos^3 x}{3}\right) = -(\pi-x)\frac{\cos^3 x}{3}|_0^\pi - \frac{1}{3}\int_0^\pi \cos^3 x\,dx\\ &= \frac{\pi}{3},\end{aligned}$$

since that $\cos^3 x$ with equally positive and negative area or using

$$\int_0^\pi \cos^3 x\,dx = \int_0^\pi \cos^2 x\,d\sin x = \int_0^\pi (1-\sin^2 x)\,d\sin x = \left[\sin x - \frac{1}{3}\sin^3 x\right]|_0^\pi = 0.$$

Rk. Variation: Evaluate the integral $\int_0^{2\pi} (\pi - x)\sin x\cos^2 x\,dx$.
The answer would be

$$\begin{aligned}\int_0^{2\pi} (\pi-x)\sin x\cos^2 x\,dx &= \int_0^{2\pi} -(\pi-x)\,d\left(\frac{\cos^3 x}{3}\right) = -(\pi-x)\frac{\cos^3 x}{3}|_0^{2\pi} - \frac{1}{3}\int_0^{2\pi} \cos^3 x\,dx\\ &= \frac{2\pi}{3},\end{aligned}$$

since that

$$\int_0^{2\pi} \cos^3 x\,dx = \int_0^{2\pi} (1-\sin^2 x)\,d\sin x = \left[\sin x - \frac{1}{3}\sin^3 x\right]|_0^{2\pi} = 0.$$

Q4. Evaluate the integral

$$\int_1^\infty \frac{6}{x^2\sqrt{x^2+8}}\,dx.$$

solution. Taking $x=\sqrt{8}\tan\theta$, we have $x: 1\to\infty$, $\theta: \tan^{-1}\frac{1}{\sqrt{8}}\to\frac{\pi}{2}$, $dx=\sqrt{8}\sec^2\theta$, and

$$\begin{aligned}\int_1^\infty \frac{6}{x^2\sqrt{x^2+8}}\,dx &= \int_{\tan^{-1}\frac{1}{\sqrt{8}}}^{\frac{\pi}{2}} \frac{6}{8\tan^2\theta\sqrt{8}\sec\theta}\cdot\sqrt{8}\sec^2\theta\,d\theta \\ &= \frac{3}{4}\int_{\tan^{-1}\frac{1}{\sqrt{8}}}^{\frac{\pi}{2}} \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{3}{4}\int_{\tan^{-1}\frac{1}{\sqrt{8}}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin^2\theta}\,d\theta \\ &= \frac{3}{4}(-\sin\theta)^{-1}\Big|_{\tan^{-1}\frac{1}{\sqrt{8}}}^{\frac{\pi}{2}} = \frac{3}{4}(-1+3)=\frac{3}{2}.\end{aligned}$$

Rk. Variation: Evaluate the integral

$$\int_2^\infty \frac{5}{x^2\sqrt{x^2+5}}\,dx.$$

Taking $x=\sqrt{5}\tan\theta$, we have $x: 2\to\infty$, $\theta: \tan^{-1}\frac{2}{\sqrt{5}}\to\frac{\pi}{2}$, $dx=\sqrt{5}\sec^2\theta$, and

$$\begin{aligned}\int_2^\infty \frac{5}{x^2\sqrt{x^2+5}}\,dx &= \int_{\tan^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \frac{5}{5\tan^2\theta\sqrt{5}\sec\theta}\cdot\sqrt{5}\sec^2\theta\,d\theta \\ &= \int_{\tan^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int_{\tan^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin^2\theta}\,d\theta \\ &= (-\sin\theta)^{-1}\Big|_{\tan^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} = -1+\frac{3}{2}=\frac{1}{2}.\end{aligned}$$

Q5. For which constant k is the improper integral

$$\int_0^\infty \left(\frac{4x+1}{2x^2+3}-\frac{kx^2+2}{2x^3+3}\right)dx$$

convergent ?

solution. Since that

$$\begin{aligned}\frac{4x+1}{2x^2+3}-\frac{kx^2+2}{2x^3+3} &= \frac{(4x+1)(2x^3+2)-(kx^2+2)(2x^2+3)}{(2x^2+3)(2x^3+3)} \\ &= \frac{(8-2k)x^4+2x^3-(3k+4)x^2+8x-4}{(2x^2+3)(2x^3+3)}\end{aligned}$$

Thus,

- if $8-2k=0$, say $k=4$, we have

$$\begin{aligned}\int_0^\infty \frac{2x^3-16x^2+8x-4}{(2x^2+3)(2x^3+3)}\,dx &< \int_0^\infty \frac{2x^3}{(2x^2+3)(2x^3+3)}\,dx \\ &< \int_0^1 \frac{2x^3}{(2x^2+3)(2x^3+3)}\,dx + \int_1^\infty \frac{2x^3}{(2x^2+3)(2x^3+3)}\,dx \\ &< c+\int_1^\infty \frac{2x^3}{2x^2\cdot 2x^3}\,dx = c+\frac{1}{2}\int_1^\infty \frac{1}{x^2}\,dx < \infty, \text{convergent},\end{aligned}$$

- if $k \neq 4$, the higher order term would be x^4 in the numerator, note that

$$\int_1^\infty \frac{1}{x}\,dx,$$

is divergent. Thus, $k = 4$ for the convergent improper integral.

Rk. Variation: For which constant k is the improper integral

$$\int_0^\infty \left(\frac{3x^2+x-1}{2x^3+1} - \frac{kx+2}{2x^2+5}\right)\,dx$$

convergent ?
Since that

$$\begin{aligned}\frac{3x^2+x-1}{2x^3+1} - \frac{kx+2}{2x^2+5} &= \frac{(3x^2+x-1)(2x^2+5)-(kx+2)(2x^3+1)}{(2x^3+1)(2x^2+5)} \\ &= \frac{(6-2k)x^4-2x^3+13x^2+(5-k)x-7}{(2x^3+1)(2x^2+5)}.\end{aligned}$$

Thus, for convergence, let's take $6-2k=0$, say, $k=3$.
Q6. Which one of the following improper integrals is convergent?

$$(a)\ \int_1^\infty \frac{\ln x^2}{x^2+4}\,dx,\quad (b)\ \int_1^\infty \frac{\sqrt{x}}{1+x}\,dx,\quad (c)\ \int_1^\infty \frac{2^x}{x+2^x}\,dx,\quad (d)\ \int_1^\infty \frac{1}{1+x\ln x}\,dx,\quad (e)\ \int_1^\infty \frac{1}{x\ln\sqrt{x}}\,dx.$$

solution. Since that

$$\int_1^\infty \frac{\ln x^2}{x^2+4}\,dx = \int_1^\infty \frac{2\ln x}{x^2+4}\,dx < \int_1^\infty \frac{2\sqrt{x}}{x^2+4}\,dx < 2\int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = 2\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx < \infty,\ \text{convergent}$$

and

$$\int_1^\infty \frac{\sqrt{x}}{1+x}\,dx > \int_1^\infty \frac{\sqrt{x}}{x+x}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{x^{\frac{1}{2}}}\,dx > \infty,\ \text{divergent},$$

and

$$\int_1^\infty \frac{2^x}{x+2^x}\,dx > \int_1^\infty \frac{2^x}{2^x+2^x}\,dx = \int_1^\infty \frac{2^x}{2^{x+1}}\,dx = \frac{1}{2}(\infty-1) = \infty,\ \text{divergent},$$

and

$$\int_1^\infty \frac{1}{1+x\ln x}\,dx > \int_3^\infty \frac{1}{2x\ln x}\,dx = \frac{1}{2}\int_3^\infty \frac{1}{\ln x}\,d\ln x = \frac{1}{2}\ln(\ln x)|3^\infty = \infty, \text{divergent},$$

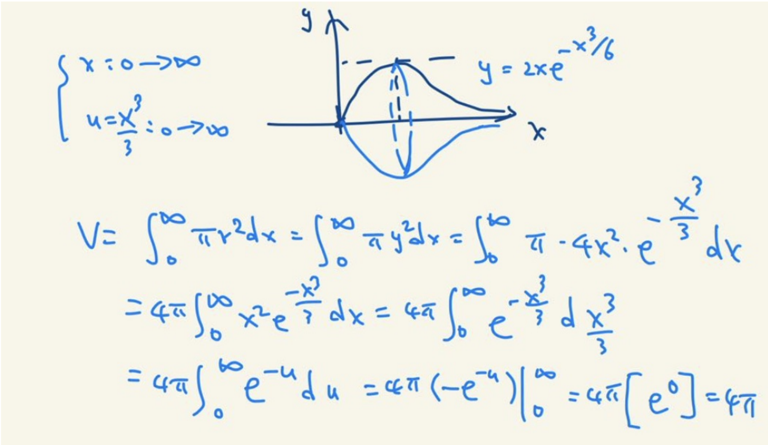
and

$$\int_1^\infty \frac{1}{x\ln\sqrt{x}}\,dx = \int_1^\infty \frac{1}{\frac{1}{2}x\ln x}\,dx = 2\ln(\ln x)|_1^\infty = \infty, \text{divergent}.$$

Q7. The region under the graph of $y = 2xe^{-x^3/6}$ over the interval $[0,\infty)$ is rotated about the x-axis to generate a solid of revolution. Find the volume of the solid.
solution. Since that

$$y' = e^{-\frac{x^3}{6}}(2-x^3),$$

we take $y'=0$, gives us $x=\sqrt[3]{2}$. Thus,



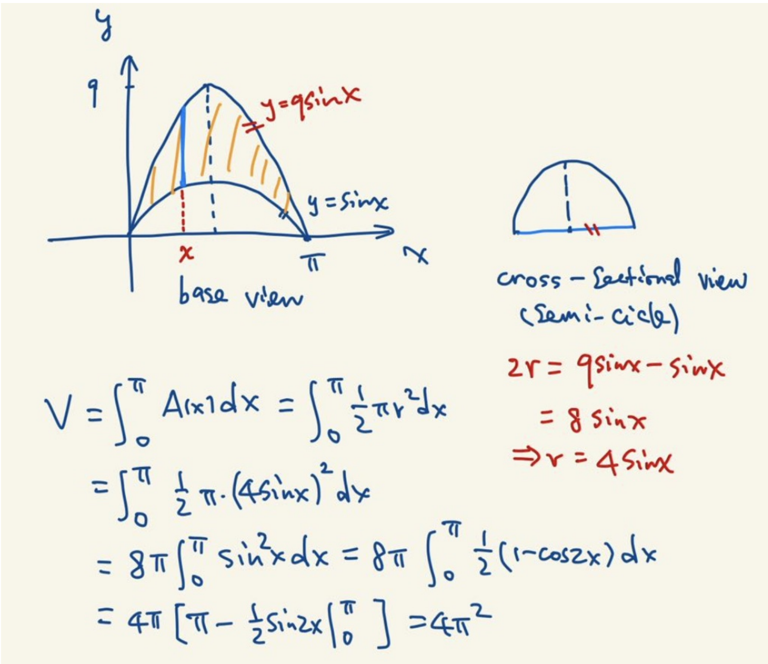
- when $0 < x < \sqrt[3]{2} \Longrightarrow 2 - x^3 > 0$, $y' > 0$, y is increasing;
- when $x > \sqrt[3]{2} \Longrightarrow 2 - x^3 < 0$, $y' < 0$, y is decreasing.

Rk. Variation: The region under the graph of $y = 3xe^{-x^3/4}$ over the interval $[0, \infty)$ is rotated about the x-axis to generate a solid of revolution. Find the volume of the solid.

$$V = \int_0^\infty \pi \cdot 9x^2 e^{\frac{-x^3}{2}}\, dx = \left[-6\pi e^{-\frac{x^3}{2}}\right]|_0^\infty = 6\pi.$$

Q8. The base of a solid sitting on the xy-plane is the region bounded between the graphs of $y = 9\sin x$ and $y = \sin x$, where $0 \le x \le \pi$. Suppose that the cross sections of the solid perpendicular to the x-axis are semi-discs. Find the volume of the solid.

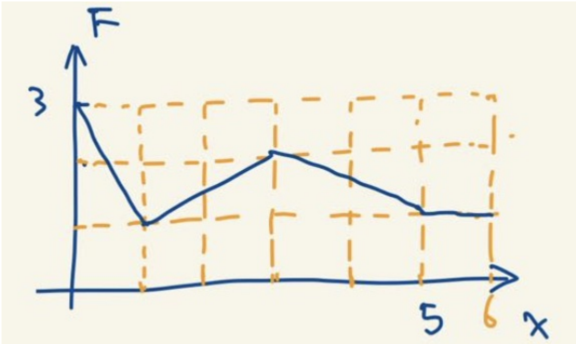
solution. The sketch is below.



Rk. Variation: The base of a solid sitting on the xy-plane is the region bounded between the graphs of $y = 5\sin x$ and $y = \sin x$, where $0 \le x \le \pi$. Suppose that the cross sections of the solid perpendicular to the x-axis are semi-discs. Find the volume of the solid.

$$V=\int_0^\pi \frac{1}{2}\pi r^2\,dx=\int_0^\pi \frac{1}{2}\pi\cdot 4\sin^2 x\,dx=\pi\int_0^\pi (1-\cos 2x)\,dx=\pi^2.$$

Q9. The graph of a force function (in newtons) is given as below. How much work (in joules) is done by the force in moving an object from $x=0$ to $x=5$ (in meters) ?



solution. The work is

$$W=\int_0^5 F(x)\,dx=1\cdot 3+5=8.$$

Rk. How much work (in joules) is done by the force in moving an object from $x=0$ to $x=3$ (in meters) ?
The work is

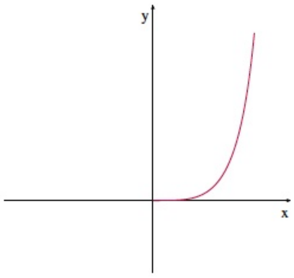
$$W=\int_0^3 F(x)\,dx=1\cdot 2+3=5.$$

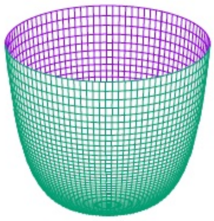
Q10. The length of the graph of a positive continuous function $y=f(x)$ over the interval is 2 units. Suppose the area of the surface of revolution obtained by rotating the graph of f about the x-axis is 2π square units. Find the area of the surface of resolution obtained by rotating the graph of $y=f(x)+1$ about the x-axis.
solution. Since that $f(x)>0$, thus $f(x)+1>0$ and the area is

$$\begin{aligned}S&=\int_a^b 2\pi\,[f(x)+1]\,\sqrt{1+[(f(x)+1)']^2}\,dx=\int_a^b 2\pi\,[f(x)+1]\,\sqrt{1+(f'(x))^2}\,dx\\&=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx+2\pi\int_a^b \sqrt{1+[f'(x)]^2}\,dx=2\pi+2\pi\cdot 2=6\pi.\end{aligned}$$

Rk. Variation: The length of the graph of a positive continuous function $y=f(x)$ over the interval is 2 units. Suppose the area of the surface of revolution obtained by rotating the graph of f about the x-axis is 3π square units. Find the area of the surface of resolution obtained by rotating the graph of $y=f(x)+1$ about the x-axis.
Since that $f(x)>0$, thus $f(x)+1>0$ and the area is

$$\begin{aligned}S&=\int_a^b 2\pi\,[f(x)+1]\,\sqrt{1+[(f(x)+1)']^2}\,dx=\int_a^b 2\pi\,[f(x)+1]\,\sqrt{1+(f'(x))^2}\,dx\\&=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx+2\pi\int_a^b \sqrt{1+[f'(x)]^2}\,dx=3\pi+2\pi\cdot 2=7\pi.\end{aligned}$$



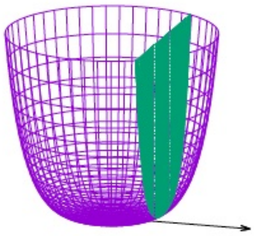


Part II

A bowl is in the shape of a surface of revolution obtained by rotating the graph of the function $y = 6\tan^2 x^2$ about the y-axis, where $0 \le x \le \frac{\sqrt{\pi}}{2}$. ($x, y$ in meters.)

(a) Find the volume of the bowl.

(b) Consider the cross sections of the solid region contained by the bowl which are perpendicular to the x-axis. Find the average value of their areas.



(c) Suppose the bowl is full of water. Express the work required to pumped all water in the bowl to an outlet at the top of the bowl by a definite integral. Do not need to evaluate the integral. (You may denote the density of water by ρ, and the gravity acceleration by g, both in SI units.)

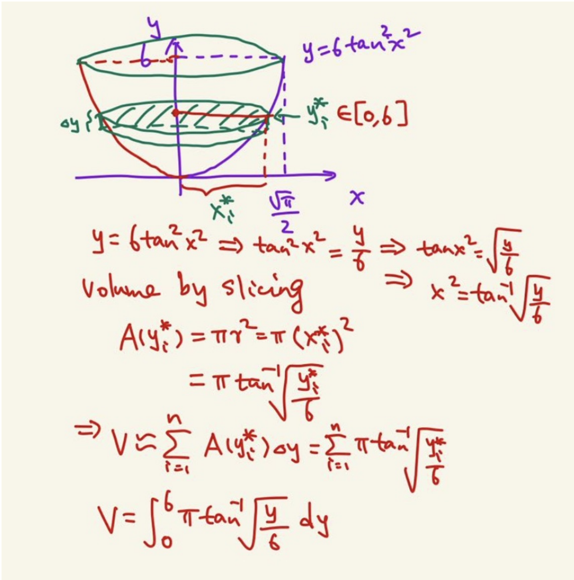
solution.

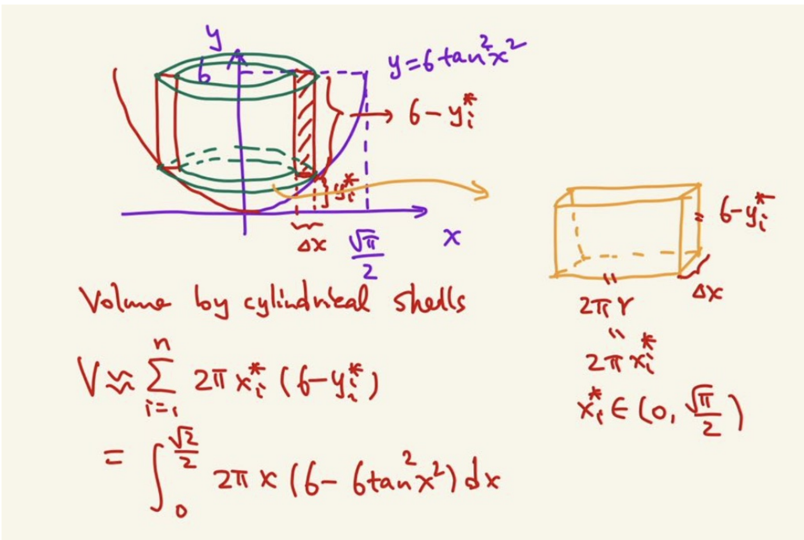
(a) Volume by slicing:

Volume by cylindrical shells: Note that using volume by slicing, we have

$$\begin{aligned}
V &= \int_0^6 \pi \tan^{-1}\sqrt{\frac{y}{6}}\, dy = \int_0^1 \pi \tan^{-1} u \cdot 12u\, du \\
&= 12\pi \int_0^1 \tan^{-1} u\, du^2 = 6\pi \int_0^1 \tan^{-1} u\, du^2 = 6\pi \left[u^2 \tan^{-1} u|_0^1 - \int_0^1 \frac{u^2+1-1}{1+u^2}\, du \right] \\
&= 6\pi \left[\frac{\pi}{4} - \left(1 - \int_0^1 \frac{1}{1+u^2}\, du \right) \right] = 6\pi \left[\frac{\pi}{4} - 1 + \tan^{-1} u|_0^1 \right] \\
&= 3\pi^2 - 6\pi
\end{aligned}$$

where the substitution $u = \sqrt{\frac{y}{6}}$, $y : 0 \to 6$ and $u : 0 \to 1$ has been used. Also, using volume


y
6
y=6 tan²x²
Δy
yᵢ* ∈[0,6]
xᵢ*
√π/2
x
y = 6tan²x² ⇒ tan²x² = y/6 ⇒ tanx² = √(y/6)
⇒ x² = tan⁻¹√(y/6)
Volume by slicing
A(yᵢ*) = πr² = π(xᵢ*)²
= π tan⁻¹√(yᵢ*/6)
⇒ V ≈ Σᵢ₌₁ⁿ A(yᵢ*)Δy = Σᵢ₌₁ⁿ π tan⁻¹√(yᵢ*/6)
V = ∫₀⁶ π tan⁻¹√(y/6) dy


y
6
y=6 tan²x²
6−yᵢ*
yᵢ*
Δx
√π/2
x
6-yᵢ*
2πr
2πxᵢ*
Δx
xᵢ* ∈ (0, √π/2)
Volume by cylindrical shells
V ≈ Σᵢ₌₁ⁿ 2π xᵢ* (6−yᵢ*)
= ∫₀^(√2/2) 2π x (6 − 6tan²x²) dx

by cylindrical shells, we have directly

$$
\begin{aligned}
V &= \int_0^{\frac{\sqrt{\pi}}{2}} 2\pi x(6 - 6\tan^2 x^2)\, dx = 12\pi \int_0^{\frac{\sqrt{\pi}}{2}} x[1 - \tan^2 x^2]\, dx \\
&= 12\pi \int_0^{\frac{\sqrt{\pi}}{2}} x[1 - (\sec^2 x^2 - 1)]\, dx = 12\pi \int_0^{\frac{\sqrt{\pi}}{2}} x[2\sec^2 x^2]\, dx \\
&= 12\pi \int_0^{\frac{\sqrt{\pi}}{2}} 2x\, dx - 12\pi \int_0^{\frac{\sqrt{\pi}}{2}} x\sec^2 x^2\, dx = 12\pi \cdot x^2\Big|_0^{\frac{\sqrt{\pi}}{2}} - 6\pi \int_0^{\frac{\sqrt{\pi}}{2}} \sec^2 x^2\, dx^2 \\
&= 12\pi \cdot \frac{\pi}{4} - 6\pi \cdot \tan x^2\Big|_0^{\frac{\sqrt{\pi}}{2}} = 3\pi^2 - 6\pi.
\end{aligned}
$$

where $1 + \tan^2 \alpha = \sec^2 \alpha$ has been used.

Of course, if one obtain

$$V_1 = \int_0^{\frac{\sqrt{\pi}}{2}} 2\pi x \cdot 6\tan^2 x^2\, dx,$$

which is the volume below the solid. Thus, the volume of solid should be

$$V = V_2 - V_1 = \pi \left(\frac{\sqrt{\pi}}{2}\right)^2 \cdot 6 - V_1 = \int_0^{\frac{\sqrt{\pi}}{2}} 2\pi x \cdot (6 - 6\tan^2 x^2)\, dx$$

where V_2 is the volume of the cylinder.

(b) Recall the the average value of a function $A(x)$ below,

$$A_{\mathsf{avg}} = \frac{1}{b-a} \int_a^b A(x)\, dx.$$

Thus, the average of the areas is

$$A_{\mathsf{avg}} = \frac{1}{\frac{\sqrt{\pi}}{2} - \left(\frac{\sqrt{\pi}}{2}\right)} \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} A(x)\, dx = \frac{1}{\sqrt{\pi}} V = \frac{1}{\sqrt{\pi}}(3\pi^2 - 6\pi) = 3\sqrt{\pi}(\pi - 2),$$

by the definition of volume by slicing.

If we want to see what is $A(x)$ here, since that the cross section area is perpendicular to x-axis, we have

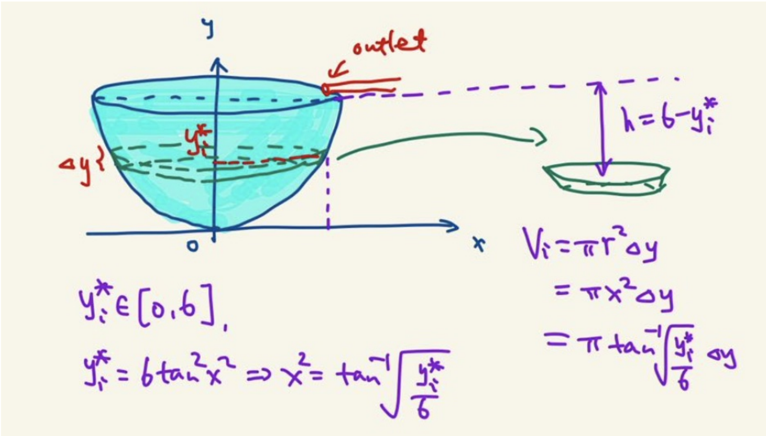
$$A(x) = 2\int_x^{\frac{\sqrt{\pi}}{2}} (6 - 6\tan^2 z^2)\, dz = 12\int_x^{\frac{\sqrt{\pi}}{2}} (2 - \sec^2 z^2)\, dz,$$

thus, the average of the area could be represented by

$$A_{\mathsf{avg}} = \frac{1}{\frac{\sqrt{\pi}}{2} - \left(\frac{\sqrt{\pi}}{2}\right)} \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} 12\int_x^{\frac{\sqrt{\pi}}{2}} (2 - \sec^2 z^2)\, dz\, dx = \frac{12}{\sqrt{\pi}} \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \int_x^{\frac{\sqrt{\pi}}{2}} (2 - \sec^2 z^2)\, dz\, dx,$$

however, $\int \sec^2 z^2\, dz$ can not be integrated by FTC, (something like $\int \cos(x^2)\, dx$). However, we have realized that V can be calculated by (a).

(c) The sketch is below.



Dividing the interval $[0, 6]$ into n subintervals, and taking y_i^* in each subinterval. Thus, the height from the top to the i-th piece is $6 - y_i^*$. The work in a subinterval is

$$W_i = m \cdot g \cdot h = \rho V_i \cdot g \cdot h = \rho g \pi \tan^{-1}\sqrt{\frac{y_i^*}{6}}\Delta y(6 - y_i^*),$$

where the tiny volume $V_i \approx \pi \tan^{-1}\sqrt{\frac{y_i^*}{6}}\Delta y$. Thus, the required work is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} W_i = \lim_{n\to\infty}\sum_{i=1}^{n}\rho g\pi\tan^{-1}\sqrt{\frac{y_i^*}{6}}\Delta y(6-y_i^*) = \pi\rho g\int_0^6 (6-y)\tan^{-1}\sqrt{\frac{y}{6}}\,dy.$$

Part III

Let $I_n = \int_0^4 \frac{1}{(x^2+16)^n}\,dx$, where $n = 1, 2, 3, \cdots$ is a positive integer.

(a) Using integration by parts, or otherwise, find $A(n)$, $B(n)$, which are expressions depending on n, such that

$$I_{n+1} = A(n)I_n + B(n).$$

(Hint: Start with I_n.)

(b) Using (a), or otherwise, evaluate the integral

$$\int_0^4 \left[\frac{32}{(x^2+16)^5} - \frac{7}{4(x^2+16)^4}\right] dx.$$

(c) If Simpson's rule on four subintervals is used to approximate

$$\pi = \int_0^4 \frac{16}{x^2+16}\,dx,$$

a rational approximate value of π can be found as

$$\pi \approx \frac{1}{3}\left[1 + \frac{64}{a} + \frac{8}{b} + \frac{64}{c} + \frac{1}{d}\right],$$

where a, b, c, d are positive integers. Find a, b, c, d.

solution.

For (a), there are four methods at least below using integration by parts.

Way 1: Note that (Starting with I_n)

$$\begin{aligned} I_n &= \int_0^4 \frac{1}{(x^2+16)^n}\,dx = x\cdot\frac{1}{(x^2+16)^n}\Big|_0^4 - \int_0^4 x\cdot\left[(x^2+16)^{-n}\right]'\,dx \\ &= \frac{4}{32^n} - \int_0^4 x\cdot(-n)(x^2+16)^{-n-1}\cdot 2x\,dx = \frac{4}{32^n} + 2n\int_0^4 \frac{x^2}{(x^2+16)^{n+1}}\,dx \\ &= \frac{4}{32^n} + 2n\int_0^4 \frac{x^2+16-16}{(x^2+16)^{n+1}}\,dx = \frac{4}{32^n} + 2nI_n - 32nI_{n+1}. \end{aligned}$$

Thus,

$$32nI_{n+1} = \frac{4}{32^n} + (2n-1)I_n,$$

say,

$$I_{n+1} = \frac{2n-1}{32n}I_n + \frac{4}{32^{n+1}n}, \quad \Longrightarrow \quad A(n) = \frac{2n-1}{32n},\ B(n) = \frac{4}{32^{n+1}n}.$$

Way 2: Note that (Starting with I_{n+1})

$$\begin{aligned} I_{n+1} &= \int_0^4 \frac{1}{(x^2+16)^{n+1}}\,dx = x\cdot\frac{1}{(x^2+16)^{n+1}}\Big|_0^4 - \int_0^4 x\cdot\left[(x^2+16)^{-(n+1)}\right]'\,dx \\ &= \frac{4}{32^{n+1}} - \int_0^4 x\cdot[-(n+1)](x^2+16)^{-(n+1)-1}\cdot 2x\,dx \\ &= \frac{4}{32^{n+1}} + 2(n+1)\int_0^4 \frac{x^2}{(x^2+16)^{n+2}}\,dx = \frac{4}{32^{n+1}} + 2(n+1)\int_0^4 \frac{x^2+16-16}{(x^2+16)^{n+2}}\,dx \\ &= \frac{4}{32^{n+1}} + 2(n+1)\int_0^4 \frac{1}{(x^2+16)^{n+1}}\,dx - 32(n+1)\int_0^4 \frac{1}{(x^2+16)^{n+2}}\,dx \\ &= \frac{4}{32^{n+1}} + 2(n+1)I_{n+1} - 32(n+1)I_{n+2}. \end{aligned}$$

Thus,

$$32(n+1)I_{n+2} = \frac{4}{32^{n+1}} + [2(n+1)-1]I_{n+1},$$

say

$$I_{n+2} = \frac{4}{32^{n+2}(n+1)} + \frac{2(n+1)-1}{32(n+1)}I_{n+1},$$

replacing $n+1$ by n, we have

$$I_{n+1} = \frac{4}{32^{n+1}\cdot n} + \frac{2n-1}{32n}I_n \quad \Longrightarrow \quad A(n) = \frac{2n-1}{32n},\ B(n) = \frac{4}{32^{n+1}n}.$$

Way 3: Note that (Starting with I_n and using the trigonometric integral and substitutions) by using substitution $x = 4\tan\theta$, we have $x: 0 \to 4$, $\theta: 0 \to \frac{\pi}{4}$, $dx = 4\sec^2\theta\,d\theta$ and

$$I_n = \int_0^{\frac{\pi}{4}} \frac{1}{16^n\cdot\sec^{2n}\theta}\cdot 4\sec^2\theta\,d\theta = \frac{4}{16^n}\int_0^{\frac{\pi}{4}} \frac{1}{\sec^{2n-2}\theta}\,d\theta = \frac{4}{16^n}\int_0^{\frac{\pi}{4}} \cos^{2n-2}\theta\,d\theta.$$

Also, note that

$$\begin{aligned}
I_{n+1} &= \frac{4}{16^{n+1}}\int_0^{\frac{\pi}{4}} \cos^{2n}\theta\, d\theta = \frac{4}{16^{n+1}}\int_0^{\frac{\pi}{4}} \cos^{2n-1}\theta\cdot(\sin\theta)'\, d\theta \\
&= \frac{4}{16^{n+1}}\int_0^{\frac{\pi}{4}} \cos^{2n-1}\theta\, d\sin\theta = \frac{4}{16^{n+1}}\left[\cos^{2n-1}\theta\cdot\sin\theta|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\sin\theta\, d\left(\cos^{2n-1}\theta\right)\right] \\
&= \frac{4}{16^{n+1}}\cos^{2n-1}\theta\cdot\sin\theta|_0^{\frac{\pi}{4}} - \frac{4}{16^{n+1}}\int_0^{\frac{\pi}{4}}\sin\theta\cdot(2n-1)\cdot\cos^{2n-2}\theta\cdot(-\sin\theta)\, d\theta \\
&= \frac{4}{16^{n+1}}\cdot\left(\frac{1}{\sqrt{2}}\right)^{2n} + \frac{4}{16^{n+1}}\cdot(2n-1)\int_0^{\frac{\pi}{4}}\cos^{2n-2}\theta\cdot(1-\cos^2\theta)\, d\theta \\
&= \frac{4}{16^{n+1}}\cdot\frac{1}{2^n} + \frac{4(2n-1)}{16^{n+1}}\left(\frac{16^n}{4}I_n - \frac{16^{n+1}}{4}I_{n+1}\right) = \frac{4}{32^n\cdot 16} + \frac{2n-1}{16}I_n - (2n-1)I_{n+1}.
\end{aligned}$$

Thus,

$$2nI_{n+1} = \frac{4}{32^n\cdot 16} + \frac{2n-1}{16}I_n,$$

say,

$$I_{n+1} = \frac{4}{32^{n+1}n} + \frac{2n-1}{32n}I_n, \quad\Longrightarrow\quad A(n) = \frac{2n-1}{32n},\ B(n) = \frac{4}{32^{n+1}n}.$$

Way 4: Note that (Starting with I_{n+1})

$$I_{n+1} = \frac{1}{16}\int_0^4 \frac{16+x^2-x^2}{(x^2+16)^{n+1}}\, dx = \frac{1}{16}\left[\int_0^4 \frac{1}{(x^2+16)^n}\, dx - \int_0^4 \frac{x^2}{(x^2+16)^{n+1}}\, dx\right],$$

and here

$$\begin{aligned}
\int_0^4 \frac{x^2}{(x^2+16)^{n+1}}\, dx &= \frac{1}{2}\int_0^4 x\cdot\frac{1}{(x^2+16)^{n+1}}\, dx^2 = -\frac{1}{2n}\int_0^4 x\, d(x^2+16)^{-n} \\
&= -\frac{1}{2n}\left[x(x^2+16)^{-n}\right]|_0^4 + \frac{1}{2n}\int_0^4 (x^2+16)^{-n}\, dx = -\frac{1}{2n}\cdot 4\cdot(32)^{-n} + \frac{1}{2n}I_n.
\end{aligned}$$

We then have

$$I_{n+1} = \frac{1}{16}\left[I_n + \frac{1}{2n}\cdot\frac{4}{32^n} - \frac{1}{2n}I_n\right] = \frac{2n-1}{32n}I_n + \frac{4}{32^{n+1}n}, \quad\Longrightarrow\quad A(n) = \frac{2n-1}{32n},\ B(n) = \frac{4}{32^{n+1}n}.$$

For (b), method 1, it follows by (a), such that

$$I_{n+1} = \frac{2n-1}{32n}I_n + \frac{4}{32^{n+1}n},$$

we can take $n = 4$, and then have

$$I_5 = \frac{7}{32\times 4}I_4 + \frac{4}{32^5\cdot 4}, \quad\Longrightarrow\quad 32I_5 = \frac{7}{4}I_4 + \frac{1}{32^4},$$

note that

$$\int_0^4 \left[\frac{32}{(x^2+16)^5} - \frac{7}{4(x^2+16)^4}\right] dx = 32I_5 - \frac{7}{4}I_4 = \frac{1}{32^4}.$$

Method 2, (compute the I_4 and I_5 separately, evaluate the result),

$$\int_0^4 \left[\frac{32}{(x^2+16)^5} - \frac{7}{4(x^2+16)^4}\right] dx = 32I_5 - \frac{7}{4}I_4,$$

by the substitution $x = 4\tan\theta$, we have $x : 0 \to 4$, $\theta : 0 \to \frac{\pi}{4}$, $dx = 4\sec^2\theta\, d\theta$ and

$$I_4 = \int_0^4 \frac{1}{(x^2+16)^4}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{16^4 \cdot \sec^8\theta} \cdot 4\sec^2\theta\, d\theta = \frac{4}{16^4}\int_0^{\frac{\pi}{4}} \frac{1}{\sec^6\theta}\, d\theta = \frac{4}{16^4}\int_0^{\frac{\pi}{4}} \cos^6\theta\, d\theta,$$

$$I_5 = \int_0^4 \frac{1}{(x^2+16)^5}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{16^5 \cdot \sec^1 0\theta} \cdot 4\sec^2\theta\, d\theta = \frac{4}{16^5}\int_0^{\frac{\pi}{4}} \frac{1}{\sec^8\theta}\, d\theta = \frac{4}{16^5}\int_0^{\frac{\pi}{4}} \cos^8\theta\, d\theta,$$

and by integration by parts,

$$\begin{aligned} I_5 &= \frac{4}{16^5}\int_0^{\frac{\pi}{4}} \cos^7\theta\, d\sin\theta = \frac{4}{16^5}\left[\cos^7\theta \cdot \sin\theta\right]|_0^{\frac{\pi}{4}} - \frac{4}{16^5}\int_0^{\frac{\pi}{4}} \sin\theta \cdot 7\cos^6\theta \cdot (-\sin\theta)\, d\theta \\ &= \frac{4}{16^5}\left(\frac{1}{\sqrt{2}}\right)^8 + \frac{4\cdot 7}{16^5}\int_0^{\frac{\pi}{4}} \cos^6\theta \cdot (1-\cos^2\theta)\, d\theta \\ &= \frac{4}{16^5\cdot 2^4} + \frac{4\cdot 7}{16^5}\left(\frac{16^4}{4}I_4 - \frac{16^5}{4}I_5\right) = \frac{4}{16^5\cdot 2^4} + \frac{7}{16}I_4 - 7I_5, \end{aligned}$$

Thus,

$$8I_5 = \frac{4}{16^5\cdot 2^4} + \frac{7}{16}I_4 \quad\Longrightarrow\quad 32I_5 - \frac{7}{4}I_4 = \frac{16}{16^5\cdot 2^4} = \frac{1}{16^4\cdot 2^4} = \frac{1}{32^4}.$$

For (c), let $f(x) = \frac{16}{x^2+16}\, dx$, the base length $\Delta x = \frac{4-0}{4} = 1$, the subintervals are then

$$[0,1], \quad [1,2] \quad [2,3], \quad [3,4],$$

and $f(0) = 1$, $f(1) = \frac{16}{17}$, $f(2) = \frac{16}{20}$, $f(3) = \frac{16}{25}$, $f(4) = \frac{16}{32}$. By Simpson's rule, we have

$$\begin{aligned} S_4 &= \sum_{i=1}^{\frac{4}{2}} \frac{1}{3}\Delta x[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})] \\ &= \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4)] \\ &= \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \\ &= \frac{1}{3}\cdot 1\left[1 + 4\cdot\frac{16}{17} + 2\cdot\frac{16}{20} + 4\cdot\frac{16}{25} + \frac{16}{32}\right] = \frac{1}{3}\left[1 + \frac{64}{17} + \frac{8}{5} + \frac{64}{25} + \frac{1}{2}\right] \\ &= \frac{1}{3}\left[1 + \frac{64}{a} + \frac{8}{b} + \frac{64}{c} + \frac{1}{d}\right], \end{aligned}$$

Thus, $a = 17, b = 5, c = 25, d = 2$ or $a = 25, b = 5, c = 17, d = 2$, where a, b, c, d are positive integers.