

# Hw#5—Asymptotics analysis on boundary layer

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Consider the following ordinary differential equation (ODE) system

$$\epsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0, \quad (1)$$

which satisfies the boundary condition over  $\Omega = [0, 1]$ , we may as well take into account firstly the Dirichlet boundary condition

$$\begin{aligned} y(0) &= \alpha, \\ y(1) &= \beta. \end{aligned}$$

We can also take the value of  $0 < |\epsilon| \ll 1$ , and we note that when  $\epsilon$  takes pass to 0, the system is changed. The formulation is as follows,

$$\frac{dy}{dx} + y = 0, \quad (2)$$

but there are two-point boundary condition, thus, the equation (2) don't have solutions in  $\Omega$ . We want to investigate the boundary layer of original system, we need to take  $\alpha = 0$  or  $\beta = 0$ . If we take  $\alpha = 0$ , we will see the left boundary layer, vise versa, we will see the right one.

We use the standard discretization scheme. Approximating the second derivative using second order centered differences,

$$\begin{aligned} \frac{d^2 y}{dx^2} &\approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \\ \frac{dy}{dx} &\approx \frac{y_{i+1} - y_i}{h}. \end{aligned} \quad (3)$$

where  $h$  the grid size,  $y_i$  the numerical solution,  $i = 0, \dots, Nx$ ,  $h = 1/Nx$ .

We solve the first system with viscosity. The discretization scheme is as follows,

$$\epsilon \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{y_{i+1} - y_i}{h} + y_i = 0 \quad (4)$$

i.e.,

$$\left( \frac{\epsilon}{h^2} + \frac{1}{h} \right) y_{i+1} + \left( 1 - \frac{2\epsilon}{h^2} - \frac{1}{h} \right) y_i + \frac{\epsilon}{h^2} y_{i-1} = 0, \quad (5)$$

i.e., the linear equations is as follows,

$$\mathbf{A}\hat{y} = \mathbf{f},$$

where

$$\mathbf{A} = \begin{pmatrix} b & a & 0 & \cdots & 0 & 0 \\ c & b & a & \cdots & \cdots & 0 \\ 0 & c & b & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & c & b & a \\ 0 & \cdots & \cdots & \cdots & c & b \end{pmatrix}_{(Nx-1) \times (Nx-1)}$$

and  $\hat{y} = (y_1, y_2, \dots, y_{Nx-1})$ ,  $\mathbf{f} = (f_1, \dots, f_{Nx-1})$ ,  $a = \epsilon/h^2 + 1/h$ ,  $b = 1 - 2\epsilon/h^2 - 1/h$ ,  $c = \epsilon/h^2$ . We need to focus on the first and final term w.r.t  $\mathbf{f}$ .

According to the matlab file **ODE\_viscosity\_left.m** as follows, we will see the phenomenon of left boundary layer.

```
% *****
% hw5
% solve the ODE with viscosity
% system with left boundary layer
% small parameter \epsilon
% *****

clear all;
format long;
%
% initialize the problem
%
xleft = 0.0;
xright = 1.0;
Nx = 1.0d5;
dx = (xright-xleft)/Nx;
x = xleft:dx:xright;
epsilon = 0.02;
alpha = 0;
beta = exp(-1);
%
% initialization
%
f = zeros(Nx-1,1); % the right term
y_num_v = zeros(1,Nx+1);
y_num_v(1) = alpha;
y_num_v(Nx+1) = beta;

tic; % start timing

e = ones(Nx-1,1);
```

```

A = spdiags([(epsilon/dx^2)*e (1-2*epsilon/dx^2-1/dx)*e (epsilon/dx^2+1/dx)*e],-1:1,Nx-1,Nx);
f(1) = -epsilon/dx^2*alpha;
f(Nx-1) = -(epsilon/dx^2 + 1/dx)*beta;
y_num_v(2:Nx) = A\f;
%
% the exact solution
%
y_exact = exp(-x);
%
% plot the numerical solution
%
figure(1);clf;
plot(x,y_num_v,'b.-');hold on;grid on;
plot(x,y_exact,'r--');hold on;
title('The viscosity system with \epsilon = 0.02')
legend('num','exact');
xlabel('x');ylabel('y');
%
% plot the error as the function of x
%
figure(2);clf;
plot(x,y_exact-y_num_v,'b.-');hold on;grid on;
title('error between the numerical and exact one');
xlabel('x');ylabel('error');
% err = max(abs(y_num_v-y_exact))
err = y_exact-y_num_v;
for i=1:Nx
if err(i) < 1e-4,
fprintf('%d\n',i);
break
end
end
x(i)
toc; % end timing

```

The result of the numerical solution and the exact solution is presented by Fig.1. We can describe the error in Fig.2. We can obtain the relationship between the thickness of the left boundary layer and the parameter  $\epsilon$ .

Thus, one has

$$\lambda \approx O(\epsilon) \tag{6}$$

where  $\lambda$  denotes the value of boundary layer.

We obtain the left boundary layer by means of matlab file **Boundary\_layer\_left.m** as follows.

```

% *****
% hw5

```

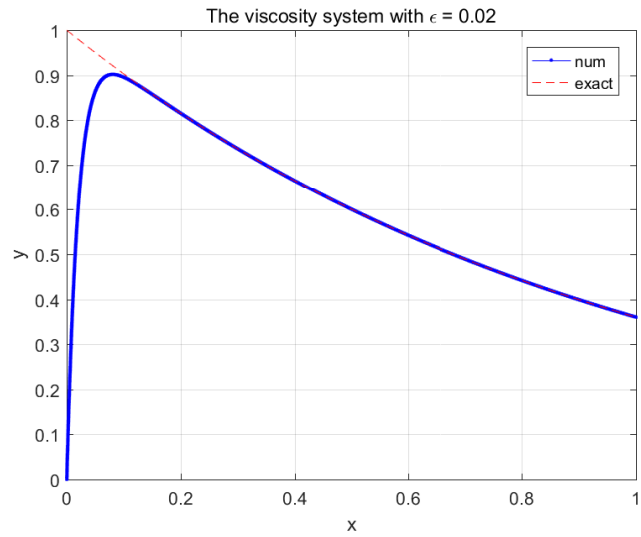


Figure 1: The numerical solution and the exact solution as the function of  $x$

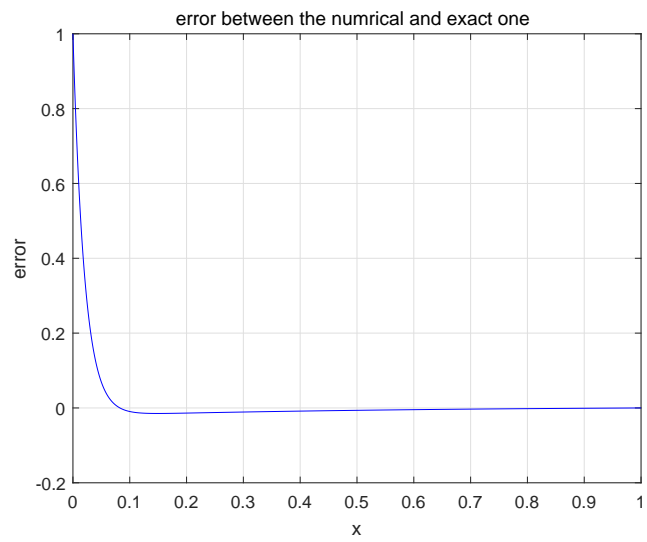


Figure 2: The error between numerical solution and the exact one as the function of  $x$

```

% plot the numerical solution between the viscosity and convection
% the order of the left boundary layer
%
% *****
format long;
Distance = [0.020000000000000 0.082610000000000
0.010000000000000 0.047250000000000
0.005000000000000 0.026780000000000
0.002500000000000 0.015010000000000];
figure(1);clf;
dx = Distance(:,1);
Dist = Distance(:,2);
loglog(dx, Dist,'b*-', 'LineWidth',1.5, 'MarkerSize',7);hold on;grid on;
title('The order of boundary layer');
xlabel('\epsilon');ylabel('Dist of boundary layer');

logh = log(dx(1:end));
logeh = log(Dist(1:end));
AL = [ length(logh) sum(logh);
sum(logh) sum(logh.*logh)];
bL = [ sum(logeh);
sum(logh.*logeh)];
xL = AL\bL;
disp('Convergence rate of Dist is ');
disp(xL(2));

xlabel('\epsilon','Interpreter','tex');
ylabel('Distance','Interpreter','tex');
legend('order: 0.820033573226420','Location','NorthWest');

```

The intuitive result is seen as Fig.3.

Again, we will do the same as previously, but we omit the details, we list the graphs as follows. You can find the detailed file in enclosed **ODE\_viscosity\_right.m**, **viscosity\_right\_error.m** and **viscosity\_boundary\_layer\_left.m**. These results are as seen as Fig.4, Fig.5 and Fig.6.

In deed, we should pay attention to the interesting phenomenon since the system degenerate into another system without the second differential term called viscosity term.

If we consider the another system into the following formulation, i.e., the first derivative is negative.

$$\epsilon \frac{d^2 y}{dx^2} - \frac{dy}{dx} + y = 0, \quad (7)$$

where the condition is the same as before. The discretization is also as above. We can investigate the code in **ODE\_viscosity\_change\_left.m** and **ODE\_viscosity\_change\_right.m**. The left boundary layer is seen as Fig.7, the error is seen as Fig.8, the right one

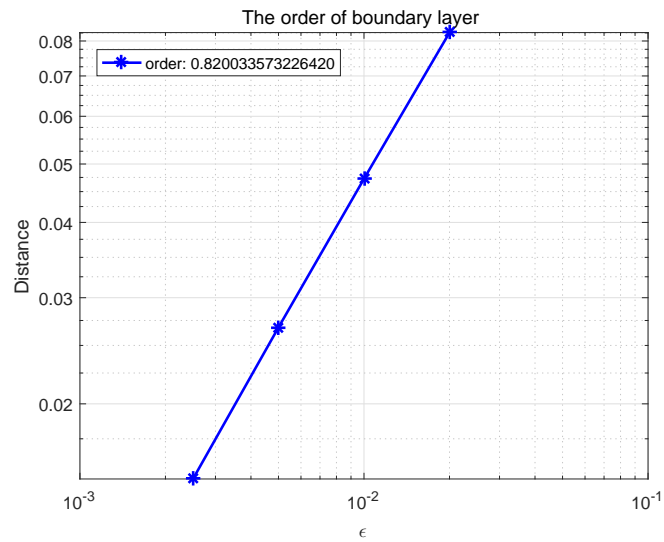


Figure 3: The order of left boundary layer as the function of  $\epsilon$

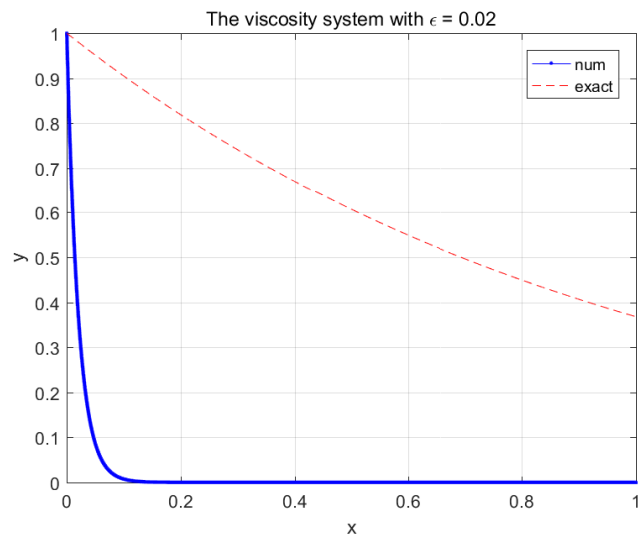


Figure 4: The numerical solution and the exact solution as the function of  $x$

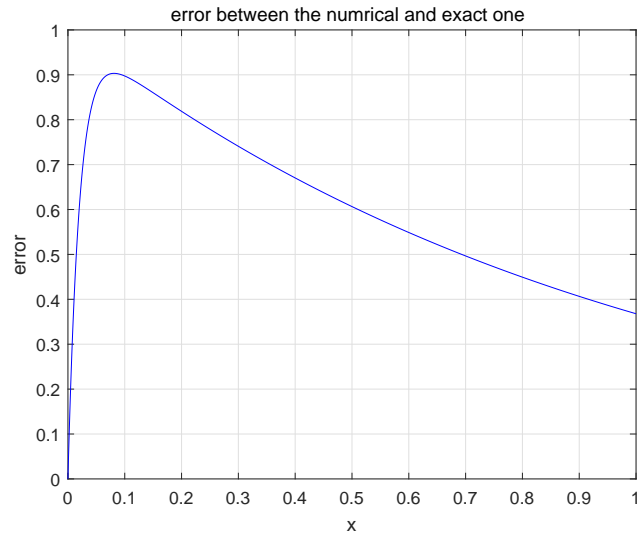


Figure 5: The error between numerical solution and the exact one as the function of  $x$

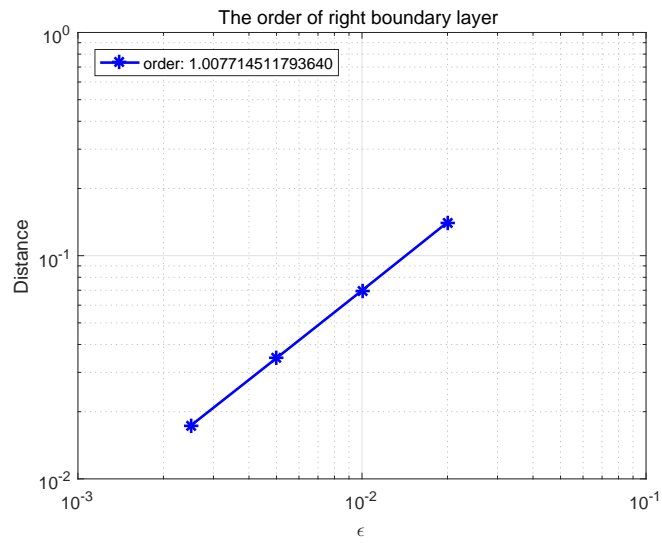


Figure 6: The order of right boundary layer as the function of  $\epsilon$

for Fig.9 and 10. For the left boundary layer, the order is seen as Fig.11. For the right boundary layer, the order is seen as Fig.12.

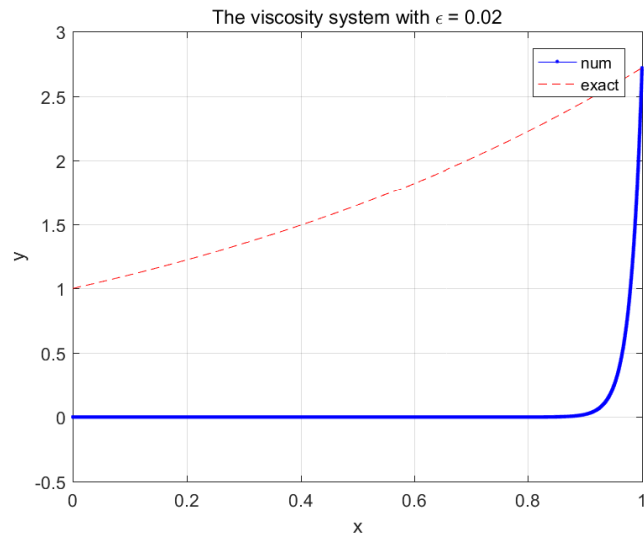


Figure 7: The numerical solution and the exact solution as the function of  $x$



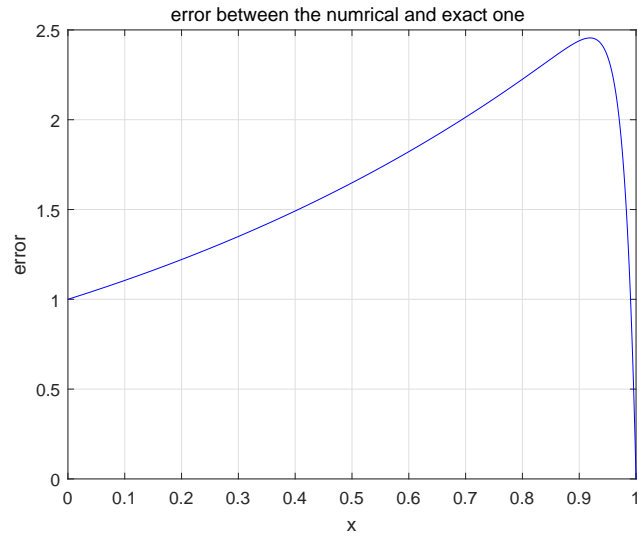


Figure 8: The error between numerical solution and the exact one as the fuction of  $x$

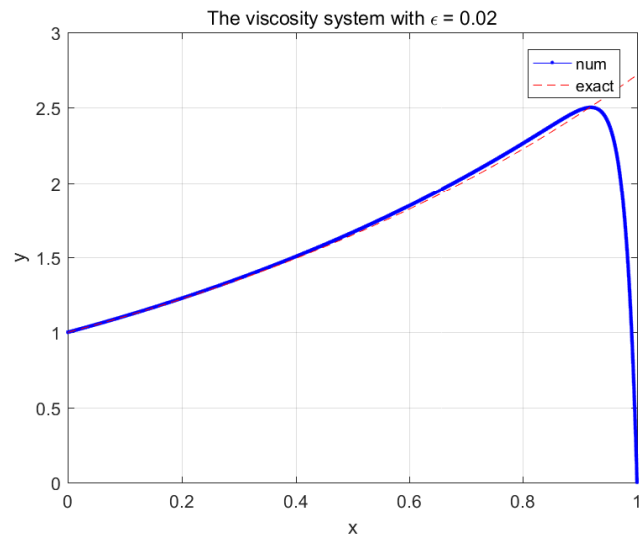


Figure 9: The numerical solution and the exact solution as the fuction of  $x$

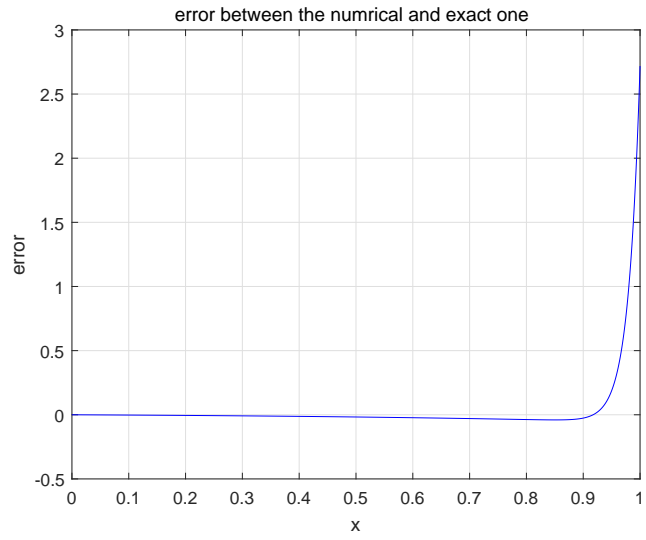


Figure 10: The error between numerical solution and the exact one as the function of  $x$

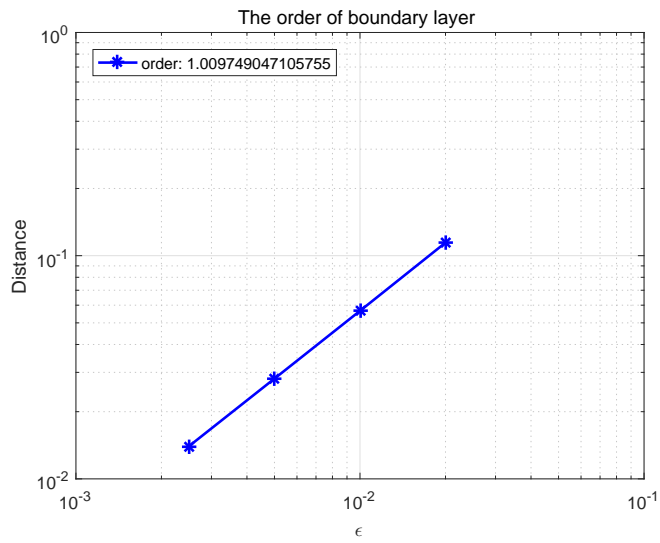


Figure 11: The order of left boundary layer as the function of  $\epsilon$

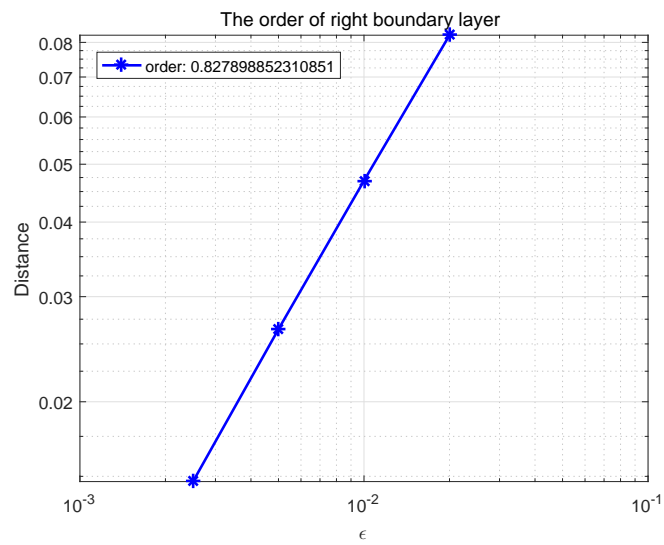


Figure 12: The order of right boundary layer as the function of  $\epsilon$