Hw ♯8 The linear response representation

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1 Spin Waves in Ferromagnets

1.1 Motion of spin waves in ferromagnets

Considering the propogations of spin waves in a uniaxial ferromagnet and assuming that

$$\overrightarrow{\mathcal{M}}(r,t) = \overrightarrow{\mathcal{M}}_0(r,t) + \overrightarrow{\boldsymbol{m}}(r,t) \tag{1}$$

$$\overrightarrow{H}^{(i)}(r,t) = \overrightarrow{H}_0^{(i)} + \overrightarrow{\boldsymbol{h}}(r,t), \tag{2}$$

where $\overrightarrow{\boldsymbol{m}}$ and $\overrightarrow{\boldsymbol{h}}$ are small pertubations from the equilibrium magnetization $\overrightarrow{\mathcal{M}}_0$ and internal field $\overrightarrow{H}_0^{(i)}$. The expression of effective field to eq. (1) and eq. (2) is given as

$$\begin{aligned}\mathcal{H} = \overrightarrow{\boldsymbol{h}} + \alpha_{ij}\frac{\partial^2 \overrightarrow{\boldsymbol{m}}}{\partial x_i \partial x_j} - \frac{1}{\overrightarrow{\mathcal{M}}_0^2}\{\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{H}_0^{(i)} \\ + \beta(\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{\boldsymbol{n}})^2\}\overrightarrow{\boldsymbol{m}} + \beta\overrightarrow{\boldsymbol{n}}(\overrightarrow{\boldsymbol{m}} \cdot \overrightarrow{\boldsymbol{n}}) - 4\overrightarrow{\mathcal{M}}_0 f''(\overrightarrow{\mathcal{M}}_0^2)(\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{\boldsymbol{m}}).\end{aligned} \tag{3}$$

The linearized equation of motion will be of the form

$$\frac{\partial \overrightarrow{\boldsymbol{m}}}{\partial t} = g[\overrightarrow{\mathcal{M}}_0, \overrightarrow{H}]. \tag{4}$$

Note that $\overrightarrow{\mathcal{M}}_0 \times \overrightarrow{\mathcal{M}}_0 = 0$ and then get

$$\begin{aligned}\frac{\partial \overrightarrow{\boldsymbol{m}}}{\partial t} = g\big[\overrightarrow{\mathcal{M}}_0, \overrightarrow{\boldsymbol{h}} + \alpha_{ij}\frac{\partial^2 \overrightarrow{\boldsymbol{m}}}{\partial x_i \partial x_j} - \frac{1}{\overrightarrow{\mathcal{M}}_0^2}\{\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{H}_0^{(i)} \\ + \beta(\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{\boldsymbol{n}})^2\}\overrightarrow{\boldsymbol{m}} + \beta\overrightarrow{\boldsymbol{n}}(\overrightarrow{\boldsymbol{m}} \cdot \overrightarrow{\boldsymbol{n}})\big].\end{aligned} \tag{5}$$

Using the Fourier repersentations

$$\overrightarrow{\boldsymbol{m}}(\overrightarrow{\boldsymbol{r}}, t) = \int \overrightarrow{\boldsymbol{m}}(\overrightarrow{\boldsymbol{k}}, \omega) e^{i(\overrightarrow{\boldsymbol{k}} \cdot \overrightarrow{\boldsymbol{r}} - \omega t)} d\overrightarrow{\boldsymbol{k}} d\omega, \tag{6}$$

$$\overrightarrow{\boldsymbol{h}}(\overrightarrow{\boldsymbol{r}}, t) = \int \overrightarrow{\boldsymbol{h}}(\overrightarrow{\boldsymbol{k}}, \omega) e^{i(\overrightarrow{\boldsymbol{k}} \cdot \overrightarrow{\boldsymbol{r}} - \omega t)} d\overrightarrow{\boldsymbol{k}} d\omega. \tag{7}$$

We have from eq. (5) that

$$-i\omega\overrightarrow{\boldsymbol{m}}(\overrightarrow{\boldsymbol{k}},\omega) = g\big[\overrightarrow{\mathcal{M}}_0, \overrightarrow{\boldsymbol{h}}(\overrightarrow{\boldsymbol{k}},\omega) - \Big\{\alpha_{ij}k_ik_j + \frac{\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{H}_0^{(i)}}{\overrightarrow{\mathcal{M}}_0^2} + \beta\frac{(\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{\boldsymbol{n}})^2}{\overrightarrow{\mathcal{M}}_0^2}\Big\}\overrightarrow{\boldsymbol{m}}(\overrightarrow{\boldsymbol{k}},\omega) \tag{8}$$

$$+\beta\overrightarrow{\boldsymbol{n}}(\overrightarrow{\boldsymbol{n}} \cdot \overrightarrow{\boldsymbol{m}}(\overrightarrow{\boldsymbol{k}},\omega))\big].$$

For simplicity, the z-axis lies along $\overrightarrow{\mathcal{M}}_0$ and the x-axis lies in the plane containing the anisotropy axis $\overrightarrow{\boldsymbol{n}}$ and $\overrightarrow{\mathcal{M}}_0$. Let ψ the angle between the vector $\overrightarrow{\boldsymbol{n}}$ and $\overrightarrow{\mathcal{M}}_0$. Thus, we let $\overrightarrow{\mathcal{M}}_0 = (0, 0, \mathcal{M}_0)^T$ and $\overrightarrow{\boldsymbol{n}} = (\sin\psi, 0, \cos\psi)^T$, where $(\cdot)^T$ denotes the transpose of the given vector. We then get

$$(\overrightarrow{\boldsymbol{m}} \cdot \overrightarrow{\boldsymbol{n}})\overrightarrow{\boldsymbol{n}} = A\overrightarrow{\boldsymbol{m}}, \quad \overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{\boldsymbol{n}} = \mathcal{M}_0\cos\psi, \tag{9}$$

where the matrix is given by

$$A = \begin{bmatrix} \sin^2\psi & 0 & \cos\psi\sin\psi \\ 0 & 0 & 0 \\ \cos\psi\sin\psi & 0 & \cos^2\psi \end{bmatrix}. \tag{10}$$

Note, however, that

$$\overrightarrow{\mathcal{M}}_0 \times \overrightarrow{\boldsymbol{h}} = B\overrightarrow{\boldsymbol{h}}, \tag{11}$$

where the matrix B is

$$B = \begin{bmatrix} 0 & -\mathcal{M}_0 & 0 \\ \mathcal{M}_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Thus, we have from eq. (8) that

$$\Big(g\Big\{\alpha_{ij}k_ik_j + \frac{\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{H}_0^{(i)}}{\overrightarrow{\mathcal{M}}_0^2} + \beta\cos^2\psi\Big\}B - i\omega I - g\beta BA\Big)\overrightarrow{\boldsymbol{m}} = gB\overrightarrow{\boldsymbol{h}}. \tag{13}$$

where I is the identity matrix. We then from eq. (13) that

$$\overrightarrow{\boldsymbol{m}} = \chi\overrightarrow{\boldsymbol{h}}, \tag{14}$$

with the susceptibility matrix

$$\chi = \Big[gpB - i\omega I - g\beta BA\Big]^{-1} gB. \tag{15}$$

where $p = \alpha_{ij}k_ik_j + \frac{\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{H}_0^{(i)}}{\overrightarrow{\mathcal{M}}_0^2} + \beta\cos^2\psi$. If we denote $C = gpB - i\omega I - g\beta BA$ and then get

$$C^{-1} = \frac{C^*}{\det(C)}, \tag{16}$$

where C^* is the adjoint matrix of C and $\det(C)$ denotes the determinant of the matrix C. Here,

$$C = \begin{bmatrix} -i\omega & -g\mathcal{M}_0 p & 0 \\ g\mathcal{M}_0 p - g\beta\mathcal{M}_0 \sin^2\psi & -i\omega & -g\beta\mathcal{M}_0 \cos\psi \sin\psi \\ 0 & 0 & -i\omega \end{bmatrix}, \tag{17}$$

and

$$C^* = \begin{bmatrix} -\omega^2 & -i\omega g\mathcal{M}_0 p & g\mathcal{M}_0 p(g\beta\mathcal{M}_0 \cos\psi \sin\psi) \\ i\omega(g\mathcal{M}_0 p - g\beta\mathcal{M}_0 \sin^2\psi) & -\omega^2 & -i\omega g\beta\mathcal{M}_0 \cos\psi \sin\psi \\ 0 & 0 & -\omega^2 + g\mathcal{M}_0 p(g\mathcal{M}_0 p - g\beta\mathcal{M}_0 \sin^2\psi) \end{bmatrix}. \tag{18}$$

Thus, we have from eq. (17) that

$$\det(C) = -i\omega[-\omega^2 + g\mathcal{M}_0 p(g\mathcal{M}_0 p - g\beta\mathcal{M}_0 \sin^2\psi)]. \tag{19}$$

It follows from eq. (15) and eq. (16) yields that

$$\begin{aligned} \chi &= \frac{g}{\det(C)} C^* B \\ &=: \frac{g}{\det(C)} D, \end{aligned} \tag{20}$$

where

$$D = \begin{bmatrix} -i\omega g\mathcal{M}_0^2 p & \mathcal{M}_0\omega^2 & 0 \\ -\mathcal{M}_0\omega^2 & -\mathcal{M}_0 i\omega(g\mathcal{M}_0 p - g\mathcal{M}_0 \sin^2\psi) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

If we introduce the notations by

$$\begin{aligned} \Omega_1 &= g\mathcal{M}_0 p = g\mathcal{M}_0\Big(\alpha_{ij}k_ik_j + \frac{\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{H}_0^{(i)}}{\overrightarrow{\mathcal{M}}_0^2} + \beta\cos^2\psi\Big) \\ \Omega_2 &= g\mathcal{M}_0 p - g\beta\mathcal{M}_0 \sin^2\psi = g\mathcal{M}_0\Big(\alpha_{ij}k_ik_j + \frac{\overrightarrow{\mathcal{M}}_0 \cdot \overrightarrow{H}_0^{(i)}}{\overrightarrow{\mathcal{M}}_0^2} + \beta\cos 2\psi\Big). \end{aligned} \tag{22}$$

We then derive that

$$\chi = \begin{bmatrix} \chi_{xx} & \chi_{xy} & 0 \\ \chi_{yx} & \chi_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned} \chi_{xx} &= \frac{g\mathcal{M}_0\Omega_1}{\Omega_1\Omega_2 - \omega^2} \\ \chi_{yy} &= \frac{g\mathcal{M}_0\Omega_2}{\Omega_1\Omega_2 - \omega^2} \\ \chi_{xy} &= -\chi_{yx} = \frac{i\omega g\mathcal{M}_0}{\Omega_1\Omega_2 - \omega^2}. \end{aligned} \tag{24}$$

1.2 Damping of spin waves

The equation of motion for $\overrightarrow{\mathcal{M}}$ will be the form

$$\frac{\partial\overrightarrow{\mathcal{M}}}{\partial t} = g[\overrightarrow{\mathcal{M}}, \mathcal{H}] + \frac{1}{\tau_2}\mathcal{H} - \frac{1}{\tau_1}[\overrightarrow{\boldsymbol{n}}, [\overrightarrow{\boldsymbol{n}}, \mathcal{H}]], \tag{25}$$

where $\overrightarrow{\boldsymbol{n}} = \frac{\mathcal{M}_0}{|\mathcal{M}_0|}$ and τ_1, τ_2 are constants which have the dimensions of time, $\tau_2 > 0$, $\frac{1}{\tau_1} + \frac{1}{\tau_2} > 0$. Here, we consider the uniaxial ferromagnetic and assume that the field $\overrightarrow{H}_0^{(i)}$ is parallel to the easy axis. The effective magnetic field is

$$\mathcal{H} = \overrightarrow{\boldsymbol{h}} - \left(\beta + \frac{\overrightarrow{H}_0^{(i)}}{\mathcal{M}_0}\right)\overrightarrow{\boldsymbol{m}} + \alpha_{ij}\frac{\partial^2\overrightarrow{\boldsymbol{m}}}{\partial x_i \partial x_j} + [\beta - 4\mathcal{M}_0^2 f''(\mathcal{M}_0^2)](\overrightarrow{\boldsymbol{m}}\cdot\overrightarrow{\boldsymbol{n}})\overrightarrow{\boldsymbol{n}}. \tag{26}$$

Note, however, that the quantity $f''(\mathcal{M}_0^2)$ in this expression for $\mathcal{H}$ can be readily be related to the static susceptibility of the ferromagnetic, that is

$$\chi_{zz}^0 = \frac{\partial\mathcal{M}_0}{\partial H_0^{(i)}}. \tag{27}$$

We have seen that in the state of equilibrium

$$H_0^{(i)} + \beta\mathcal{M}_0 - 2\mathcal{M}_0 f'(\mathcal{M}_0^2). \tag{28}$$

We have from eq. (28) that

$$\beta = 2f'(\mathcal{M}_0^2) - \frac{H_0^{(i)}}{\mathcal{M}_0}. \tag{29}$$

We take derivative for $\mathcal{M}_0$ regarding H_0 and get

$$4\mathcal{M}_0^2 f''(\mathcal{M}_0^2) = \frac{1}{\chi_{zz}^0} - \frac{H_0^{(i)}}{\mathcal{M}_0} \tag{30}$$

The linearized equation of motion is

$$\frac{\partial\overrightarrow{\boldsymbol{m}}}{\partial t} = g[\overrightarrow{\mathcal{M}}_0, \mathcal{H}] + \frac{1}{\tau_2}\mathcal{H} - \frac{1}{\tau_1}[\overrightarrow{\boldsymbol{n}}, [\overrightarrow{\boldsymbol{n}}, \mathcal{H}]]. \tag{31}$$

Using the Fourier transform eq. (6), then we get

$$\begin{aligned} -i\omega\overrightarrow{\boldsymbol{m}}(\overrightarrow{\boldsymbol{k}}, \omega) &= g\Big[\overrightarrow{\mathcal{M}}_0, \overrightarrow{\boldsymbol{h}} - \big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0}\big)\overrightarrow{\boldsymbol{m}} - \alpha_{ij}k_ik_j\overrightarrow{\boldsymbol{m}} + \beta(\overrightarrow{\boldsymbol{m}}\cdot\overrightarrow{\boldsymbol{n}})\overrightarrow{\boldsymbol{n}}\Big] \qquad (32) \\ &+ \frac{1}{\tau_2}\{\overrightarrow{\boldsymbol{h}} - \big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0}\big)\overrightarrow{\boldsymbol{m}} - \alpha_{ij}k_ik_j\overrightarrow{\boldsymbol{m}} + (\beta - 4\mathcal{M}_0^2 f''(\mathcal{M}_0^2))(\overrightarrow{\boldsymbol{m}}\cdot\overrightarrow{\boldsymbol{n}})\overrightarrow{\boldsymbol{n}}\} \\ &- \frac{1}{\tau_1}\big[\overrightarrow{\boldsymbol{n}}, [\overrightarrow{\boldsymbol{n}}, \overrightarrow{\boldsymbol{h}} - \big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0}\big)\overrightarrow{\boldsymbol{m}} - \alpha_{ij}k_ik_j\overrightarrow{\boldsymbol{m}} + (\beta - 4\mathcal{M}_0^2 f''(\mathcal{M}_0^2))(\overrightarrow{\boldsymbol{m}}\cdot\overrightarrow{\boldsymbol{n}})\overrightarrow{\boldsymbol{n}}]\big]. \end{aligned}$$

Rewrite the above representation in matrix form as

$$\Big[g(\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}+\alpha_{ij}k_ik_j)B-g\beta BA-i\omega I+\frac{1}{\tau_2}\big(\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}+\alpha_{ij}k_ik_j\big)I \tag{33}$$

$$-\frac{1}{\tau_2}\big(\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}-\frac{1}{\chi_{zz}^0}\big)A+\frac{1}{\tau_1}\big[\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}-\alpha_{ij}k_ik_j\big]F\Big]\overrightarrow{\boldsymbol{m}}$$

$$=(gB+\frac{1}{\tau_2}I-\frac{1}{\tau_1}F)\overrightarrow{\boldsymbol{h}}. \tag{34}$$

Here, the matrix I is the identity matrix, and We then derive that

$$A=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},B=\begin{bmatrix}0&-\mathcal{M}_0&0\\\mathcal{M}_0&0&0\\0&0&1\end{bmatrix},F=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&0\end{bmatrix}. \tag{35}$$

Thus, BA is the full-zero matrix. We introduce the notation

$$G=g(\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}+\alpha_{ij}k_ik_j)B-g\beta BA-i\omega I+\frac{1}{\tau_2}\big(\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}+\alpha_{ij}k_ik_j\big)I \tag{36}$$

$$-\frac{1}{\tau_2}\big(\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}-\frac{1}{\chi_{zz}^0}\big)A+\frac{1}{\tau_1}\big[\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}-\alpha_{ij}k_ik_j\big]F,$$

and

$$J=gB+\frac{1}{\tau_2}I-\frac{1}{\tau_1}F. \tag{37}$$

We then obtain

$$\overrightarrow{\boldsymbol{m}}=\chi\overrightarrow{\boldsymbol{h}}, \tag{38}$$

where

$$\chi=G^{-1}J=\frac{G^*}{\det(G)}J. \tag{39}$$

After calculation, we get

$$G=\begin{bmatrix}G_{11}&G_{12}&0\\G_{21}&G_{22}&0\\0&0&G_{33}\end{bmatrix},J=\begin{bmatrix}J_{11}&J_{12}&0\\J_{21}&J_{22}&0\\0&0&J_{33}\end{bmatrix}, \tag{40}$$

where

$$G_{11} = -i\omega + \frac{1}{\tau_2}\big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0} + \alpha_{ij}k_ik_j\big) - \frac{1}{\tau_1}\big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0} - \alpha_{ij}k_ik_j\big)$$
$$G_{12} = -g\mathcal{M}_0\big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0} + \alpha_{ij}k_ik_j\big), \quad G_{13} = 0$$
$$G_{21} = g\mathcal{M}_0\big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0} + \alpha_{ij}k_ik_j\big)$$
$$G_{22} = -i\omega + \frac{1}{\tau_2}\big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0} + \alpha_{ij}k_ik_j\big) + \frac{1}{\tau_1}\big(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0} - \alpha_{ij}k_ik_j\big)$$
$$G_{23} = 0 \quad G_{31} = 0 \quad G_{32} = 0$$
$$G_{33} = -i\omega + \frac{1}{\tau_2}\big(\alpha_{ij}k_ik_j + \frac{1}{\chi_{zz}^0}\big),$$

and

$$J_{11} = \frac{1}{\tau_1} + \frac{1}{\tau_2} \quad J_{12} = -g\mathcal{M}_0 \quad J_{13} = 0$$
$$J_{21} = g\mathcal{M}_0 \quad J_{22} = \frac{1}{\tau_2} - \frac{1}{\tau_1} \quad J_{23} = 0$$
$$J_{31} = 0 \quad J_{32} = 0 \quad J_{33} = \frac{1}{\tau_2},$$

and

$$G^* = \begin{bmatrix} G_{22}G_{33} & -G_{12}G_{33} & 0 \\ -G_{21}G_{33} & G_{11}G_{33} & 0 \\ 0 & 0 & G_{11}G_{22} - G_{12}G_{21}. \end{bmatrix}. \tag{41}$$

We then derive the following expression for the high-frequency susceptibility tensor:

$$\chi = \begin{bmatrix} \chi_{xx} & \chi_{xy} & 0 \\ \chi_{yx} & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{bmatrix}, \tag{42}$$

in which

$$\chi_{xx} = \Big[G_{22}G_{33}\big(\frac{1}{\tau_1} + \frac{1}{\tau_2}\big) - g\mathcal{M}_0G_{12}G_{33}\Big] / \det(G)$$
$$\chi_{xy} = \Big[- G_{22}G_{33}g\mathcal{M}_0 - G_{12}G_{33}\big(\frac{1}{\tau_2} - \frac{1}{\tau_1}\big)\Big] / \det(G)$$
$$\chi_{yx} = \Big[- G_{21}G_{33}\big(\frac{1}{\tau_1} + \frac{1}{\tau_2}\big) + G_{11}G_{33}g\mathcal{M}_0\Big] / \det(G)$$
$$\chi_{yy} = \Big[G_{21}G_{33}g\mathcal{M}_0 + G_{11}G_{33}\big(\frac{1}{\tau_2} - \frac{1}{\tau_1}\big)\Big] / \det(G)$$
$$\chi_{zz} = \Big[(G_{11}G_{22} - G_{12}G_{21})\frac{1}{\tau_2}\Big] / \det(G),$$

where $\det(G) = G_{33}[G_{11}G_{22} - G_{12}G_{21}]$. We finally get the expression of the susceptibility tensor in which the entries as

$$\chi_{xx} = \chi_{yy} = \frac{g\mathcal{M}_0\Omega - (i\omega/\tau) + (\Omega/g\mathcal{M}_0\tau^2)}{\Omega^2 - (\omega + i\Omega/g\mathcal{M}_0\tau)^2}$$
$$\chi_{zz} = \frac{\chi_{zz}^0}{1 + \chi_{zz}^0(\alpha_{ij}k_ik_j - i\omega\tau_2)},$$
$$\chi_{xy} = -\chi_{yx} = \frac{i\omega g\mathcal{M}_0}{\Omega^2 - (\omega + i\Omega/g\mathcal{M}_0\tau)^2},$$

where

$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2}, \quad \Omega = g\mathcal{M}_0\big(\alpha_{ij}k_ik_j + \frac{H_0^{(i)}}{\mathcal{M}_0} + \beta\big).$$