Project 1
Due on Friday, March 31th, 2017

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Consider the following ordinary differential equation for u:

$$-u''(x) + \pi^2 \cos^2(\pi x) u(x) = f(x), \quad x \in [0,1], \tag{1}$$

with boundary conditions:

$$\begin{aligned} u(0) &= 0 \\ u(1) &= 0, \end{aligned} \tag{2}$$

i) Consider $f(x) = \pi^2 \sin(\pi x) \cosh(\sin(\pi x))$, and check that the function $u(x) = \sinh(\sin(\pi x))$ is the solution to the boundary value problem (BVP) (1)+(2).

ii) We want to solve this BVP numerically. We begin by discretizing the interval $[0,1]$. For this, consider the gridpoints:

$$x_i = ih, \quad i = 0, 1, \ldots, n+1, \quad h = \frac{1}{n+1}. \tag{3}$$

Note that $h_i = x_{i+1} - x_i = h$ for all i. Now we approximate the second derivative. Show that if g has four continuous derivatives, then

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i'' + O(h^2) \tag{4}$$

where $g_i = g(x_i)$.

iii) Consider now the linear system of equations

$$-\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + \pi^2 \cos^2(\pi x_i)\, g_i = f(x_i), \quad i = 1, 2, \ldots, n. \tag{5}$$

Show that this can be rewritten in matrix form as

$$\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f},$$

where $\boldsymbol{g} = (g_1, \ldots, g_n)^T$, $\boldsymbol{f} = (f_1, \ldots, f_n)^T$, and the matrix $\boldsymbol{A}$ is tridiagonal, with entries:

$$a_{i,j} = \begin{cases} -\frac{1}{h^2} & |i-j| = 1; \\ \frac{2}{h^2} + \pi^2 \cos^2(\pi x_i) & i = j; \\ 0 & Otherwise. \end{cases} \tag{6}$$

iv) Solve the system of equations(5). Use the following values: $n = 10$, 20, 40, 80, 160, 320. For each $h = 1/(n+1)$, compute the error

$$e(h) = \sup_{1\leq i\leq n} |g_i - u(x_i)| \tag{7}$$

and do a log-log plot of $e(h)$, that is, plot $\log(e(h))$ as a function of $\log(h)$. Show, using this plot, that $e(h) = O(h^2)$, consistent with (4).

Solution.

i) Consider $f(x) = \pi^2 \sin(\pi x)\cosh(\sin(\pi x))$, if $u(x) = \sinh(\sin(\pi x))$, first of all, $u(x)$ satisfies boundary conditions (2). And we note that

$$\begin{aligned} u'(x) &= \pi\cos(\pi x)\cdot\cosh(\sin(\pi x)), \\ u''(x) &= \pi^2 \sinh(\sin(\pi x)\cdot\cos^2(\pi x) - \pi^2\sin(\pi x)\cdot\cosh(\sin(\pi x)). \end{aligned}$$

Then,

$$\begin{aligned} -u''(x) + \pi^2\cos^2(\pi x)u(x) &= \pi^2\sin(\pi x)\cosh(\sin(\pi x)) \\ &\quad - \pi^2\sinh(\sin(\pi x))\cos^2(\pi x) \\ &\quad + \pi^2\cos^2(\pi x)\sinh(\sin(\pi x)) \\ &= \pi^2\sin(\pi x)\cosh(\sin(\pi x)) \\ &= f(x). \end{aligned}$$

On the one hand, $u(x)$ satisfies ordinary differential equation (ODE)(1), i.e., the function $u(x)$ is the analytical solution to the boundary value problem (BVP)(1)+(2). On the other hand, we can take advantage of BVP solver of Matrix Lab and call deval function to check $u(x)$ is the solution of BVP.

ii) Suppose $g(x)$ is the solution of BVP(1)+(2), we discrete the interval $[0, 1]$. For this, consider the gridpoints:

$$x_i = ih, \quad i = 0, 1, \ldots, n+1, \quad h = \frac{1}{n+1}.$$

Due to $g(x)$ has four continuous derivatives, then we will give the Taylor expansion of $g(x_{i+1})$ and $g(x_{i-1})$ with respect to x_i.

$$\begin{aligned} g(x_{i+1}) &= g(x_i+h) \\ &= g(x_i) + hg'(x_i) + \frac{1}{2}h^2 g''(x_i) + \frac{1}{3!}h^3 g^{(3)}(x_i) + \frac{1}{4!}h^4 g^{(4)}(\xi_i'), \end{aligned} \tag{8}$$

where $x_i < \xi_i' < x_{i+1}$.

$$\begin{aligned} g(x_{i-1}) &= g(x_i-h) \\ &= g(x_i) - hg'(x_i) + \frac{1}{2}h^2 g''(x_i) - \frac{1}{3!}h^3 g^{(3)}(x_i) + \frac{1}{4!}h^4 g^{(4)}(\xi_i''), \end{aligned} \tag{9}$$

where $x_{i-1} < \xi_i'' < x_i$.
From (8) and (9), we deduce that

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i'' + \frac{1}{12}h^2 g^{(4)}(\xi_i) \quad x_{i-1} < \xi_i < x_i, \tag{10}$$

i.e.,

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i'' + O(h^2),$$

where $g_i = g(x_i)$.

iii) Consider now the linear system of equations

$$-\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + \pi^2 \cos^2(\pi x_i)\, g_i = f_i \quad i = 1, 2, \ldots, n, \tag{11}$$

where $f_i = f(x_i)$. We can deserve that

$$-\frac{1}{h^2}g_{i-1} + (\frac{2}{h^2} + \pi^2 \cos^2(\pi x_i))g_i - \frac{1}{h^2}g_{i+1} = f_i \quad i = 1, 2, \ldots, n, \tag{12}$$

we can plug $g_0 = 0$, $g_{n+1} = 0$ into systems (12), one has $n \times n$ order linear equations

$$\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f},$$

in other words, we can deduce the process in detail, i.e.,

$$\begin{cases} -\frac{1}{h^2}g_0 + (\frac{2}{h^2} + \pi^2 cos^2(\pi x_1))g_1 - \frac{1}{h^2}g_2 & = f_1, \\ -\frac{1}{h^2}g_1 + (\frac{2}{h^2} + \pi^2 cos^2(\pi x_2))g_2 - \frac{1}{h^2}g_3 & = f_2, \\ -\frac{1}{h^2}g_2 + (\frac{2}{h^2} + \pi^2 cos^2(\pi x_3))g_3 - \frac{1}{h^2}g_4 & = f_3, \\ \quad \cdots \qquad \cdots \qquad \cdots & \\ -\frac{1}{h^2}g_{n-1} + (\frac{2}{h^2} + \pi^2 cos^2(\pi x_n))g_n - \frac{1}{h^2}g_{n+1} & = f_n, \end{cases}$$

note that $g_0 = 0$, $g_{n+1} = 0$, hence

$$\boldsymbol{A} = \begin{pmatrix} \frac{2}{h^2} + \pi^2\cos^2(\pi x_1) & -\frac{1}{h^2} & 0 & \cdots & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2} + \pi^2\cos^2(\pi x_2) & -\frac{1}{h^2} & \cdots & \vdots \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2} + \pi^2\cos^2(\pi x_3) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{1}{h^2} \\ 0 & 0 & \cdots & -\frac{1}{h^2} & \frac{2}{h^2} + \pi^2\cos^2(\pi x_{n-1}) \end{pmatrix}$$

then, (12) can be rewritten in matrix form as

$$\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f},$$

where $\boldsymbol{g} = (g_1, \ldots, g_n)^T$, $\boldsymbol{f} = (f_1, \ldots, f_n)^T$, and the matrix $\boldsymbol{A}$ is tridiagonal, with entries:

$$a_{i,j} = \begin{cases} -\frac{1}{h^2} & |i-j| = 1; \\ \frac{2}{h^2} + \pi^2\cos^2(\pi x_i) & i = j; \\ 0 & \textit{Otherwise}. \end{cases} \tag{13}$$

iv) Solve the system of equations $\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f}$, that is, how to solve the tridiagonal equations. To begin with, according to the following algorithm, we can save the elements of $\boldsymbol{A}$ and $\boldsymbol{f}$, and then solve the formula equation $\boldsymbol{g} = \boldsymbol{A} \setminus \boldsymbol{f}$ by Matrix Lab. Consider the algorithms as the following that:

step1. Input $N = n + 1$, output the approximate value g_i of solution $g(x)$ in $x_i, \quad i = 1, \ldots, n$.

$$\begin{aligned} h &= \frac{1}{N}; \\ x &= h; \\ d_1 &= \frac{2}{h^2} + \pi^2\cos^2(\pi x); \\ c_1 &= -\frac{1}{h^2}; \\ b_1 &= f(x); \end{aligned}$$

step2. For $i = 2, \ldots, n$

$$\begin{aligned} x &= ih; \\ d_i &= \frac{2}{h^2} + \pi^2 \cos^2(\pi x); \\ c_i &= -\frac{1}{h^2}; \\ a_i &= -\frac{1}{h^2}; \\ b_i &= f(x); \end{aligned}$$

step3.

$$\begin{aligned} x &= 1 - h; \\ d_n &= \frac{2}{h^2} + \pi^2 \cos^2(\pi x); \\ a_n &= -\frac{1}{h^2}; \\ b_n &= f(x). \end{aligned}$$

Where $\mathbf{d}$=$(d_1, d_2, \ldots, d_n)$ represents diagonal elements, $\mathbf{a}$ represents lower-semi-diagonal elements, $\mathbf{c}$ represents upper-semi-diagonal elements, clearly, $\mathbf{a}$=$\mathbf{c}$, $\mathbf{b}$ represents right term. Hence, we save datas of $\boldsymbol{A}$ and $\boldsymbol{f}$.
On the one hand, $\boldsymbol{g} = \boldsymbol{A} \setminus \boldsymbol{f}$, on the other hand, we can solve $\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f}$ by tri-diagonal algorithm or speedup method, ie., we define function as $f = tridiag(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, n)$ and construct circulation as for $i = 2 : n$, i.e.,

step4.

$$\begin{aligned} r &= \frac{a(i)}{b(i-1)}; \\ b(i) &= b(i) - r * c(i-1); \\ d(i) &= d(i) - r * d(i-1); \\ d(n) &= \frac{d(n)}{b(n)}. \end{aligned}$$

Consider another circulation as for $i = n-1 : -1 : 1$,

$$d(i) = \frac{d(i) - c(i) * d(i+1)}{b(i)},$$

may as well suppose the solution is $g_1, g_2, \ldots, g_n$.

step5. As for $i = 1, \ldots, n$, $x = ih$, output (x, y_i). The procedure is complete.

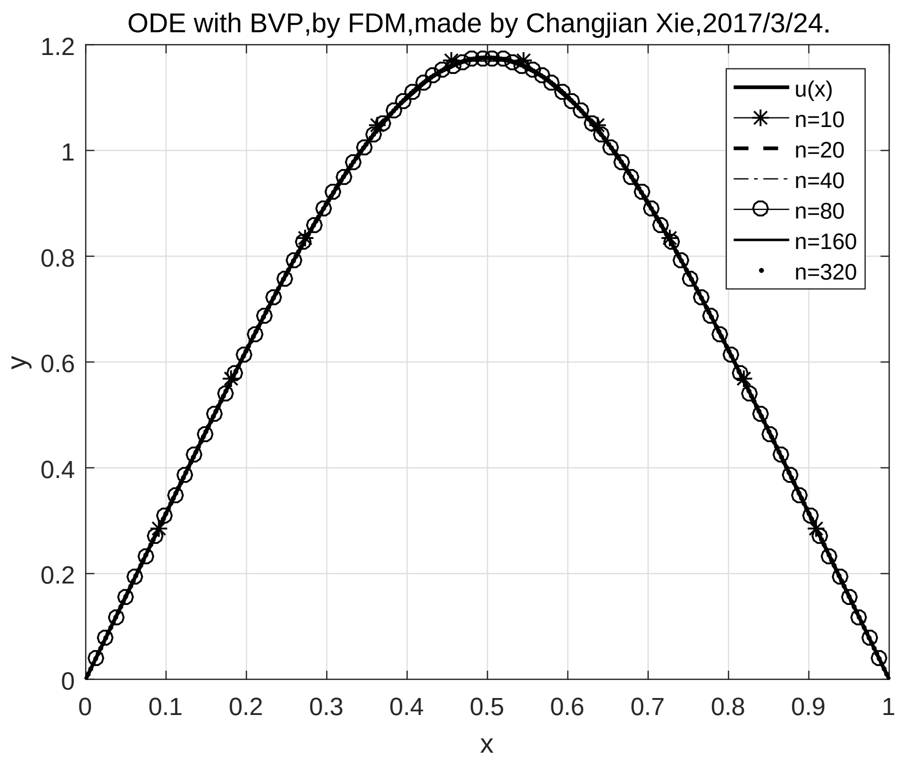

Figure 1: Consider to call function with respect to $n = 10$, $n = 20$,$n = 40$,$n = 80$,$n = 160$,$n = 320$, we deserve numerical solution of BVP, compare with analytical solution, note that the numerical solution is an effective approximation of analytical solution.

In order to see clearly, we give the analytical solution and each graph concerning n.

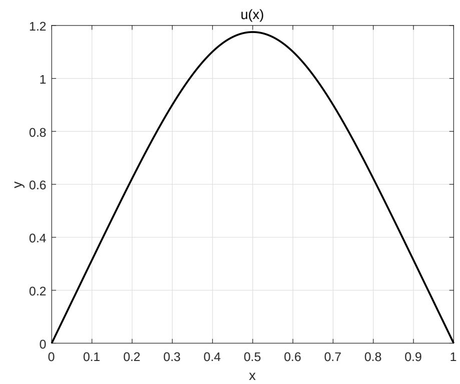

Figure 2: Analytical solution.

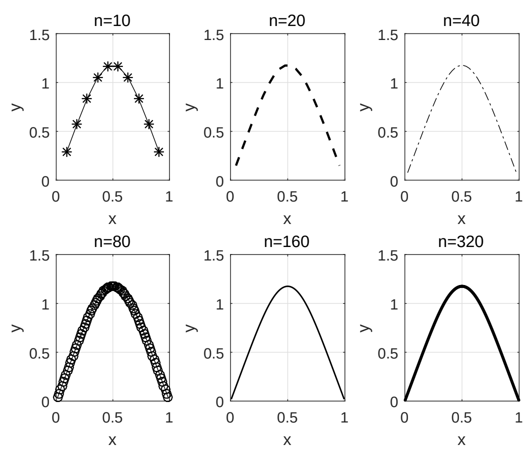

Figure 3: Numerical solution.

When $n = 10$, $n = 20$, $n = 40$, $n = 80$, $n = 160$, $n = 320$, we compute respectively the error equipped with L^∞ norm, $e(h_1) = 0.0093614382288$, $e(h_2) = 0.00260910867557$, $e(h_3) = 0.00068740375$, $e(h_4) = 0.00017631744$, $e(h_5) = 0.00004464145$, $e(h_6) = 0.0000112308$.

Consider $n = 10, 20, 40, 80, 160, 320$, we deserve $\log e(h)$ is the function of $\log h$. From the following figure, the two straight lines have the same slope 2. In other words, they parallel with each other.

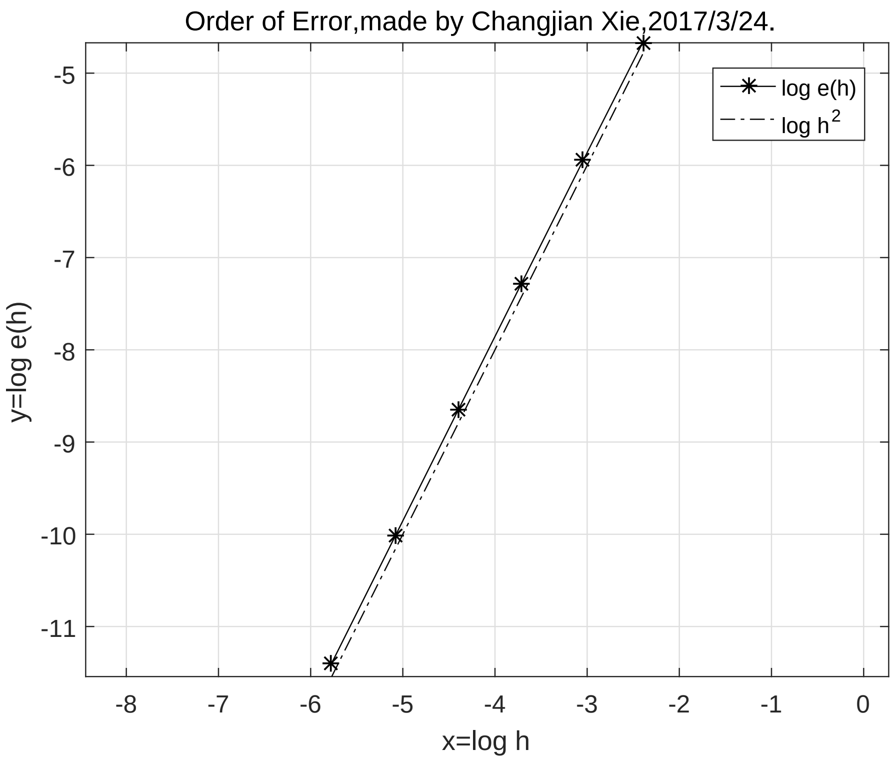

Figure 4: The order of error equipped with L^∞ norm.

Note that we can draw the $\log h^2$ with respect to $\log h$. comparing two graphs, we have the conclusion that the difference between $\log e(h)$ and $\log h^2$ controlled by constant, so $e(h) = O(h^2)$ holds uniformly concerning $\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f}$.

Notation. Refer to the least square method, the method of least squares is a standard approach in regression analysis. We can transfer the original question into a easier case and take advantage of linear least squares. The model as following that: $\{h_i, e(h_i)\}_{i=1}^n$, $e(h) = ch^\alpha$, $e(h_i) = ch_i^\alpha$, $i = 1, \cdots, n$, where $n \gg 1$. In the sequel, n is different from the above n which discrete the interval $[0, 1]$, where n represents $\{h_i, e(h_i)\}_{i=1}^n$, $i = 1, \cdots, n$, $e(h_i) \approx ch_i^2$, the motivation stems from

$$\min_{c,\alpha} \frac{1}{n} \sum_{i=1}^{n} (e(h_i) - ch_i^\alpha)^2,$$

we take log which have same bottom on the two-side. Then, the problem transfers into

$$\log e(h) = \log c + \alpha \log h,$$

for $\{x_i, y_i\}_{i=1}^n$, we need to get

$$\min_{\alpha,\beta} \frac{1}{n} \sum_{i=1}^{n} (y_i - (\alpha x_i + \beta))^2,$$

we define

$$f(\alpha, \beta) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (\alpha x_i + \beta))^2,$$

then, we take partial derivative for the above formula as following that

$$\begin{cases} \frac{\partial f}{\partial \alpha} = 0, \\ \\ \frac{\partial f}{\partial \beta} = 0, \end{cases}$$

i.e.,

$$\begin{cases} \frac{2}{n} \sum_{i=1}^{n} (y_i - \alpha x_i - \beta)(-x_i) & = 0, \\ \\ \frac{2}{n} \sum_{i=1}^{n} (y_i - \alpha x_i - \beta)(-1) & = 0, \end{cases}$$

i.e.,

$$\begin{cases} (\sum_{i=1}^n x_i^2)\alpha + \sum_{i=1}^n x_i\beta & = \sum_{i=1}^n x_i y_i, \\ \\ (\sum_{i=1}^n x_i)\alpha + n\beta & = \sum_{i=1}^n y_i, \end{cases}$$

it follows that

$$\alpha = \frac{\sum_{i=1}^n y_i \sum_{i=1}^n x_i - n \sum_{i=1}^n x_i y_i}{(\sum_{i=1}^n x_i)^2 - n \sum_{i=1}^n x_i^2},$$

$$\beta = \frac{\sum_{i=1}^n x_i \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i}{(\sum_{i=1}^n x_i)^2 - n \sum_{i=1}^n x_i^2},$$

we can compute the value of α and β by the method of linear least squares, i.e.,

α	β
1.9947585	0.1198744

Conclusion: The above method called finite difference method (FDM), the error is defined usually as the difference between the approximation and the exact analytical solution. To use a finite difference method to approximate the solution to a problem, one must first discretize the problem's domain. The remainder term of a Taylor polynomial is convenient for analyzing the local truncation error. Refer to application of FDM, we can consider the normalized heat equation in one dimension, with homogeneous Dirichlet boundary conditions, etc.