Hw♯7–The Optimized Problem of Homogenization Coefficients

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Let $\Omega = [d_1, d_2]$ be an inteval in R and consider the problem

$$\begin{cases} -\frac{d}{dx}\left(a^\epsilon \frac{du^\epsilon}{dx}\right) = f \text{ in } [d_1, d_2] \\ u^\epsilon(d_1) = u^\epsilon(d_2) = 0. \end{cases} \tag{1}$$

We assume here that a is a positive function in $L^\infty(0, \ell_1)$ such that

$$\begin{cases} a \text{ is } \ell_1\text{-periodic,} \\ 0 < \alpha \leq a(x) \leq \beta < +\infty, \end{cases} \tag{2}$$

where α and β are constants. The a^ϵ from (1) is the function defined by

$$a^\epsilon(x) = a\left(\frac{x}{\epsilon}\right). \tag{3}$$

We have the following satement:

Theorem 1 *Let $f \in L^2(d_1, d_2)$ and a^ϵ be defined by (2) and (3). Let $u^\epsilon \in H_0^1(d_1, d_2)$ be the solution of problem (1). Then,*

$$u^\epsilon \rightharpoonup u^0 \textit{ weakly in } H_0^1(d_1, d_2),$$

where u^0 is the unique solution in $H_0^1(d_1, d_2)$ of the problem

$$\begin{cases} -\frac{d}{dx}\left(\frac{1}{M_{(0,\ell_1)}\left(\frac{1}{a}\right)}\frac{du^0}{dx}\right) = f \textit{ in } [d_1, d_2], \\ u^0(d_1) = u^0(d_2) = 0. \end{cases} \tag{4}$$

Here, we need to solve the following optimized cell problem for simplicity,

$$\max_{0\leq z\leq \ell} \frac{1}{M_{(0,\ell)}\left(\frac{1}{a}\right)}, \tag{5}$$

where ℓ is the cell, the picewise homogenization coefficients

$$a(x) = \begin{cases} \alpha, \ , 0 \leq x \leq z \\ \beta, \ z \leq x \leq \ell, \end{cases} \tag{6}$$

the big scale is L, the small parameter $\epsilon = \ell/L$.

That is to say, we need to minimize $M_{(0,\ell)}\left(\frac{1}{a}\right)$, we note that

$$M_{(0.\ell)}\left(\frac{1}{a(x)}\right) = \frac{1}{\ell}\int_0^\ell \frac{1}{a(x)}\,dx. \tag{7}$$

Then, one has

$$\begin{aligned}\min_{0\le z\le \ell} M_{(0,\ell)}\left(\frac{1}{a(x)}\right) &= \min_{0\le z\le \ell}\left(\frac{z}{\alpha} + (\ell - z)\frac{1}{\beta}\right)\frac{1}{\ell} \\ &= \min_{0\le z\le \ell}\frac{z}{\ell}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) + \frac{1}{\beta}.\end{aligned}$$

Note that $\ell > 0$, then, we have three cases as follows.

- If $\frac{1}{\alpha} - \frac{1}{\beta} > 0$, we will obtain that the minimization value of the mean is $\frac{1}{\beta}$ when $z = 0$.
- If $\frac{1}{\alpha} - \frac{1}{\beta} < 0$, we will obtain that the minimization value of the mean is $\frac{1}{\alpha}$ when $z = \ell$.
- If $\frac{1}{\alpha} - \frac{1}{\beta} = 0$, we will obtain that the minimization value of the mean is $\frac{1}{\beta}$ when z is arbitrary.

That is to say, when $\frac{1}{\alpha} - \frac{1}{\beta} \neq 0$, one has there is only one material. But when $\frac{1}{\alpha} - \frac{1}{\beta} = 0$, the possibility is arbitary w.r.t. the two mixed mterials.

If we take the homogenization coefficient of continous media

$$a(x) = \begin{cases}\alpha x + 1, \\ \beta x + 1,\end{cases} \tag{8}$$

then, one has

$$\begin{aligned}\min_{0\le z\le \ell} M_{(0,\ell)}\left(\frac{1}{a(x)}\right) &= \min_{0\le z\le \ell}\left(\int_0^z (\alpha x + 1)\,dx + \int_z^\ell (\beta x + 1)\,dx\right)\frac{1}{\ell} \\ &= \min_{0\le z\le \ell}\left(\frac{\alpha z^2}{2} + z + \frac{\beta}{2}(\ell^2 - z^2) + \ell - z\right)\frac{1}{\ell} \\ &= \min_{0\le z\le \ell}\left(z^2(\frac{\alpha}{2} - \frac{\beta}{2}) + \frac{\beta}{2}\ell^2\right)\frac{1}{\ell} \\ &= \min_{0\le z\le \ell}\left(z^2(\alpha - \beta) + \frac{\beta}{2}\ell^2\right)\frac{1}{2\ell}.\end{aligned}$$

Note that $\ell > 0$, then, we have three cases as follows.

- If $\alpha - \beta > 0$, we will obtain that the minimization value of the mean is $\frac{\beta}{2}\ell$ when $z = 0$.

- If $\alpha - \beta < 0$, we will obtain that the minimization value of the mean is $\frac{\alpha}{2}\ell - \frac{\beta}{4}\ell$ when $z = \ell$.
- If $\alpha - \beta = 0$, we will obtain that the minimization value of the mean is $\frac{\beta}{2}\ell$ when z is arbitrary.

The conclusion is same as the picewise case.