Homework 1
Due on Monday, March 23th, 2017

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Consider the propoties of weak convergence, as a consequence of the principle of uniform boundedness, every weakly convergent sequence is bounded. The norm is weakly lower-semicontinuous, if x_n converges weakly to x, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$. Let $\{x_n\}$ be a sequence weakly convergent to x in E.

(i) $\{x_n\}$ is a bounded sequence in E, i.e., there exists a constant C independent of n such that

$$\forall n \in N, \qquad \|x_n\|_E \leqslant C.$$

(ii) The norm on E is lower semi-continuous with respect to the weak convergence, i.e.,

$$\|x\|_E \leq \liminf_{n\to\infty} \|x_n\|_E.$$

Proof.

(i) Let $\{x_n\}$ be a sequence in E, such that

$$x_n \rightharpoonup x \qquad \text{weakly in } E.$$

Hence, for any $x' \in E'$, one has $< x', x_n >_{E',E} \longrightarrow < x', x >_{E',E}$.
For x fixed in E, set the map

$$f : x' \in E \longmapsto < x', x >_{E',E} .$$

Then, $f \in E''$. Notice that from $\|x\|_E = \|f\|_E$ and the definition of weak convergence, one has $f(x_n) \to f(x)$. For any $f_n \in E'$ and $x'' \in E''$, we finally get

$$x''(f_n) = f(x_n) \to f(x) = x''(f).$$

Observe that $\{x''_n\} \subset E''$ and from the principle of uniform boundedness for metric space, we have $\{x''_n\}$ a bounded sequence in E.
Since $\|x''_n\|_E = \|x_n\|_E$, which implies $\{x_n\}$ a bounded sequence in E.
The following lemma plays an important role in the proof of (ii):

Lemma. Suppose that $\mathscr{X}, \mathscr{Y}$ are both Banach space, $A_n \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, where $n = 1, 2, \cdots$ for any $x \in \mathscr{X}$, $\{A_n(x)\}$ is covergent sequence in $\mathscr{Y}$, we can prove the result, i.e., there exists $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, such that A_n is strongly covergent to A, moreover,

$$\|A\| \leq \varliminf_{n\to\infty} \|A_n\|.$$

In fact, for any $x \in \mathscr{X}$, $\{A_n(x)\}$ is covergent sequence in $\mathscr{Y}$, thus, we can define

$$Ax = \lim_{n\to\infty} A_n x,$$

observe that $\{A_n x\}$ is bounded in $\mathscr{Y}$, i.e.,

$$\sup_{n\geq 1} \|A_n x\| < \infty, \quad \forall x \in \mathscr{X}.$$

From the principle of uniform boundedness, there exists $M > 0$, such that $\|A_n\| \leq M$, where $n \geq 1$ however,

$$\|Ax\| = \varliminf_{n\to\infty} \|A_n x\| \leq \varliminf_{n\to\infty} \|A_n\|\|x\| \leq M\|x\|, \quad \forall x \in \mathscr{X}.$$

Then,

$$A \in \mathscr{L}(\mathscr{X}, \mathscr{Y}) \quad \text{and} \quad \|A\| \leq \varliminf_{n\to\infty} \|A_n\|.$$

Notation. The lemma is similar to Banach-Steinhass Theorem. This inequality holds strictly, whenever the convergence is not strong. For example, infinite orthonormal sequences converge weakly to zero, as demonstrated below.

(ii) From the definition of weak convergence, we know that

$$\lim_{n\to\infty} < f, x_n >= \lim_{n\to\infty} f(x_n) = f(x) =< f, x >, \quad \forall f \in E'.$$

There exists a natural map $\mathcal{J}\colon\ E' \mapsto E''$, such that $\hat{x_n} = \mathcal{J} x_n$, $\hat{x} = \mathcal{J} x$. Then,

$$\begin{aligned} \|\hat{x_n}\| &= \|x_n\|, \\ \|\hat{x}\| &= \|x\|, \end{aligned}$$

and

$$\begin{aligned} < \hat{x_n}, f > &= < f, x_n >, \\ < \hat{x}, f > &= < f, x >, \end{aligned}$$

From the lemma, it is easy to obtain

$$| < \hat{x}, f > | = \lim_{n\to\infty} | < \hat{x_n}, f > | \leq \underline{\lim}_{n\to\infty} \|\hat{x_n}\| \|f\|.$$

This proves

$$\|\hat{x}\| \leq \underline{\lim}_{n\to\infty} \|\hat{x_n}\|.$$

Consequently,

$$\|x\| \leq \underline{\lim}_{n\to\infty} \|x_n\|.$$

i.e.,

$$\|x\|_E \leq \liminf_{n\to\infty} \|x_n\|_E.$$

□

In particular, if we take account of Hilbert space, we can prove (ii) easily.
In deed, we can choose an orthonormal basis e_k and observe that

$$\sum_{k\leq p} \|x, e_k\|^2 = \lim_{n\to\infty} \|x_n, e_k\|^2.$$

Now, The sequence on the right is bounded by $\|x_n\|^2$ is independently of p, so

$$\sum_{k\leq p} \|x, e_k\|^2 \leq \liminf_{n\to\infty} \|x_n\|^2.$$

From the definition of $lim\ inf$, we take $p \to \infty$ to conclude that

$$\|x\|^2 \leq \liminf_{n\to\infty} \|x_n\|^2.$$

Then,

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Example. The Hilbert space $L^2[0, 2\pi]$ is the space of the square-integrable functions on the interval $[0, 2\pi]$ equipped with the inner product defined by

$$\langle f, g \rangle = \int_0^{2\pi} f(x) \cdot g(x)\, dx,$$

the sequence of functions $f_1, f_2, \ldots$ defined by $f_n(x) = \sin(nx)$ converges weakly to the zero function in $L^2[0, 2\pi]$, as the integral $\int_0^{2\pi} \sin(nx) \cdot g(x)\, dx$ tends to zero for any square-integrable function g on $[0, 2\pi]$ when n goes to infinity, i.e., $\langle f_n, g \rangle \to \langle 0, g \rangle = 0$. We can take advantage of Matrix Lab to draw the first 4 functions in the sequence $f_n(x) = \sin(nx)$ on $[0, 2pi]$. As $n \to \infty$, f_n converges weakly to $f = 0$.

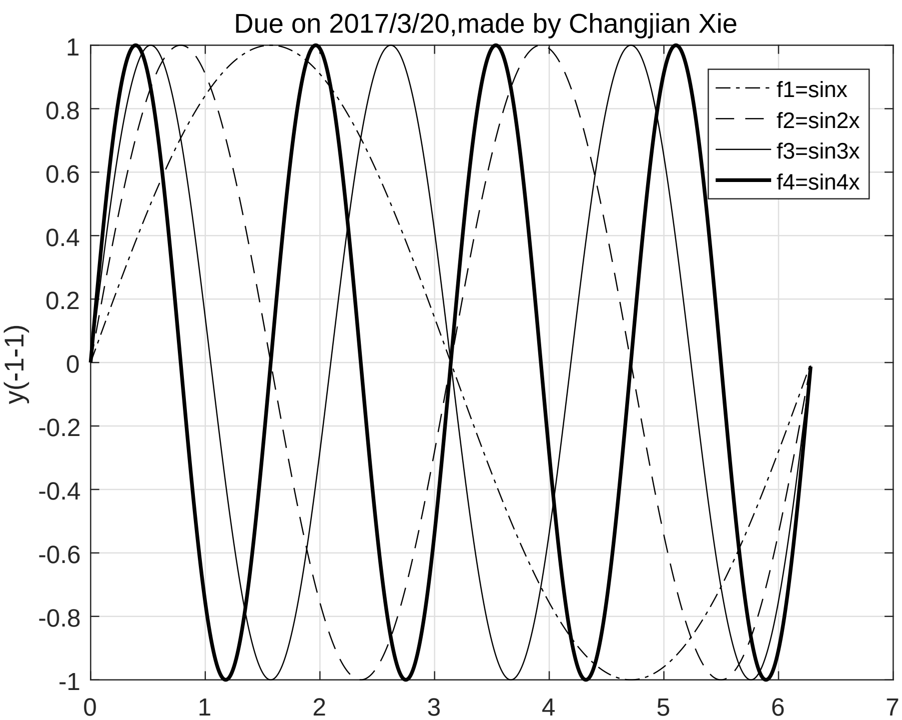

Figure 1: Consider $f_n(x) = \sin(nx)$ on $[0, 2pi]$ with respect to $n = 1, 2, 3, 4$, we deserve the curves and note that n larger, f_n converges weakly to $f = 0$.