Homework 2
Due on Monday, March 23th, 2017
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Consider there are many sets of functions which are dense in L^p-spaces. First of all, recall the notion of dense set in metric space. A subset $\mathscr{A}$ of a topological space $\mathscr{X}$ is dense in $\mathscr{X}$ if for any point x in $\mathscr{X}$, any neighborhood of x contains at least one point from $\mathscr{A}$, i.e., $\mathscr{A}$ has non-empty intersection with every non-empty open subset of $\mathscr{X}$. Ω is the open set in R^N, $f(x)$ defined in Ω which is measurable function. Its function value only take a limited number of real numbers, this function is called a simple function defined on Ω. Some lemmas in Lebesgue integral theory are proved without proof.
To begin with, the following lemmas are essential to the theorem.

Lemma 1. *Let Ω an open set in R^N, $f(x)$ defined in Ω which is non-negative measurable function. Then, there exists a sequence of non-negative monotonically increasing simple function $\{S_n(x)\}$ which convergent to $f(x)$.*

Lemma 2. *Let Ω an bounded open set in R^N, $f(x)$ defined in Ω which is almost everywhere bounded measurable function. Then, for any $\varepsilon > 0$, there exists a function $g(x) \in C_0(\Omega)$ which subject to*

$$\sup_{x\in\Omega} |g(x)| \leq \sup_{x\in\Omega} |f(x)|,$$

$$\mu(\{x \in \Omega | f(x) \neq g(x)\}) < \varepsilon,$$

where $\mu(G)$ denote by Lebesgue measure of G.

Lemma 3. *Ω is a area of R^N, $\mu(\Omega)$ is a infinite number, for $1 \leq p < q \leq \infty$. Then,*

1. if $f \in L^q(\Omega)$, it follows that $f \in L^p(\Omega)$ and

$$\|f; L^p(\Omega)\| \leq (\mu(\Omega))^{\frac{1}{p}-\frac{1}{q}} \|f; L^q(\Omega)\|, \tag{1}$$

2. if $f \in L^\infty(\Omega)$, it follows that

$$\lim_{p\to\infty} \|f; L^p(\Omega)\| = \|f; L^\infty(\Omega)\|, \tag{2}$$

3. if for any $p \in [1,\infty)$, $f \in L^p(\Omega)$, and there exists a constant K with

$$\|f; L^p(\Omega)\| \le K, \quad \forall p \in [1,\infty), \tag{3}$$

it follows that $f \in L_\infty(\Omega)$, and

$$\|f; L^\infty(\Omega)\| \le K. \tag{4}$$

Proof.

1. For $p < \infty$, from Hölder inequality, one has

$$\begin{aligned} \|f; L^p(\Omega)\|_p &= \int_\Omega |f|^p dx \\ &\le [\int_\Omega (|f|^p)^{\frac{q}{p}} dx]^{\frac{p}{q}} [\int_\Omega dx]^{1-\frac{p}{q}} \\ &= \|f; L^q(\Omega)\|^p (\mu(\Omega))^{1-\frac{p}{q}}. \end{aligned}$$

Note that the term is infinite number of right hand above the formula, hence, we can take power as $\frac{1}{p}$, then we obtain (1). Next, we consider $q = \infty$, it follows that

$$\|f; L^p(\Omega)\| = [\int_\Omega |f|^p dx]^{\frac{1}{p}} \le \|f; L^\infty(\Omega)\| (\mu(\Omega))^{\frac{1}{p}},$$

2. if $f \in L_\infty(\Omega)$, q in the right hand of (1) takes ∞. We take $q \to \infty$, one has

$$\overline{\lim_{p\to\infty}} \|f; L^p(\Omega)\| \le \|f; L^\infty(\Omega)\|. \tag{5}$$

In addition, from the definition of function norm in $L^\infty(\Omega)$, for all $\varepsilon > 0$, there exists a set A with $\mu(A) > 0$, it follows

$$|f(x)| > \|f; L^\infty(\Omega)\| - \varepsilon, \qquad x \in A. \tag{6}$$

Then,

$$\int_\Omega |f|^p dx \ge \int_A |f|^p dx \ge (\|f; L^\infty(\Omega)\| - \varepsilon)^p \mu(A).$$

we can take power as $\frac{1}{p}$, then

$$[\int_\Omega |f|^p dx]^{\frac{1}{p}} \ge (\|f; L^\infty(\Omega)\| - \varepsilon)(\mu(A))^{\frac{1}{p}}.$$

Thus, one has

$$\varliminf_{p\to\infty}[\int_\Omega |f|^p dx]^{\frac{1}{p}} \geq \|f; L^\infty(\Omega)\| - \varepsilon.$$

Due to ε is arbitrary, it follows that

$$\varliminf_{p\to\infty}[\int_\Omega |f|^p dx]^{\frac{1}{p}} \geq \|f; L^\infty(\Omega)\|. \tag{7}$$

From (5) and (7), we obtain (2).

3. If $f(x)$ doesn't belong to $L^\infty(\Omega)$ or doesn't satisfy (4), anyway, we always find a subset A in Ω with $\mu(A) > 0$ and constant $K_1 > K$, it follows that

$$|f(x)| > K_1, \qquad \forall x \in A. \tag{8}$$

The next process is same as the one which (6) deduce (7). From (8), one has

$$\varliminf_{p\to\infty}[\int_\Omega |f(x)|^p dx]^{\frac{1}{p}} \geq K_1 > K. \tag{9}$$

Clearly, (9) is contrary to (3). Hence, one has (4). □

Lemma 4. *$C_0(\Omega)$ is dense in $L^p(\Omega)$, for $1 \leq p < \infty$.*

Proof. Let $f \in L^p(\Omega)$, we denote by $f_1 = \max(0, f)$, $f_2 = \max(0, -f)$, then f_1 and f_2 are both non-negative function in $L^p(\Omega)$, moreover, $f = f_1 - f_2$. If we have found two sequences $\{g_{1n}\}$ and $\{g_{2n}\}$ in $C_0(\Omega)$, they are convergent to f_1 and f_2 respectively, then from the Minkowski inequality, it follows that

$$\|f - (g_{1n} - g_{2n})\|_p \leq \|f_1 - g_{1n}\|_p + \|f_2 - g_{2n}\|_p.$$

Consequently, the sequence $\{g_n\} = \{g_{1n} - g_{2n}\}$ in $C_0(\Omega)$ convergent to $L^p(\Omega)$. So, we consider the non-negative function in $L^p(\Omega)$. It's easy to know that there exists monotonically increasing function sequences as the following that

$$0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq s_n(x) \leq \cdots \leq f(x), \quad \forall x \in \Omega,$$

with

$$\lim_{n\to\infty} s_n(x) = f(x).$$

Obviously, $s_n(x) \in L^p(\Omega)$ and

$$0 \leq (f(x) - s_n(x))^p \leq (f(x))^p.$$

From Lebesgue theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega (f(x)-s_n(x))^p dx = \int_\Omega \lim_{n\to\infty}(f(x)-s_n(x))^p dx = 0.$$

Then, for any $\varepsilon > 0$, there exists simple function $s_{n_0}(x)$ which satisfies

$$\|f - s_{n_0}\|_p \leq \frac{\varepsilon}{2}. \tag{10}$$

May as well let $supps_n(x)$ a bounded set, then, there exists $g(x) \in C_0(\Omega)$ with

$$|g(x)| \leq \|s_{n_0}(x)\|_\infty, \qquad \forall x \in \Omega,$$

note that

$$\mu(\{x\in\Omega | s_{n_0}(x) \neq g(x)\}) < \frac{\varepsilon}{4\max(|s_{n_0}(x)|_\infty, 1)}.$$

Then,

$$\begin{aligned}[\int_\Omega |g-s_n|^p dx]^{\frac{1}{p}} &\leq \max_{x\in\Omega}|g-s_n|\mu(\{x\in\Omega|s_{n_0}(x)\\ &\leq 2\|s_{n_0}\|_\infty \frac{\varepsilon}{4\max(|s_{n_0}(x)|_\infty, 1)} \leq \frac{\varepsilon}{2}.\end{aligned}$$

From Minkowski inequality, it follows that

$$\|f-g\|_p \leq \|f-s_{n_0}\|_p + \|s_{n_0} - g\|_p \leq \varepsilon. \tag{11}$$

In general, for each $\varepsilon > 0$, there exists $g(x) \in C_0\Omega$ which satisfies (2). □

We consider $C_0^\infty(R^N)$, let $\gamma = \int_{|x|<1}\exp(-\frac{1}{1-|x|^2})dx$, we define

$$j(x) = \begin{cases} \frac{1}{\gamma}\exp(-\frac{1}{1-|x|^2}), & |x|<1, \\ 0, & otherwise, \end{cases}$$

with $j(x) \in C_0^\infty(R^N)$, $\int_{R^N} j(x)dx = 1$, $j(x) \geq 0$, $suppj(x) = \{x \in R^N| \quad |x| \leq 1\}$. For any real number $\delta > 0$, we construct the function as

$$j_\delta(x,y) = \delta^{-N} j(\frac{x-y}{\delta}), \tag{12}$$

it follows that

$$\int_{R^N} j_\delta(x,y)dy = 1.$$

For $f \in L^1(\Omega)$, we construct

$$\tilde{f} = \begin{cases} f, & x \in \Omega, \\ 0, & otherwise, \end{cases}$$

then, $\tilde{f} \in L^1(R^N)$. We define

$$J_\delta f(x) = \int_{R^N} \tilde{f}(x - \delta y) j(y) dy = \int_{R^N} \tilde{f}(y) j_\delta(x,y) dy, \tag{13}$$

where J_δ is called by regularized operator. The properties of function J_δ as following that

Lemma 5.

1. If $f \in L^1(\Omega)$, then $J_\delta f \in C^(\infty)(R^N)$;

2. If $K \subset\subset \Omega$, $f \in L^1(\Omega)$ and $f(x) = 0, \quad x \in \Omega \setminus K$, for $\delta < dist(K, \Gamma)$, then $J_\delta f \in C_0^\infty(\Omega)$;

3. If $f \in L^p(\Omega)$, $1 \le p < \infty$, then $J_\delta f(x)$ is convergent to f, with

$$\|J_\delta f; L^p(\Omega)\| \le \|f; L^p(\Omega). \tag{14}$$

4. If $f \in C(\Omega) \cap L^1(\Omega)$, then $J_\delta f(x)$ is uniformly convergent to f in $K \subset\subset \Omega$.

Theorem 1. *$C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, for $1 \le p < \infty$.*

Proof. From lemma 4 and 5, we can obtain theorem 1. In brief, the progress as following that:

(1) if $u \in L^p(\Omega)$, where $1 < p < \infty$, $u_\delta(x)$ construct as above, then from Young inequality, it follows that

$$\|u_\delta\|_p \le \|u\|_p.$$

(2) From Lusin theorem, we prove $C_0^0(\Omega)$ that dense in $L^p(\Omega)$.

(3) For $u \in L^p(\Omega)$, we need to find $u_\delta \in C_0^\infty(\Omega)$ which subject to

$$\|u_\delta - u\|_p < \varepsilon,$$

next, we can prove for three steps:

step 1. we need to find $\varphi \in C_0^0(\Omega)$ which satisfies with $\|u - \varphi\|_p < \frac{\varepsilon}{3}$,
step 2. we need to find $\varphi_\delta \in C_0^\infty(\Omega)$ which satisfies with $\|\varphi - \varphi_\delta\|_p < \frac{\varepsilon}{3}$,
step 3. we need to find $u_\delta \in C_0^\infty(\Omega)$ which satisfies with $\|\varphi\delta - u_\delta\|_p < \frac{\varepsilon}{3}$. $\square$