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An Introduction to Homogenization

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Preface

The aim of homogenization theory is to establish the macroscopic behaviour of a system which is 'microscopically' heterogeneous, in order to describe some characteristics of the heterogeneous medium (for instance, its thermal or electrical conductivity). This means that the heterogeneous material is replaced by a homogeneous fictitious one (the 'homogenized' material), whose global (or overall) characteristics are a good approximation of the initial ones. From the mathematical point of view, this signifies mainly that the solutions of a boundary value problem, depending on a small parameter, converge to the solution of a limit boundary value problem which is explicitly described.

During the last ten years, we have both had the opportunity to give courses on homogenization theory for graduate and postgraduate students in several universities and schools of engineers. We realized that, while at the research level many excellent books have been written in the past, for the graduate level there was a lack of elementary reference books which could be used as an introduction to the field. Also, many classical and known results in linear homogenization, though currently taught, are not really available in the literature, either in books or in research articles. This lack naturally led to the idea to extend the material of our courses into the book we present here.

When teaching, we had to take into account that often the audience was not really familiar with the variational approach of partial differential equations (PDEs), which is the natural framework for homogenization theory. This is why we started the book with this topic. It is the subject of the first four chapters.

We have deliberately chosen not to present too many results, but to have those included all well explained. We focus our attention on the periodic homogenization of linear partial differential equations. A periodic distribution of the heterogeneities is a realistic assumption for a large class of applications. From the mathematical point of view, it contains the main difficulties arising in the study of composite materials.

Chapter 1 deals with two notions of convergence, the weak and the weak* one. This allows us to describe, in Chapter 2, the asymptotic behaviour of rapidly oscillating periodic functions.

In Chapter 3 we introduce the distributions and give the basic notions and theorems of Sobolev spaces. We pay particular attention to Sobolev spaces of periodic functions. The results of this chapter, as well as those of Chapter 1, are classical and are the necessary prerequisites for the variational approach of PDEs. We do not give their proofs but detailed references are quoted.

In Chapter 4 the variational approach to classical second order linear elliptic equations is introduced. Existence and uniqueness results for solutions of these equations with various boundary conditions are proved. Again we treat in detail the case of periodic boundary conditions.

From Chapter 5 to Chapter 12 we treat the periodic homogenization of several kinds of second order boundary value problems with rapidly oscillating periodic coefficients. We are concerned with elliptic equations, the linearized system of elasticity, the heat and the wave equations.

The model case is the Dirichlet problem for elliptic equations. The results concerning this case are the object of Chapters 5 and 6. In Chapter 5 we formulate the problem and list some physical examples. We also study two particular cases: the one-dimensional case and the case of layered materials. In Chapter 6 we state the general homogenization result and prove some properties of the homogenized coefficients.

The main homogenization methods for proving the general result are presented in Chapters 7–9. Thus, the multiple-scale method is described in Chapter 7. Chapter 8 is devoted to the oscillating test functions method. Finally, in Chapter 9 we introduce the two-scale convergence method.

In Chapter 8 we also prove some important related results, as for instance the convergence of energies and the existence of correctors. The convergence of eigenvalues and eigenvectors is also proved.

Chapters 10, 11 and 12 are devoted to the linearized system of elasticity, the heat equation and the wave equations respectively. In each chapter, we start by proving the existence and uniqueness of a solution. Then, we study the homogenization of the problem.

We conclude this book with a short overview of some general approaches to the study of the non-periodic case.

The idea of writing this book was to provide detailed proofs and tools adapted to the level we have in mind. Our hope is to give a background of homogenization theory not only to students, but also to researchers –in mathematics as well as in engineering, mechanics, or physics- who are interested in a mathematical introduction to the field.

Special thanks go to three of our colleagues. We thank Petru Mironescu for many helpful suggestions concerning the first four chapters. We also express our gratitude to Olivier Alvarez for his accurate reading of the manuscript and for his useful remarks and suggestions. Finally, we thank Thomas Lanchand-Robert for his valuable and patient help while we were typing this book in TEX.

This book represents for us the ultimate 'joint venture', which would have never been possible without a truly deep friendship and mutual understanding.

Paris D.C.
Rouen P.D.
March 1999

Contents

Introduction

The aim of this book is to present the mathematical theory of the homogenization. This theory has been introduced in order to describe the behaviour of composite materials.

Composite materials are characterized by the fact that they contain two or more finely mixed constituents. They are widely used nowadays in industry, due to their properties. Indeed, they have in general a 'better' behaviour than the average behaviour of their individual constituents. Well-known examples are the superconducting multifilamentary composites which are used in the composition of optical fibres.

Generally speaking, in a composite the heterogeneities are small compared to its global dimension. So, two scales characterize the material, the microscopic one, describing the heterogeneities, and the macroscopic one, describing the global behaviour of the composite. From the macroscopic point of view, the composite looks like a 'homogeneous' material. The aim of 'homogenization' is precisely to give the macroscopic properties of the composite by taking into account the properties of the microscopic structure.

As a model case, let us fix our attention on the problem of the steady heat conduction in an isotropic composite.

Consider first a homogeneous body occupying Ω with thermal conductivity γ. For simplicity, we assume that the material is isotropic, which means that γ is a scalar. Suppose that f represents the heat source and g the temperature on the surface $\partial\Omega$ of the body, which we can assume to be equal to zero.

Then the temperature $u = u(x)$ at the point $x \in \Omega$ satisfies the following homogeneous Dirichlet problem:

$$\begin{cases} -\text{div}\,(\gamma \nabla u(x)) = f(x) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where ∇u denotes the gradient of u defined by

$$\nabla u = \text{grad}\, u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right).$$

Since γ is constant, this can be rewritten in the form

$$\begin{cases} -\gamma\, \Delta u = f & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

where $\Delta u = \text{div}(\text{grad } u)$. The flux of the temperature is defined by

$$q = \gamma \;\; \text{grad } u. \tag{0.3}$$

This is a classical elliptic boundary value problem and it is well known that if f is sufficiently smooth, it admits a unique solution u which is twice differentiable and solves system (0.2) at any point x in Ω.

If now we consider a heterogeneous material occupying Ω, then the thermal conductivity takes different values in each component of the composite. Hence, γ is now a function, which is discontinuous in Ω, since it jumps over surfaces which separate the constituents. To simplify, suppose we are in presence of a mixture of two materials, one occupying the subdomain Ω_1 and the second one the subdomain Ω_2, with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2 \cup (\partial\Omega_1 \cap \partial\Omega_2)$.

Suppose also that the thermal conductivity of the body occupying Ω_1 is γ_1 and that of the body occupying Ω_2 is γ_2, i.e.

$$\gamma(x) = \begin{cases} \gamma_1 & \text{if } x \in \Omega_1 \\ \gamma_2 & \text{if } x \in \Omega_2. \end{cases}$$

Then the temperature and flux of the temperature in a point $x \in \Omega$ of the composite take respectively, the values

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega_1 \\ u_2(x) & \text{if } x \in \Omega_2 \end{cases}$$

and

$$q = \begin{cases} q_1 = \gamma_1 \text{ grad } u_1 & \text{in } \Omega_1 \\ q_2 = \gamma_2 \text{ grad } u_2 & \text{in } \Omega_2. \end{cases}$$

The usual physical assumptions are the continuity of the temperature u and of the flux q at the interface of the two materials, i.e.

$$\begin{cases} u_1 = u_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2 \\ q_1 \cdot n_1 = q_2 \cdot n_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2. \end{cases} \tag{0.4}$$

where n_i is the outward normal unit vector to $\partial\Omega_i$, $i = 1, 2$ and $n_1 = -n_2$ on $\partial\Omega_1 \cap \partial\Omega_2$. Therefore, the temperature u is solution of the stationary thermal problem. Then the corresponding system (0.1) reads

$$\begin{cases} -\text{div } (\gamma(x) \text{ grad } u(x)) = f(x) & \text{in } \Omega_1 \cup \Omega_2 \\ u = 0 \quad \text{on } \partial\Omega \\ u_1 = u_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2 \\ q_1 \cdot n_1 = q_2 \cdot n_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2. \end{cases} \tag{0.5}$$

Formally, we can write this system in the form

$$\begin{cases} -\text{div } (\gamma(x) \text{ grad } u(x)) = f(x) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{0.6}$$

Observe that from (0.4), it follows that the gradient of u is discontinuous. Moreover, in general, the flux q is not differentiable.

Taking into account these discontinuities, the question is what is the appropriate mathematical formulation of this problem and in which functional space can one have a solution (since one can not expect to have solutions of class C^1)?

An answer to these questions can be given by introducing a weak notion of solution. It is built on the notion of weak derivative, the so-called derivative in the sense of distributions. This is defined in Chapter 3, where we also introduce the Sobolev spaces which constitute the natural functional framework for weak solutions.

In the definition of a weak solution, problem (0.6) (or (0.5)) is replaced by a variational formulation, namely

$$\begin{cases} \text{Find } \ u \in H \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f\, v\, dx, \quad \forall v \in H, \end{cases} \tag{0.7}$$

where H is an appropriate Sobolev space taking into account the boundary conditions on u. In (0.7) the derivatives are taken in the sense of distributions.

Of course, if u were sufficiently smooth, (0.7) and (0.6) would be equivalent. As seen above, this is not the case for a composite material, so the sense to be given to (0.6) is only that u solves (0.7). Let us point out that the equation in (0.7) is checked for any v belonging to the space H. This is why v is usually called a test function.

Existence and uniqueness results of a weak solution of (0.7) are proved in Chapter 4, where we also treat other kinds of boundary value problems.

Let us turn back to the question of the macroscopic behaviour of the composite material occupying Ω. Suppose that the heterogeneities are very small with respect to the size of Ω and that they are evenly distributed. This is a realistic assumption for a large class of applications.

From the mathematical point of view, one can modelize this distribution by supposing that it is a periodic one (see Fig. 0.1).

This periodicity can be represented by a small parameter, 'ε'.

Then the coefficient γ in (0.7) depends on ε and (0.7) reads

$$\begin{cases} \text{Find } \ u^\varepsilon \in H \ \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f\, v\, dx. \qquad \forall v \in H. \end{cases} \tag{0.8}$$

A natural way to introduce the periodicity of γ^ε in (0.8) is to suppose that it has the form

$$\gamma^\varepsilon(x) = \gamma\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N. \tag{0.9}$$

where γ is a given periodic function of period Y. This means that we are given a reference period Y, in which the reference heterogeneities are given. By definition (0.9), the heterogeneities in Ω are periodic of period εY and their size is

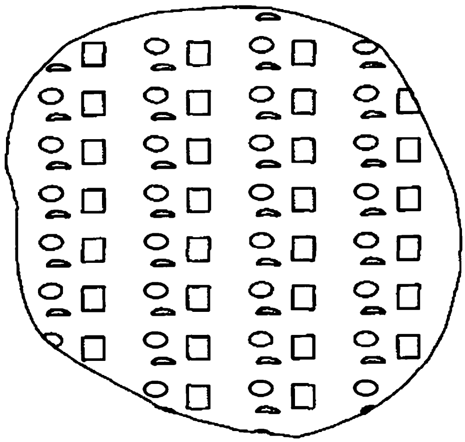


Fig. 0.1

of order of ε. Problem (0.8) is then written as follows:

$$\begin{cases} \text{Find } u^\varepsilon \in H \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma\Big(\frac{x}{\varepsilon}\Big) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f\, v\, dx. \qquad \forall v \in H. \end{cases} \tag{0.10}$$

and Fig. 0.2 shows the periodic structure of Ω. Observe that two scales characterize our model problem (0.10), the macroscopic scale x and the microscopic one $\frac{x}{\varepsilon}$, describing the micro-oscillations.

The discontinuities of this problem make the model very difficult to treat, in particular from the numerical point of view. Also, the pointwise knowledge of the characteristic of the material does not provide in a simple way any information on its global behaviour.

Observe also that making the heterogeneities smaller and smaller means that we 'homogenize' the mixture and from the mathematical point of view this means that ε tends to zero. Taking $\varepsilon \to 0$ is the mathematical 'homogenization' of problem (0.10).

Many natural questions arise:

(1) Does the temperature u^ε converge to some limit function u^0?

(2) If that is true, does u^0 solve some limit boundary value problem?

(3) Are then the coefficients of the limit problem constant?

(4) Finally, is u^0 a good approximation of u^ε?

Answering these questions is the aim of the mathematical theory of 'homogenization'.

These questions are very important in the applications since, if one can give positive answers, then the limit coefficients, as it is well known from engineers

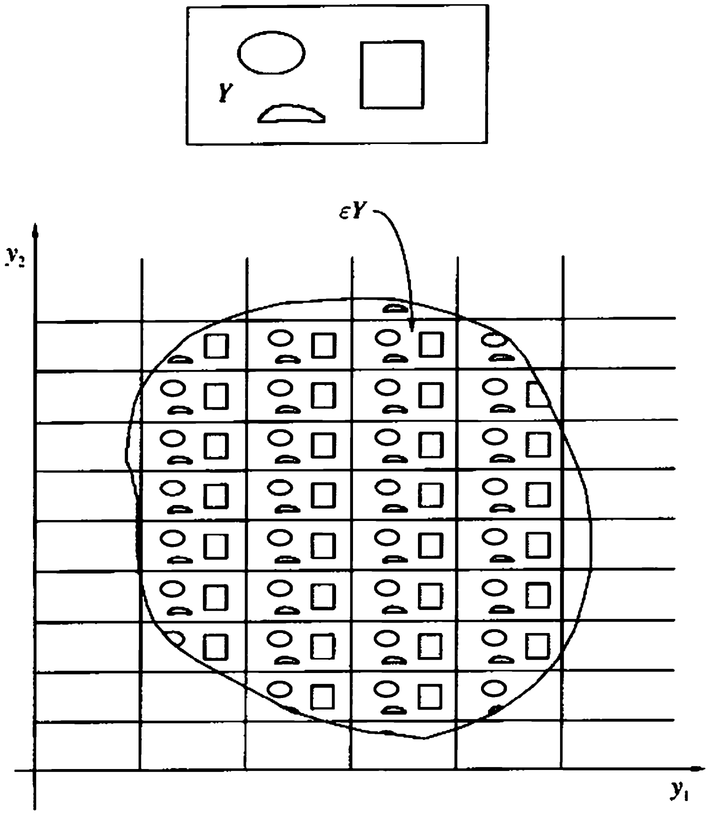


Fig. 0.2

and physicians, are good approximations of the global characteristics of the composite material, when regarded as an homogeneous one. Moreover, replacing the problem by the limit one allows us to make easy numerical computations.

The first remark is that the function γ^ε converges in a weak sense to the mean value of γ , i.e. one has

$$\int_\Omega \gamma^\varepsilon(x)\, v(x)\, dx \longrightarrow \int_\Omega \mathcal{M}_Y(\gamma)\, v(x)\, dx. \tag{0.11}$$

for any integrable function v. Here the mean value $\mathcal{M}_Y(\gamma)$ is defined by

$$\mathcal{M}_Y(\gamma) = \frac{1}{|Y|}\int_Y \gamma(y)\, dy.$$

This result on the convergence of periodic functions is proved in Chapter 2. The notion of weak convergence and related properties are presented in Chapter 1.

One can also (thanks to weak-compactness results stated in Chapter 1) show that u^ε converges to some function u^0 and that ∇u^ε weakly converges to ∇u^0.

The question is whether these convergences and convergence (0.11) are sufficient to homogenize problem (0.10). To do that, one has to pass to the limit in the product $\gamma^\varepsilon \nabla u^\varepsilon$. This is the main difficulty in homogenization theory. Actually, in general (see Chapters 1 and 2), the product of two weakly convergent sequences does not converge to the product of the weak limits. In Section 5.1 we show that there is a vector function ξ, weak limit of the product $\gamma^\varepsilon \nabla u^\varepsilon$ and satisfying the equation

$$-\text{div } \xi = f. \tag{0.12}$$

But

$$\xi \neq \mathcal{M}_Y(\gamma)\nabla u^0,$$

so that from (0.12) one cannot easily deduce an equation satisfied by u^0. This already occurs in the one-dimensional case where Ω is some interval $]d_1, d_2[$. One has (see Section 5.3)

$$\xi = \frac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)} \frac{du^0}{dx}.$$

Moreover, u^0 is the unique solution of the homogenized problem

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)} \dfrac{du^0}{dx}\right) = f \quad \text{in }]d_1, d_2[\\ u^0(d_1) = u^0(d_2) = 0. \end{cases}$$

Clearly, $\xi \neq \mathcal{M}_Y(\gamma)\nabla u^0$, since

$$\frac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)} \neq \mathcal{M}_Y(\gamma).$$

Even for the one-dimensional case this homogenization result is not trivial. The situation is of course, more complicated in the general N-dimensional case. The one-dimensional result could suggest that in the N-dimensional case the limit problem can be described in terms of the mean value of γ^{-1}. This is not true, as can already be seen in the case of layered materials studied in Section 5.4, where γ depends only on one variable, say x_1. In this case, the homogenized problem of (0.10) is

$$\begin{cases} -\text{div } (A^0 \nabla u^0) = f \quad \text{in } \Omega \\ u^0 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.13}$$

where the homogenized matrix A^0 is constant, diagonal and given by

$$A^0 = \begin{pmatrix} \dfrac{1}{\mathcal{M}_Y(\gamma^{-1})} & 0 & \dots & 0 \\ 0 & \mathcal{M}_Y(\gamma) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{M}_Y(\gamma) \end{pmatrix}.$$

Let us point out that the homogenized material is no longer isotropic, since A^0 is not of the form $a^0 I$.

Observe also that in these particular examples of the one-dimensional case and of layered materials, the homogenized coefficients are algebraic formulas involving γ.

For the general N-dimensional case, as seen in Chapter 6, the homogenized problem is still of the form (0.13). The coefficients of A^0 are defined by means of some periodic functions which are the solutions of some boundary value problems of the same type as (0.10) posed in the reference cell Y. The coefficients a^0_{ij} of the matrix A^0 are defined by

$$a^0_{ij} = \frac{1}{|Y|}\int_Y \gamma\, \delta_{ij}\, dy - \frac{1}{|Y|}\int_Y \gamma \frac{\partial \chi_j}{\partial y_i}\, dy, \quad \forall\, i,j = 1,\dots,N, \tag{0.14}$$

where δ_{ij} is the Kronecker symbol. The function χ_j for $j = 1,\dots,N$ is the solution of the problem

$$\begin{cases} -\text{div}\,(\gamma(y)\nabla\chi_j) = -\dfrac{\partial\gamma}{\partial y_j} & \text{in } Y \\ \chi_j \quad Y\text{-periodic} \\ \mathcal{M}_Y(\chi_j) = 0. \end{cases} \tag{0.15}$$

This result can be proved by different methods. We present in this book three of them.

In Chapter 7 we use the multiple-scale method, which consists of searching for u^ε in the form

$$u^\varepsilon(x) = u_0\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon^2 u_2\Big(x, \frac{x}{\varepsilon}\Big) + \cdots, \tag{0.16}$$

where $u_j = u_j(x, y)$ are Y-periodic in the second variable y.

The multiple-scale method is a classical one, widely used in mechanics and physics for problems containing several small parameters describing different scalings. It is well adapted to the periodic framework in which we work in this book. Its interest is that in general, it permits us to obtain formally the homogenized problem.

Chapter 8 is devoted to the oscillating test functions method introduced by L. Tartar. As we have seen above, in problem (0.10) the function u^ε is continuous at the interface $\partial\Omega_1 \cap \partial\Omega_2$ but its gradient is not, and behaves in such a way that the flux $\gamma\nabla u^\varepsilon$ remains continuous. The idea of Tartar's method is to construct test functions $v = w^\varepsilon_j \varphi$ for (0.10) having the same kind of discontinuities as u^ε and having a known limit. For our example, one has

$$w^\varepsilon_j(x) = -\chi_j\Big(\frac{x}{\varepsilon}\Big) + x_j, \quad j = 1,\dots,N. \tag{0.17}$$

and φ is a smooth function vanishing on $\partial\Omega$. Using these test functions in the variational formulation (0.10), one is able to pass to the limit and identify ξ in terms of u^0. Actually, one obtains ξ in the form

$$\xi = A^0 \nabla u^0.$$

This together with (0.12) gives the homogenized problem.

In Chapter 8 we also prove a corrector result which for the model problem (0.10) is the following. Let us introduce the (corrector) matrix $C^\varepsilon = (C^\varepsilon_{ij})_{1\le i,j\le N}$ defined by

$$C^\varepsilon_{ij}(x) = \frac{\partial w_j}{\partial y_i}\left(\frac{x}{\varepsilon}\right),$$

where w_j is given by (0.17). Then,

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \to 0$$

in a usual (strong) convergence.

Moreover, let us observe that, when applying the multiple-scale method one finds

$$u_1(x, y) = -\sum_{j=1}^{N} \chi_j(y) \frac{\partial u_0}{\partial x_j}.$$

Therefore

$$\begin{aligned}\nabla u^\varepsilon(x) &= \nabla u^0(x) - \sum_{k=1}^{N} \nabla_y \chi_k\left(\frac{x}{\varepsilon}\right) \frac{\partial u^0}{\partial x_k}(x) - \varepsilon \sum_{k=1}^{N} \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right) \nabla\left(\frac{\partial u^0}{\partial x_k}\right)(x) + \cdots \\ &= C^\varepsilon(x)\nabla u^0(x) - \varepsilon \sum_{k=1}^{N} \chi_k\left(\frac{x}{\varepsilon}\right) \nabla\left(\frac{\partial u^0}{\partial x_k}\right)(x) + \cdots.\end{aligned}$$

Hence $C^\varepsilon(x)\nabla u^0(x)$ is the first term in the asymptotic expansion (0.16) of ∇u^ε.

In the same chapter we also give further properties of the homogenized problem.

In Chapter 9 we prove again the convergence result by the two-scale method which takes into account the two scales of the problem and introduces the notion of 'two-scale convergence'. This convergence is tested on functions of the form $\psi(x, x/\varepsilon)$. One of the interests of the two-scale method is that it justifies mathematically the formal asymptotic development (0.16).

In Chapters 10, 11, and 12 we treat respectively the linearized system of elasticity, the heat equation and the wave equation. For each problem, we first prove the existence and uniqueness of the solution, then we study their homogenization.

Finally, Chapter 13 contains a short overview of some methods used in the general non-periodic case. In particular, we fix our attention on G-convergence and H-convergence.

1

Weak and weak* convergences in Banach spaces

We recall in this chapter the main properties of weak and weak* convergence in a Banach space. We also detail these notions for the particular case of L^p-spaces.

Let us begin by recalling the notions of a Banach and a Hilbert space which are the functional spaces in which we work in this book. The spaces we consider in this book are all real.

Definition 1.1. A mapping

$$\|\cdot\| : x \in E \longmapsto \|x\| \in \mathbb{R}_+$$

is called a *norm* on the vector space E iff

$$\begin{cases} \|x\| = 0 \Longleftrightarrow x = 0 \\ \|\lambda x\| = |\lambda|\|x\|, \quad \text{for any } \lambda \in \mathbb{R},\ x \in E \\ \|x + y\| \le \|x\| + \|y\|, \quad \text{for any } x,\ y \in E. \end{cases}$$

Then E is called a *normed space* and its norm is denoted by $\|\cdot\|_E$.

Moreover, E is called a Banach space iff it is complete with respect to the following convergence (called strong convergence):

$$x_n \longrightarrow x \text{ in } E \Longleftrightarrow \|x_n - x\|_E \longrightarrow 0.$$

Definition 1.2. Let H be a real linear space. A mapping

$$(\cdot,\cdot)_H : (x, y) \in H \times H \longmapsto (x, y)_H \in \mathbb{R}$$

is called a *(real) scalar product* iff

$$\begin{cases} (x, x)_H > 0 \Longleftrightarrow x \neq 0, \\ (x, y)_H = (y, x)_H, \quad \text{for any } x,\ y \in H \\ (\lambda x + \mu y, z)_H = \lambda(x, z)_H + \mu(y, z)_H, \quad \text{for any } \lambda,\ \mu \in \mathbb{R},\ x,\ y,\ z \in H. \end{cases}$$

Moreover, if H is a Banach space with respect to the norm associated to this scalar product, i.e. with

$$\|x\|_H = (x, x)_H^{\frac{1}{2}},$$

then H is called a Hilbert space.

Here we present in particular, the properties of Banach spaces needed in the study of homogenization problems. We refer the reader for proofs and more details concerning Banach spaces to Yosida (1964), Edwards (1965), Rudin (1966, 1973), Zeidler (1980), and Brezis (1987).

1.1 Linear forms on Banach spaces

In this section, we give some basic properties of mappings on Banach spaces and in particular, we will introduce the notion of dual space. In the sequel, E and F denote two Banach spaces.

Definition 1.3. Let $\Lambda : E \longmapsto F$ be a linear map (i.e. such that for any x, $y \in E$ and for α, $\beta \in \mathbb{R}$, one has $\Lambda(\alpha x + \beta y) = \alpha\Lambda(x) + \beta\Lambda(y)$). Then, Λ is bounded iff

$$\sup_{x \in E \setminus \{0\}} \frac{\|\Lambda(x)\|_F}{\|x\|_E} < +\infty.$$

One denotes by $\mathcal{L}(E, F)$ the set of linear and bounded maps from E to F.

The main property of bounded linear maps is given by the following result:

Proposition 1.4. *The quantity*

$$\|\Lambda\|_{\mathcal{L}(E,F)} = \sup_{x \in E \setminus \{0\}} \frac{\|\Lambda(x)\|_F}{\|x\|_E},$$

defines a norm on $\mathcal{L}(E, F)$, which is a Banach space for this norm.
Then one has

$$\|\Lambda(x)\|_F \le \|\Lambda\|_{\mathcal{L}(E,F)} \|x\|_E, \quad \forall x \in E, \tag{1.1}$$

where $\|\Lambda\|_{\mathcal{L}(E,F)}$ is the smallest number for which (1.1) holds.

Moreover, the linearity implies that

$$\|\Lambda\|_{\mathcal{L}(E,F)} = \sup_{\substack{x \in E \setminus \{0\} \\ \|x\|_E \le 1}} \frac{\|\Lambda(x)\|_F}{\|x\|_E} = \sup_{\substack{x \in E \\ \|x\|_E = 1}} \|\Lambda(x)\|_F,$$

for $\Lambda \in \mathcal{L}(E, F)$.

The following result characterizes the space $\mathcal{L}(E, F)$:

Theorem 1.5. *Let Λ a linear map from E to F. Then, the following three assertions are equivalent:*

i) Λ *is bounded,*

ii) Λ *is continuous,*

iii) Λ *is continuous at a point $v_0 \in E$.*

Proof. Let us prove first that $(i) \Rightarrow (ii)$.

If Λ is bounded, from (1.1) and the linearity of Λ, one immediately has

$$\|\Lambda u - \Lambda v\|_F \leq \|\Lambda\|_{\mathcal{L}(E,F)} \|u - v\|_E, \quad \forall u, v \in E,$$

whence the continuity of Λ.

The implication $(ii) \Rightarrow (iii)$ is obvious.

Let now prove that $(iii) \Rightarrow (i)$. If Λ is continuous in $v_0 \in E$, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\|v - v_0\|_E < \delta \Longrightarrow \|\Lambda(v - v_0)\|_F < \varepsilon,$$

which, if $w = v - v_0$, reads

$$\|w\|_E < \delta \Longrightarrow \|\Lambda w\|_F < \varepsilon.$$

Consequently, setting $z = \frac{2}{\delta} w$, we can write

$$\sup_{\|z\|_E = 1} \|\Lambda z\|_F = \frac{2}{\delta} \sup_{\|w\|_E = \delta/2} \|\Lambda w\|_F \leq \frac{2}{\delta} \varepsilon.$$

and this implies (i), in view of Proposition 1.4. □

Definition 1.6. If E is a Banach space, the set of the linear and continuous maps from E into $\mathbb{R}$ is called the dual space of E and is denoted E'. If $x' \in E'$, the image $x'(x)$ of $x \in E$ is denoted by $\langle x', x \rangle_{E',E}$. The bracket $\langle \cdot, \cdot \rangle_{E',E}$ is called the duality pairing between E' and E.

The dual space $E'' = (E')'$ of E' is called the bidual of E.

An immediate consequence of Proposition 1.4 and Theorem 1.5 is the result:

Corollary 1.7. *The dual space E' is characterized as follows:*

$$E' = \mathcal{L}(E, \mathbb{R}).$$

and it is a Banach space for the norm

$$\|x'\|_{E'} = \sup_{x \in E \setminus \{0\}} \frac{|\langle x', x \rangle_{E',E}|}{\|x\|_E}, \quad \forall x' \in E'.$$

Moreover, one has

$$|\langle x', x \rangle_{E',E}| \leq \|x'\|_{E'} \|x\|_E, \quad \forall x \in E.$$

From this corollary it is obvious that E'' is a Banach space too. Generally, E can be identified with a subspace of E'' through a canonical isometry. Indeed,

Proposition 1.8. *Let $x \in E$ be fixed and introduce the map*

$$f_x : x' \in E' \longmapsto \langle x', x \rangle_{E',E} \in \mathbb{R}.$$

Then $f_x \in E''$ and the map

$$\mathcal{F} : x \in E \longmapsto f_x \in E''$$

is an isometry, i.e.

$$\|x\|_E = \|\mathcal{F}(x)\|_{E''} = \|f_x\|_{E''}.$$

Thanks to this result, one identifies x with f_x, and then E with its image $\mathcal{F}(E) \subset E''$.

Definition 1.9. Let $\mathcal{F}$ be the map defined by Proposition 1.8. The space E is called reflexive iff $\mathcal{F}(E) = E''$.

If E is reflexive, due to the above properties, we identify E and E''.

1.2 Weak convergence

In all this section E is a (real) Banach space equipped with the norm $\|\cdot\|_E$. In Definition 1.1 we introduced the notion of strong convergence with respect to this norm. Other notions of convergence can be defined on E. We are concerned here with that of weak convergence.

Definition 1.10. A sequence $\{x_n\}$ in E is said to *converge weakly* to x iff

$$\forall x' \in E', \quad \langle x', x_n \rangle_{E',E} \quad \to \quad \langle x', x \rangle_{E',E}.$$

This weak convergence is denoted

$$x_n \rightharpoonup x \quad \text{weakly in } E.$$

Remark 1.11. Let us mention that the uniqueness of the weak limit is a consequence of the Hahn–Banach theorem (see for instance Yosida, 1964, Chapter 6). ◊

Proposition 1.12. *Strong convergence implies weak convergence.*

Proof. Let $\{x_n\}$ be a sequence in E such that

$$x_n \to x \quad \text{strongly in } E.$$

Then, for any $x' \in E'$, thanks to Corollary 1.7 one has

$$\lim_{n\to\infty} |\langle x', x_n \rangle_{E',E} - \langle x', x \rangle_{E',E}| \leq \lim_{n\to\infty} \|x'\|_{E'} \|x_n - x\|_E = 0.$$

□

Proposition 1.13. *If* $\dim E = N < +\infty$, *the strong and the weak convergences are equivalent.*

Proof. Let $\{x_n\}$ be a sequence in E such that

$$x_n \rightharpoonup x_0 \quad \text{weakly in } E.$$

Let $(e_i)_{i=1}^N$ be a basis of E with $\|e_i\|_E = 1$ for any $i = 1, \dots, N$. Then, for any $y \in E$, there exist $y^1, \dots, y^N$ uniquely determined in $\mathbb{R}$ such that $y = \sum_{i=1}^N y^i e_i$. Hence, we can define the maps

$$f_i : y \longmapsto y^i \in \mathbb{R}, \qquad i = 1, \dots, N,$$

which are N elements in E'. From Definition 1.10 one has, in particular,

$$\lim_{n\to\infty} \langle f_i, x_n - x_0 \rangle_{E',E} = 0, \qquad i = 1, \dots, N. \tag{1.2}$$

But

$$\|x_n - x_0\|_E = \left\| \sum_{i=1}^n e_i \langle f_i, x_n - x_0 \rangle_{E',E} \right\|_E \leq \sum_{i=1}^N \|e_i \langle f_i, x_n - x_0 \rangle_{E',E}\|_E$$
$$\leq \sum_{i=1}^N |\langle f_i, x_n - x_0 \rangle_{E',E}|,$$

which with (1.2), gives the strong convergence of x_n to x_0.

This, together with Proposition 1.12, ends the proof. □

Proposition 1.13 is not true if $\dim E = +\infty$. The easiest way to see that is to construct some counterexamples. For instance, Examples 2.4 and 2.5 from Chapter 2 exhibit sequences which are weakly but not strongly convergent.

The following result is a particular case of the Banach–Steinhaus theorem. We refer to Yosida (1964) for a proof.

Proposition 1.14. *Let* $\{x_n\}$ *be a sequence weakly convergent to* x *in* E. *Then*

i) $\{x_n\}$ *is a bounded sequence in* E, *i.e. there exists a constant* C *independent of* n *such that*

$$\forall n \in \mathbb{N}, \qquad \|x_n\|_E \leq C.$$

ii) the norm on E *is lower semi-continuous with respect to the weak convergence, i.e.*

$$\|x\|_E \leq \liminf_{n\to\infty} \|x_n\|_E.$$

To investigate further properties of the weak convergence, we need the following definition:

Definition 1.15. We say that the Banach space E is *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left(x, y \in E,\ \|x\|_E \leq 1,\ \|y\|_E \leq 1,\ \|x-y\|_E > \varepsilon \right) \implies \left(\left\| \frac{x+y}{2} \right\|_E < 1 - \delta \right).$$

Remark 1.16. It can be proved (see for instance Rudin, 1966) that any uniformly convex Banach space is reflexive, the converse being not true. It is also easy to check that any Hilbert space is uniformly convex (and therefore reflexive). ◊

Proposition 1.17. *Let $\{x_n\}$ be a sequence in E. One has*

$$(a)\ \left(x_n \to x \quad \text{strongly in } E\right) \implies \quad (b) \begin{cases} i) & x_n \rightharpoonup x \quad \text{weakly in } E \\ ii) & \|x_n\|_E \to \|x\|_E. \end{cases}$$

Moreover, if E is uniformly convex, then $(a) \iff (b)$.

Proof. $(a) \implies (b)$. Proposition 1.12 gives b(i). Convergence b(ii) follows from the inequality

$$\left| \|x\|_E - \|y\|_E \right| \leq \|x-y\|_E.$$

$(b) \implies (a)$. Suppose now that E is uniformly convex. Clearly, if $x = 0$, from b(ii) one has the convergence to zero of $\|x_n\|_E$ which is by definition the strong convergence of $\{x_n\}$.

Let now $x \neq 0$. We will argue by contradiction. Suppose that

$$\limsup_{n\to\infty} \|x_n - x\|_E > 0.$$

Then, there exists a subsequence (for simplicity, still denoted by x_n) such that

$$\lim_{n\to\infty} \|x_n - x\|_E = \ell > 0. \tag{1.3}$$

We may assume that $\|x_n\|_E \neq 0$ and set $z_n = x_n/\|x_n\|_E$, $z = x/\|x\|_E$. Observe that, by definition, $\|z_n\|_E = \|z\|_E = 1$. Moreover, since for any $x' \in E'$,

$$\langle x', z_n \rangle_{E',E} = \frac{1}{\|x_n\|_E} \langle x', x_n \rangle_{E',E},$$

one has from (b) that $z_n \rightharpoonup z$. Consequently, $(z_n + z)/2 \rightharpoonup z$ weakly in E.

From Proposition 1.14(ii) it follows that

$$\begin{aligned} 1 = \|z\|_E \leq \liminf_{n\to\infty} \left\| \frac{z_n + z}{2} \right\|_E &\leq \limsup_{n\to\infty} \left\| \frac{z_n + z}{2} \right\|_E \\ &\leq \limsup_{n\to\infty} \left\| \frac{z_n}{2} \right\|_E + \left\| \frac{z}{2} \right\|_E = 1. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \left\| \frac{z_n + z}{2} \right\|_E = 1. \tag{1.4}$$

Let now

$$\varepsilon < \frac{\ell}{2\|x\|}.$$

Then, from (1.3) there exists n_0 such that, for $n > n_0$, one has

$$\|x_n - x\|_E > \ell - (\ell - 2\varepsilon\|x\|_E) = 2\varepsilon\|x\|_E.$$

Also, from hypothesis (b)(ii), there exists n_1 such that, for $n > n_1$, one has

$$\left|\|x_n\|_E - \|x\|_E\right| \leq \varepsilon\|x\|_E.$$

Then, for any $n > \max\{n_0, n_1\}$, we have successively

$$\begin{aligned} 2\varepsilon < \frac{\|x_n - x\|_E}{\|x\|_E} &\leq \left\| \frac{x_n}{\|x_n\|_E} - \frac{x}{\|x\|_E} \right\|_E + \left\| \frac{x_n}{\|x\|_E} - \frac{x_n}{\|x_n\|_E} \right\|_E \\ &= \|z_n - z\|_E + \frac{\left|\|x_n\|_E - \|x\|_E\right|}{\|x\|_E} \leq \|z_n - z\|_E + \varepsilon. \end{aligned}$$

Hence, one has for any $n > \max\{n_0, n_1\}$

$$\|z_n - z\|_E > \varepsilon.$$

Consequently, from Definition 1.15 we deduce the existence of some $\delta > 0$ such that

$$\lim_{n\to\infty} \left\| \frac{z_n + z}{2} \right\|_E \leq 1 - \delta < 1.$$

which is in contradiction with (1.4). □

The following theorem states one of the main properties of the weak convergence in reflexive Banach spaces. For the proof, which is rather technical, we refer again to Yosida (1964) or to Zeidler (1980).

Theorem 1.18 (Eberlein–Šmuljan). *Assume that E is reflexive and let $\{x_n\}$ be a bounded sequence in E. Then*

i) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that, as $k \to \infty$,

$$x_{n_k} \rightharpoonup x \quad \text{weakly in } E.$$

ii) *if each weakly convergent subsequence of* $\{x_n\}$ *has the same limit* x*, then the whole sequence* $\{x_n\}$ *weakly converges to* x*, i.e.*

$$x_n \rightharpoonup x \quad \text{weakly in } E.$$

The last result of this section will be used frequently throughout this book. It enables us to pass to the limit in products of 'weak–strong' convergent sequences.

Proposition 1.19. *Let* $\{x_n\} \subset E$ *and* $\{y_n\} \subset E'$ *such that*

$$\begin{cases} x_n \rightharpoonup x & \text{weakly in } E \\ y_n \to y & \text{strongly in } E'. \end{cases}$$

Then

$$\lim_{n\to\infty} \langle y_n, x_n\rangle_{E',E} = \langle y, x\rangle_{E',E}.$$

Proof. From Corollary 1.7 one has

$$\begin{aligned} &\lim_{n\to\infty} |\langle y_n, x_n\rangle_{E',E} - \langle y, x\rangle_{E',E}| \\ &\quad = \lim_{n\to\infty} |\langle y_n - y, x_n\rangle_{E',E} + \langle y, x_n - x\rangle_{E',E}| \\ &\quad \leq \lim_{n\to\infty} \|y_n - y\|_{E'} \|x_n\|_E + \lim_{n\to\infty} |\langle y, x_n - x\rangle_{E',E}| = 0, \end{aligned}$$

where, to pass to the limit, we have used Proposition 1.14(i). □

1.3 Weak* convergence

As can be seen from Definition 1.10, to check the weak convergence for a sequence of E, one needs to know what is the space E'. It may happen that E' is 'too big' a space. This renders the verification of the weak convergence condition too difficult. Moreover, in this case, there are too 'few' weakly convergent sequences. This situation leads to the more general following notion of weak* convergence:

Definition 1.20. Let F be a Banach space and set $E = F'$. A sequence $\{x_n\}$ in E is said to *converge weakly** to x iff

$$\langle x_n, x'\rangle_{F',F} \to \langle x, x'\rangle_{F',F}, \quad \forall x' \in F. \tag{1.5}$$

This weak* convergence is denoted

$$x_n \rightharpoonup x \qquad \text{weakly* in } E.$$

Remark 1.21. The uniqueness of the weak* limit is immediate. Indeed, if the sequence $\{x_n\}$ has two weak* limits x and y, then from Definition 1.20 one must have $\langle x, x'\rangle_{F',F} = \langle y, x'\rangle_{F',F}$, for all $x' \in F$, which implies that $x = y$ in F'. ◊

Proposition 1.22. *Let* F *be a Banach space and* $E = F'$*. Then any weakly convergent sequence in* E *is also weakly* convergent.*

Proof. Let $\{x_n\}$ be a sequence of $E = F'$ such that

$$x_n \rightharpoonup x \quad \text{weakly in } E.$$

Then, by definition

$$\langle x', x_n \rangle_{F'', F'} \to \langle x', x \rangle_{F'', F'}, \quad \forall x' \in F''.$$

This implies (1.5), since $F \subset F''$. □

Remark 1.23. From Definition 1.20 and the proof of Proposition 1.22, it is clear that the two notions of convergence are a priori not equivalent since in general, the inclusion $F \subset F''$ is strict (see for details Akilov and Kantorovich, 1981). Obviously, if the space F is reflexive, weak convergence and weak* convergence are equivalent. ◊

The main properties of weak convergence are still valid for weak* convergence with analogous proofs. In particular, the results from Section 1.1 read as follows:

Proposition 1.24. *Let $\{x_n\}$ be a sequence weakly* convergent to x in $E = F'$ where F is a Banach space. Then*

i) *$\{x_n\}$ is a bounded sequence in E, i.e. there exists a constant C independent of n such that*

$$\forall n \in \mathbb{N}, \qquad \|x_n\|_E \leq C,$$

ii) *the norm is lower semi-continuous with respect to the weak* convergence, i.e.*

$$\|x\|_E \leq \liminf_{n \to \infty} \|x_n\|_E.$$

To give the equivalent of Theorem 1.18 for weak* convergence, we need to introduce another definition.

Definition 1.25. We say that the Banach space F is *separable* if there exists a set, at most countable, which is dense in F.

Then, the following result holds true:

Theorem 1.26. *Let F be a separable Banach space and let $E = F'$. If $\{x_n\}$ is a bounded sequence in E, then*

i) *there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that, as $k \to \infty$,*

$$x_{n_k} \rightharpoonup x \quad \text{weakly* in } E,$$

ii) *if each weakly* convergent subsequence of $\{x_n\}$ has the same limit x, then the whole sequence $\{x_n\}$ weakly* converges to x, i.e.*

$$x_n \rightharpoonup x \quad \text{weakly* in } E.$$

One also has a result concerning products of 'weak*–strong' convergent sequences.

Proposition 1.27. *Let $\{x_n\}$ be a sequence in $E = F'$ where F is a Banach space and $\{y_n\}$ a sequence in E' such that*

$$\begin{cases} y_n \to y & \text{strongly in } F \\ x_n \rightharpoonup x & \text{weakly* in } E. \end{cases}$$

Then

$$\lim_{n\to\infty} \langle x_n, y_n\rangle_{F',F} = \langle x, y\rangle_{F',F}.$$

1.4 Some properties of L^p-spaces

In this section we give a short presentation of L^p-spaces. We suppose known the basic properties of Lebesgue measure and integration theory (for details we refer the reader to Rudin, 1966). In the sequel, $\mathbb{R}^N$ is equipped with the Lebesgue measure dx. We will denote by $|\omega|$ the Lebesgue measure of a measurable set ω. As usual, we will identify two integrable functions which are almost everywhere equal.

From now on, $\mathcal{O}$ and Ω denote respectively, an open set and a bounded open set in $\mathbb{R}^N$. Let us recall in this section the definition of the space $\mathcal{D}(\mathcal{O})$. For more properties of this space, we refer the reader to Chapter 3.

Definition 1.28. For any function $\varphi : \mathcal{O} \mapsto \mathbb{R}$, the *support* of φ, denoted by $\operatorname{supp}\varphi$, is defined as the following closed set of $\mathcal{O}$:

$$\operatorname{supp}\ \varphi = \overline{\{x \in \mathcal{O},\ \varphi \neq 0\}} \cap \mathcal{O}.$$

We denote by $\mathcal{D}(\mathcal{O})$ the set of indefinitely differentiable functions whose support is a compact set of $\mathbb{R}^N$ contained in $\mathcal{O}$.

We denote also by $C_c^0(\mathcal{O})$ the set of continuous functions whose support is a compact set of $\mathbb{R}^N$ contained in $\mathcal{O}$.

Remark 1.29. Let us observe that one can construct functions in $\mathcal{D}(\mathcal{O})$ having an arbitrarily small support. For instance, for any $a > 0$, the function φ defined on $\mathbb{R}^N$ by

$$\varphi(x) = \begin{cases} e^{-\frac{a^2}{a^2-|x|^2}} & \text{if } |x| < a \\ 0 & \text{if } |x| \geq a, \end{cases}$$

is clearly in $C^\infty(\mathbb{R}^N)$ and its support is the closed ball centred at the origin and of radius a.

Notice that in the literature $\mathcal{D}(\mathcal{O})$ is often denoted by $C_0^\infty(\mathcal{O})$. ◇

Definition 1.30. Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$. Define

$$L^p(\mathcal{O}) = \left\{ f \mid f : \mathcal{O} \longmapsto \mathbb{R}, f \text{ measurable and such that } \int_{\mathcal{O}} |f(x)|^p \, dx < +\infty \right\}$$

$$L^\infty(\mathcal{O}) = \left\{ f \mid f : \mathcal{O} \longmapsto \mathbb{R}, f \text{ measurable and such that there exists } C \in \mathbb{R} \right.$$
$$\left. \text{with } |f| \leq C, \text{ a.e. on } \mathcal{O} \right\}.$$

Define also

$$L^p_{\text{loc}}(\mathcal{O}) = \{ f \mid f \in L^p(\omega), \text{ for any open bounded set } \omega \text{ with } \overline{\omega} \subset \mathcal{O} \}.$$

The next two propositions give the main properties of L^p-spaces.

Proposition 1.31. *Let $p \in \mathbb{R}$ with $1 \leq p \leq +\infty$. The set $L^p(\mathcal{O})$ is a Banach space for the norm*

$$\|f\|_{L^p(\mathcal{O})} = \begin{cases} \left[\int_{\mathcal{O}} |f(x)|^p \, dx \right]^{\frac{1}{p}} & \text{if } p < +\infty \\ \inf\{C, \ |f| \leq C \text{ a.e. on } \mathcal{O}\} & \text{if } p = +\infty. \end{cases}$$

If $p = 2$, the space $L^2(\mathcal{O})$ is a Hilbert space for the scalar product

$$(f, g)_{L^2(\mathcal{O})} = \int_{\mathcal{O}} f(x) \, g(x) \, dx.$$

Proposition 1.32. *The space $L^p(\mathcal{O})$ is separable for $1 \leq p < +\infty$, and is uniformly convex for $1 < p < +\infty$.*

Remark 1.33. Taking into account Remark 1.16, it follows that $L^p(\mathcal{O})$ for $1 < p < +\infty$, is reflexive. Note that the space $L^1(\mathcal{O})$ is not reflexive and also, that $L^\infty(\mathcal{O})$ is neither reflexive nor separable. ◊

Proposition 1.34 (Hölder inequality). *Let $1 \leq p \leq +\infty$ and p' be its conjugate, i.e.*

$$\begin{cases} \dfrac{1}{p'} = 1 - \dfrac{1}{p} & \text{if } 1 < p < +\infty \\ p' = 1 & \text{if } p = +\infty \\ p' = +\infty & \text{if } p = 1. \end{cases}$$

Then,

$$\int_{\mathcal{O}} |f(x) \, g(x)| \, dx \leq \|f\|_{L^p(\mathcal{O})} \|g\|_{L^{p'}(\mathcal{O})},$$

for any $f \in L^p(\mathcal{O})$ and $g \in L^{p'}(\mathcal{O})$. For $p = 2$, this inequality is called the Cauchy-Schwarz inequality.

An easy consequence of this inequality are the following inclusions:

Corollary 1.35. *Let $1 \leq p \leq q \leq +\infty$. Then,*

$$L^q(\Omega) \subset L^p(\Omega)$$

with

$$\|f\|_{L^p(\Omega)} \leq c\|f\|_{L^q(\Omega)},$$

where the constant c depends on $|\Omega|$, p and q.

Theorem 1.36 (Representation theorem). *Let $1 \leq p < +\infty$ and p' be its conjugate. Let $f \in [L^p(\mathcal{O})]'$. Then, there exists a unique $g \in L^{p'}(\mathcal{O})$ such that*

$$\langle f, \varphi \rangle_{[L^p(\mathcal{O})]', L^p(\mathcal{O})} = \int_{\mathcal{O}} g(x)\, \varphi(x)\, dx, \quad \forall \varphi \in L^p(\mathcal{O}).$$

Moreover

$$\|g\|_{L^{p'}(\mathcal{O})} = \|f\|_{[L^p(\mathcal{O})]'}.$$

Remark 1.37. Due to this theorem, the space $[L^p(\mathcal{O})]'$ can be identified with $L^{p'}(\mathcal{O})$ for $1 \leq p < +\infty$, so that in particular, $[L^1(\mathcal{O})]' = L^\infty(\mathcal{O})$. Let us point out that, on the contrary $[L^\infty(\mathcal{O})]' \neq L^1(\mathcal{O})$. One has in fact, $L^1(\mathcal{O}) \subset [L^\infty(\mathcal{O})]'$ strictly. (We refer to Brezis, 1987, for the proof of this result). The space $[L^\infty(\mathcal{O})]'$ has a complicate structure (see Akilov and Kantorovich, 1981, for a characterization of this space). ◇

There are many sets of functions, useful in applications, which are dense in L^p-spaces. In particular, we will make use in the sequel of the following result:

Theorem 1.38. *$\mathcal{D}(\mathcal{O})$ is dense in $L^p(\mathcal{O})$, for $1 \leq p < +\infty$.*

Remark 1.39. This density result does not hold true for $p = \infty$. Indeed, if $\{f_n\}$ is a sequence in $\mathcal{D}(\mathcal{O})$ that strongly converges to f in $L^\infty(\mathcal{O})$, then necessarily, f would be continuous, since the uniform convergence preserve the continuity at the limit. But, obviously, a function in $L^\infty(\mathcal{O})$ is not necessarily continuous. ◇

To state another very important density result, we need to introduce the following two definitions:

Definition 1.40. Let A be a measurable set in $\mathbb{R}^N$. The *characteristic function* of A is the function χ_A defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \mathbb{R}^N \setminus A. \end{cases}$$

Definition 1.41. A function $f : \mathbb{R}^N \mapsto \mathbb{R}$ is called a *step function* if

$$f(x) = \sum_{k=1}^{m} \alpha_k \chi_{I_k},$$

with $m \in \mathbb{N}$, $\alpha_k \in \mathbb{R}$ and where I_k is an interval in $\mathbb{R}^N$, for any $k \in \{1, \dots, m\}$.

If $\Omega \subset \mathbb{R}^N$ is a bounded open set, we denote by $\mathcal{S}(\Omega)$ the set of step functions of the form $\sum_{k=1}^{m} \alpha_k \chi_{I_k}$, such that $I_k \subset \Omega$, for any $k \in \{1, \dots, m\}$.

Obviously, $\mathcal{S}(\Omega) \subset L^p(\Omega)$ for any p such that $1 \le p \le +\infty$. Furthermore, we have the following density result (see Rudin, 1966):

Theorem 1.42. *If $1 \le p < +\infty$, $\mathcal{S}(\Omega)$ is dense in $L^p(\Omega)$.*

Remark 1.43. Observe that this theorem implies in particular that $L^p(\Omega)$ is dense in $L^1(\Omega)$. ◊

We end this section by a result which will be widely used in the sequel.

Theorem 1.44. *If $f \in L^1_{\text{loc}}(\mathcal{O})$ is such that*

$$\int_{\mathcal{O}} f(x)\, \varphi(x)\, dx = 0, \quad \forall \varphi \in \mathcal{D}(\mathcal{O}),$$

then $f = 0$, a.e. on $\mathcal{O}$.

1.5 Weak convergence in L^p for $1 < p < \infty$

Let $\{u_n\}$ be a sequence in $L^p(\Omega)$ with $1 < p < \infty$. In this case, due to Definition 1.10 and Remark 1.37, the weak convergence

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega)$$

signifies that

$$\int_\Omega u_n\, \varphi\, dx \longrightarrow \int_\Omega u\, \varphi\, dx, \qquad \forall\, \varphi \in L^{p'}(\Omega),$$

with $1/p + 1/p' = 1$.

Remark 1.45. Since for $1 < p < \infty$, $L^p(\Omega) = \big(L^{p'}(\Omega)\big)'$, the weak convergence is equivalent to the weak* convergence. This follows from Remarks 1.23 and 1.33. Moreover, again for $1 < p < \infty$, Theorem 1.18 shows that any bounded set in $L^p(\Omega)$ is weakly compact. ◊

The next result of this section gives a characterization of the weak convergence in the space $L^p(\Omega)$. It will be often used in the study of periodic oscillating functions, which are discussed in Chapter 2.

Proposition 1.46. *Let $1 < p < \infty$ and $\{u_n\}$ be a sequence in $L^p(\Omega)$. Then, the following equivalence holds true:*

$$(a)\ \left(u_n \rightharpoonup u \text{ weakly in } L^p(\Omega)\right) \iff$$
$$(b) \begin{cases} i)\ \|u_n\|_{L^p(\Omega)} \le C & \text{(independently of } n), \\ ii)\ \displaystyle\int_I u_n\, dx \longrightarrow \int_I u\, dx, & \text{for any interval } I \subset \Omega. \end{cases}$$

Proof. Suppose that (a) holds. Then, (i) follows from Proposition 1.14 and (ii) is obtained by testing the weak convergence for the function $\varphi = \chi_I$. Hence $(a) \Longrightarrow (b)$.

Assume now that (b) holds. Let $\varphi \in L^{p'}(\Omega)$, with $1/p + 1/p' = 1$. From Theorem 1.42, for any positive η, there exists a step function φ_η such that

$$\|\varphi - \varphi_\eta\|_{L^{p'}(\Omega)} \le \eta,$$

with

$$\varphi_\eta = \sum_{k=1}^{m} \alpha_k \chi_{I_k},$$

where $m \in \mathbb{N}$, $\alpha_k \in \mathbb{R}$ and I_k is an open interval in Ω. for any $k \in \{1, \dots, m\}$. Then,

$$\int_\Omega (u_n - u)\, \varphi\, dx = \int_\Omega (u_n - u)\, \varphi_\eta\, dx + \int_\Omega (u_n - u)\, (\varphi - \varphi_\eta)\, dx. \tag{1.6}$$

From (ii) we have, as $n \to \infty$,

$$\int_\Omega (u_n - u)\, \varphi_\eta\, dx = \sum_{k=1}^{m} \alpha_k \int_{I_k} (u_n - u)\, dx \longrightarrow 0.$$

From (i), the definition of φ_η and the Hölder inequality, one easily has that

$$\int_\Omega (u_n - u)\, (\varphi - \varphi_\eta)\, dx \le C_1 \eta.$$

where C_1 is independent of n and η. Then, (a) follows from (1.6) by making first $n \to \infty$ and then $\eta \to 0$. □

1.6 Weak convergence in L^1

Due to Definition 1.10 and Remark 1.37, the weak convergence

$$u_n \rightharpoonup u \quad \text{weakly in } L^1(\Omega)$$

means that

$$\int_\Omega u_n\, \varphi\, dx \longrightarrow \int_\Omega u\, \varphi\, dx, \qquad \forall\, \varphi \in L^\infty(\Omega).$$

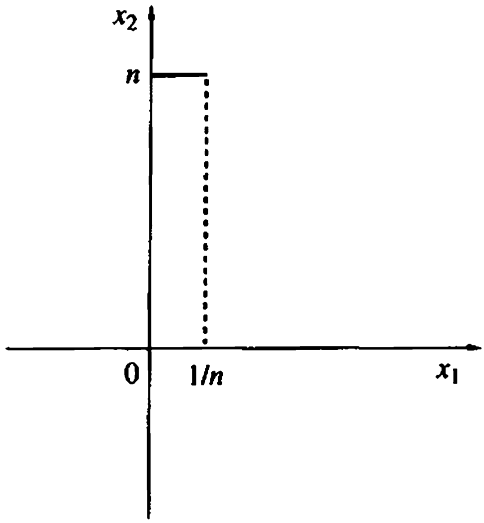


Fig. 1.1

Since $L^1(\Omega)$ is not reflexive (see Remark 1.33), the weak compactness Theorem 1.18 does not apply. This makes the study of bounded sequences in $L^1(\Omega)$ quite difficult. The following example exhibits a bounded sequence in $L^1(\Omega)$ from which one can not extract any weakly convergent sequence in $L^1(\Omega)$.

Example 1.47. Let u_n be the function defined by (see Fig. 1.1)

$$u_n(x) = \begin{cases} n & 0 \le x \le \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, u_n is in $L^1(-1,1)$ with

$$\|u_n\|_{L^1(-1,1)} = 1.$$

Let $\varphi \in C_c^0(-1,1)$ (see Definition 1.28). Then,

$$\langle u_n, \varphi \rangle_{L^1(-1,1),L^\infty(-1,1)} = n \int_0^{\frac{1}{n}} \varphi(x)\, dx \longrightarrow \varphi(0),$$

due to the mean value theorem. One can show (see Remark 1.49 below) that there is no function $u_0 \in L^1(-1,1)$, such that

$$\int_{-1}^{1} u_0(x)\, \varphi(x)\, dx = \varphi(0), \quad \forall \varphi \in C_c^0(-1,1).$$

This means that $\{u_n\}$ does not converge weakly in $L^1(-1,1)$. ◊

Let us now consider the dual space of $C_c^0(\Omega)$ introduced in Definition 1.28. It is known that $[C_c^0(\Omega)]' = M(\Omega)$, where $M(\Omega)$ is the set of positive measures (called Radon measures) on the bounded domain Ω. We have the following result which characterizes the limit points of a bounded sequence in $L^1(\Omega)$:

Proposition 1.48. *Let $\{u_n\}$ be a bounded sequence in $L^1(\Omega)$. Then, $\{u_n\}$ is weakly* compact in $M(\Omega)$, i.e. there exists a subsequence $\{u_{n_k}\}$ and $u \in M(\Omega)$ such that*

$$\lim_{k\to\infty} \int_\Omega u_{n_k}\varphi\, dx = \langle u, \varphi\rangle_{M(\Omega), C_c^0(\Omega)}, \quad \forall \varphi \in C_c^0(\Omega).$$

Proof. The result is a consequence of the fact that $L^1(\Omega)$ can be identified with a subspace of $M(\Omega)$. Indeed, the map

$$T: f \in L^1(\Omega) \longmapsto Tf \in M(\Omega),$$

with Tf defined by

$$\langle Tf, \varphi\rangle_{M(\Omega), C_c^0(\Omega)} = \int_\Omega f\varphi\, dx, \quad \forall \varphi \in C_c^0(\Omega),$$

satisfies

$$\|Tf\|_{M(\Omega)} = \|f\|_{L^1(\Omega)}.$$

This means that T is an isometry, so that Tf is identified with f and we can write $L^1(\Omega) \subset M(\Omega)$. Consequently, we can apply Proposition 1.24 to $\{u_n\}$ which is also a bounded sequence in $M(\Omega)$. □

Remark 1.49. We can now make more precise the comment on the limit of the sequence $\{u_n\}$ introduced in Example 1.47. Indeed, Proposition 1.48 shows that $\{u_n\}$ is weakly* convergent in $M(-1,1)$ to the measure δ_0, defined by

$$\langle \delta_0, \varphi\rangle_{M(-1,1), [C_c^0(-1,1)]'} = \varphi(0).$$

It can be shown that δ_0, called the Dirac function at the origin, is not in $L^1(-1,1)$. ◊

Remark 1.50. Since L^1 cannot be characterized as the dual of some Banach space, the notion of weak* convergence is not interesting in this space. ◊

At this point, one can ask under which conditions a bounded sequence in $L^1(\Omega)$ is weakly compact. To answer this question, we need the following definition:

Definition 1.51. Let Ω be a bounded open set in $\mathbb{R}^N$ and $\{u_n\}$ a sequence in $L^1(\Omega)$. The functions u_n are *equi-integrable* if, for any $\eta > 0$, there exists $\delta > 0$ such that

$$\forall n \in \mathbb{N}, \qquad \int_E |u_n(x)|\, dx < \eta, \quad \text{for any } E \subset \Omega \text{ with } |E| < \delta.$$

Then the answer to the above question is

Proposition 1.52 (Dunford–Pettis). *Let $\{u_n\}$ be a sequence in $L^1(\Omega)$. Then,*

($\{u_n\}$ is weakly compact in $L^1(\Omega)$) *is equivalent to:*

$$\begin{cases} i) & \{u_n\} \text{ is bounded in } L^1(\Omega) \\ ii) & \text{the functions } u_n \text{ are equi-integrable.} \end{cases}$$

For the proof we refer to Dunford and Schwartz (1958) or Edwards (1965).

1.7 Weak* convergence in L^∞

From Definition 1.10, the weak convergence

$$u_n \rightharpoonup u \quad \text{weakly in } L^\infty(\Omega)$$

means that

$$\langle \varphi, u_n \rangle_{[L^\infty(\Omega)]', L^\infty(\Omega)} \longrightarrow \langle \varphi, u \rangle_{[L^\infty(\Omega)]', L^\infty(\Omega)}, \quad \forall\, \varphi \in [L^\infty(\Omega)]'.$$

As mentioned in Remark 1.37, the space $[L^\infty(\Omega)]'$ has a complicated structure so that it is very difficult to check this convergence. On the other hand, since $[L^1(\Omega)]' = L^\infty(\Omega)$, weak* convergence is the convenient notion for this case.

From Definition 1.20, it follows that

$$u_n \rightharpoonup u \quad \text{weakly* in } L^\infty(\Omega)$$

is equivalent to

$$\int_\Omega u_n\, \varphi\, dx \to \int_\Omega u\, \varphi\, dx, \quad \forall\, \varphi \in L^1(\Omega).$$

Since $L^1(\Omega)$ is not reflexive, weak convergence and weak* convergence in $L^\infty(\Omega)$ are not equivalent.

Remark 1.53. From Corollary 1.35 and Theorem 1.18, it follows that the weak* convergence of a sequence $\{u_n\}$ in $L^\infty(\Omega)$ to some element $u \in L^\infty(\Omega)$, implies the weak convergence of $\{u_n\}$ to u in any $L^p(\Omega)$ with $1 \le p < +\infty$. ◊

Remark 1.54. The space $L^1(\Omega)$ being separable (see Proposition 1.32), Theorem 1.26 implies that from any bounded sequence in $L^\infty(\Omega)$ one can extract a subsequence weakly* convergent in $L^\infty(\Omega)$. ◊

A last result in this section is the equivalent of Proposition 1.46 for the case $p = \infty$. We have

Proposition 1.55. *Let $\{u_n\}$ be a sequence in $L^\infty(\Omega)$. Then, one has the following equivalence:*

(a) $\left(u_n \rightharpoonup u \ \text{ weakly* in } L^\infty(\Omega)\right) \Longleftrightarrow$

$$(b)\begin{cases} i) \ \|u_n\|_{L^\infty(\Omega)} \le C & \text{(independently of } n) \\ ii) \ \displaystyle\int_I u_n\, dx \longrightarrow \int_I u\, dx & \text{for any interval } I \subset \Omega. \end{cases}$$

Proof. Suppose that (a) holds. Then, (i) follows from Proposition 1.24 and (ii) is obtained by testing the weak convergence for the function $\varphi = \chi_I$. Hence $(a) \Longrightarrow (b)$. The implication $(b) \Longrightarrow (a)$ follows by the same argument as that used in the proof of Proposition 1.46. □

2

Rapidly oscillating periodic functions

In this chapter we study a relevant class of periodic oscillating functions, which plays an essential role in homogenization theory. We turn our attention, in particular, to functions of the form

$$a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$$

where a is a periodic function and where, from now on, $\varepsilon > 0$ takes its values in a sequence which tends to zero. Let us point out that if a is Y-periodic (see Definition 2.1 below), then a_ε is εY-periodic. Moreover, as can be seen in the examples below, the smaller ε is, the more rapid are the oscillations. Therefore, a natural question is to describe the behaviour of the sequence $\{a_\varepsilon\}$ as $\varepsilon \to 0$. This is the aim of Section 2.3.

2.1 Periodic functions in L^1

Throughout this book, Y will denote the interval in $\mathbb{R}^N$ defined by

$$Y =]0, \ell_1[\times \cdots \times]0, \ell_N[. \tag{2.1}$$

where $\ell_1, \ldots, \ell_N$ are given positive numbers. We will refer to Y as the reference period.

The following definition introduces the notion of periodicity for functions which are defined almost everywhere.

Definition 2.1. Let Y be defined by (2.1) and f a function defined a.e. on $\mathbb{R}^N$. The function f is called *Y-periodic* iff

$$f(x + k\,\ell_i\, e_i) = f(x) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall k \in \mathbb{Z}, \quad \forall i \in \{1, \ldots, N\},$$

where $\{e_1, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^N$.

In the case $N = 1$, we simply say that f is ℓ_1-periodic.

The mean value of a periodic function is essential when studying periodic oscillating functions. Let us recall its definition.

Definition 2.2. Let Ω be a bounded open set of $\mathbb{R}^N$ and f a function in $L^1(\Omega)$. The *mean value* of f over Ω is the real number $\mathcal{M}_\Omega(f)$ given by

$$\mathcal{M}_\Omega(f) = \frac{1}{|\Omega|} \int_\Omega f(y)\, dy.$$

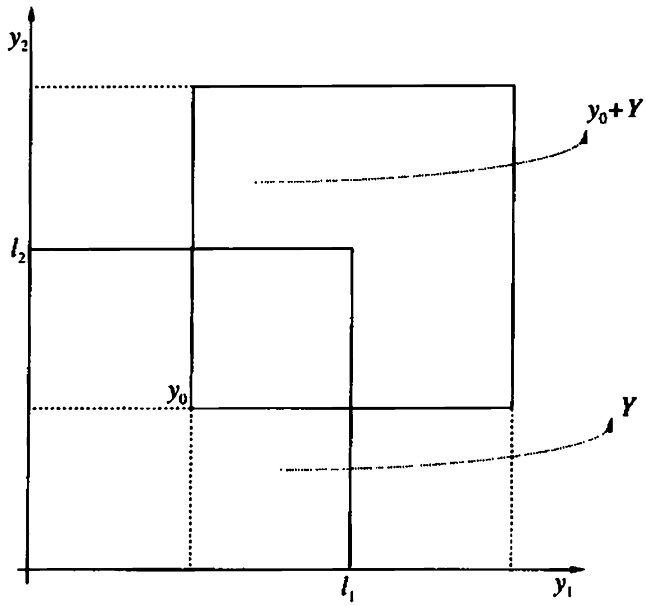


Fig. 2.1

The following lemma shows that the mean value of a periodic function can be computed on any translated set of the reference period:

Lemma 2.3. *Let f be a Y-periodic function in $L^1(Y)$. Let y_0 be a fixed point in $\mathbb{R}^N$ and denote by Y_0 the translated set of Y, defined by*

$$Y_0 = y_0 + Y.$$

Set

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N.$$

Then

$$\begin{cases} i) & \displaystyle\int_{Y_0} f(y)\,dy = \int_Y f(y)\,dy, \\ ii) & \displaystyle\int_{\varepsilon Y_0} f_\varepsilon(x)\,dx = \int_{\varepsilon Y} f_\varepsilon(x)\,dx = \varepsilon^N \int_Y f(y)\,dy. \end{cases} \tag{2.2}$$

Proof. If $y_0 = (y_0^1, \ldots, y_0^N)$, then one has $Y_0 =]y_0^1, y_0^1 + \ell_1[\times \cdots \times]y_0^N, y_0^N + \ell_N[$ (see Fig. 2.1).

Let $i \in \{1, \ldots, N\}$ be fixed and set

$$\Phi(y_i) = \int_{\Pi_{j \neq i}]y_0^j, y_0^j + \ell_j[} f\,dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_N,$$

which is an ℓ_i-periodic function.

Then, one has by periodicity

$$\begin{aligned}\int_{Y_0} f\,dy &= \int_{y_0^1}^{y_0^1+\ell_1} \Phi(y_1)\,dy_1 \\ &= -\int_0^{y_0^1} \Phi(y_1)\,dy_1 + \int_{\ell_1}^{y_0^1+\ell_1} \Phi(y_1)\,dy_1 + \int_0^{\ell_1} \Phi(y_1)\,dy_1 \\ &= \int_0^{\ell_1} \Phi(y_1)\,dy_1 = \int_0^{\ell_1} \int_{\Pi_{j\neq 1}]y_0^j, y_0^j+\ell_j[} f\,dy.\end{aligned}$$

Observe now that

$$\int_{\Pi_{j\neq 1}]y_0^j, y_0^j+\ell_j[} f\,dy_2\cdots dy_N = \int_{y_0^2}^{y_0^2+\ell_2} \int_{\Pi_{j>2}]y_0^j, y_0^j+\ell_j[} f\,dy_2\cdots dy_N.$$

Making a similar computation in the direction y_2, it is easily seen that

$$\int_{\Pi_{j\neq 1}]y_0^j, y_0^j+\ell_j[} f\,dy_2\cdots dy_N = \int_0^{\ell_2} \int_{\Pi_{j>2}]y_0^j, y_0^j+\ell_j[} f\,dy_2\cdots dy_N.$$

Hence,

$$\int_{Y_0} f\,dy = \int_0^{\ell_1}\int_0^{\ell_2} \int_{\Pi_{j>2}]y_0^j, y_0^j+\ell_j[} f\,dy_2\cdots dy_N.$$

Then (i) follows repeating successively the same argument in the directions $y_3, \ldots, y_N$. By a change of variables, assertion (ii) is straightforward. □

2.2 Examples

The following classical examples are very significant:

Example 2.4. Let $v(y)$ be the periodic function of period 1, defined on $\mathbb{R}$ by

$$v(y) = \sin(2\pi y)$$

and set

$$v_\varepsilon(x) = v\left(\frac{x}{\varepsilon}\right) = \sin\left(2\pi\frac{x}{\varepsilon}\right), \quad x \in]a, b[,$$

where $a, b \in \mathbb{R}$. Observe that if for instance, $a = 0$, $b = 2$ and ε takes its values in the sequence $\{1/2^n\}$ where $n \in \mathbb{N}$. for $n = 0, 1, 2$, we have the pictures drawn in Figs 2.2–2.4. From the figures it is clear that, as $\varepsilon \to 0$, $\sin(2\pi x/\varepsilon)$ cannot converge in almost any point. Applying Proposition 1.46, one shows in particular that

$$v_\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(a, b). \tag{2.3}$$

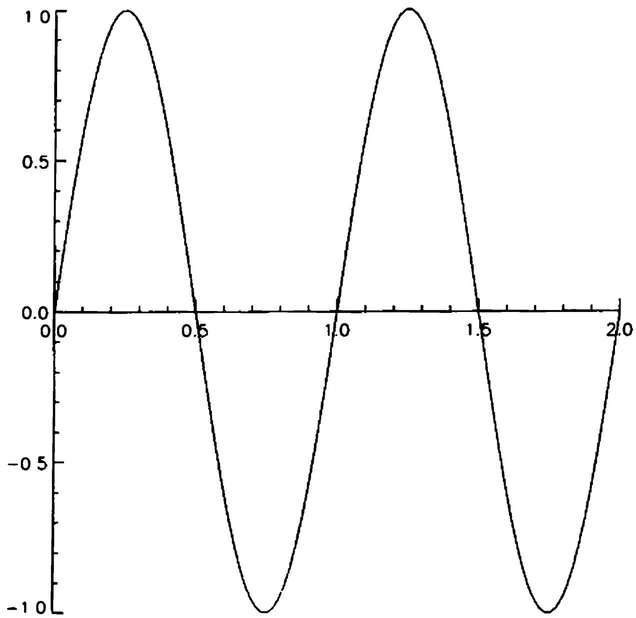


Fig. 2.2 ($n = 0$)

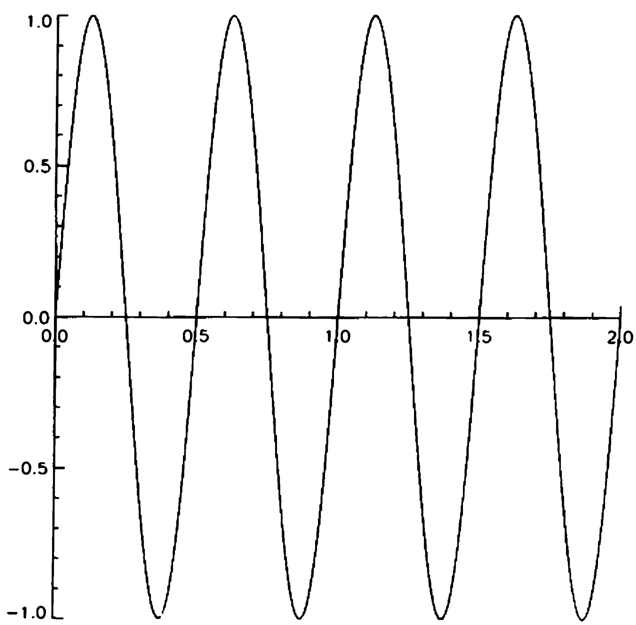


Fig. 2.3 ($n = 1$)

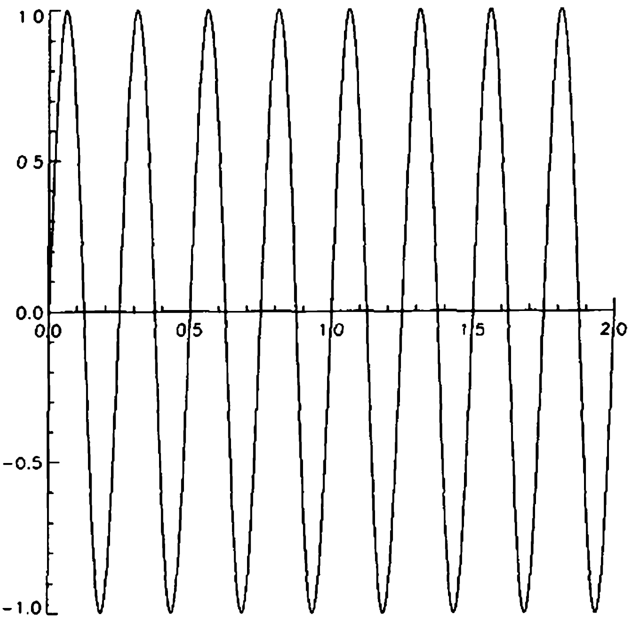


Fig. 2.4 ($n = 2$)

Indeed, it is obvious that the sequence $\{v_\varepsilon\}$ is bounded independently of ε in $L^2(a, b)$. On the other hand, for any interval $I =]a_1, b_1[\subset]a, b[$, one has

$$\int_{a_1}^{b_1} \sin\left(2\pi\frac{x}{\varepsilon}\right) dx = -\frac{\varepsilon}{2\pi}\cos\left(2\pi\frac{x}{\varepsilon}\right)\Big|_{a_1}^{b_1} \longrightarrow 0.$$

Hence, (*b*) from Proposition 1.46 holds, and so (2.3) is proved.

Let us remark that this convergence is not strong in $L^2(a, b)$. Indeed,

$$\begin{aligned} \|v_\varepsilon - 0\|^2_{L^2(a,b)} &= \int_a^b \sin^2\left(2\pi\frac{x}{\varepsilon}\right) dx = \frac{\varepsilon}{2\pi}\int_{\frac{2\pi a}{\varepsilon}}^{\frac{2\pi b}{\varepsilon}} \sin^2 y\, dy \\ &= \frac{\varepsilon}{2\pi}\int_{\frac{2\pi a}{\varepsilon}}^{\frac{2\pi b}{\varepsilon}} \frac{1 - \cos 2y}{2}\, dy = \frac{b-a}{2} + \frac{\varepsilon}{8\pi}\left[-\sin\frac{4\pi b}{\varepsilon} + \sin\frac{4\pi a}{\varepsilon}\right], \end{aligned}$$

so that, as $\varepsilon \to 0$,

$$\|v_\varepsilon - 0\|^2_{L^2(a,b)} \longrightarrow \frac{b-a}{2} \neq 0.$$

◇

Example 2.5. Let $v(y)$ be the periodic function of period 2, defined on $]0, 2[$ by

$$v(y) = \begin{cases} \alpha & \text{if } y \in (0, \frac{2}{3}) \\ \beta & \text{otherwise,} \end{cases}$$

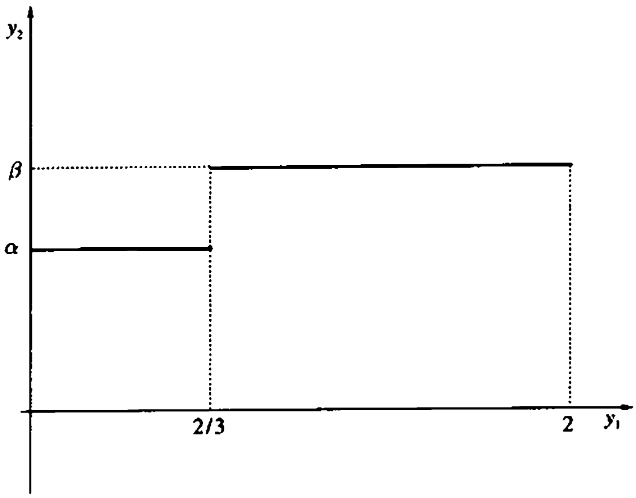


Fig. 2.5 ($n = 0$)

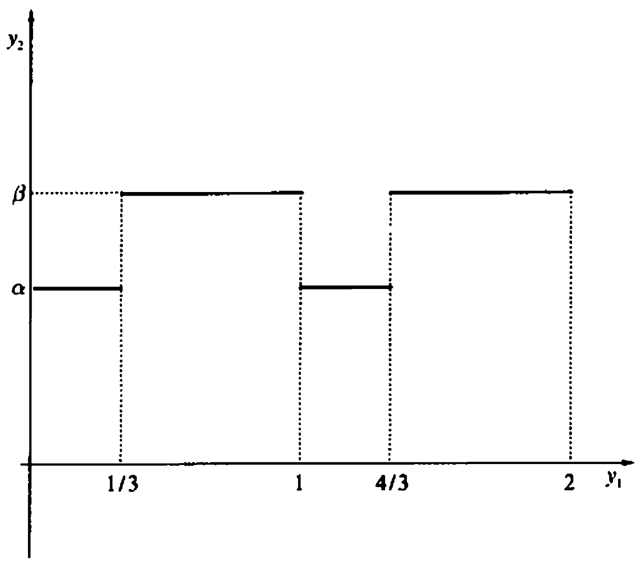


Fig. 2.6 ($n = 1$)

with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$. As in the previous example, we set

$$v_\varepsilon(x) = v\left(\frac{x}{\varepsilon}\right), \quad x \in]a, b[,$$

where $a, b \in \mathbb{R}$. Let us draw again (Figs 2.5–2.7) its graph for $n = 0, 1, 2$, in the case where $a = 0$, $b = 2$ and ε takes its values in the sequence $\{1/2^n\}$, $n \in \mathbb{N}$. Here also, one can easily see that if $\varepsilon \to 0$, v_ε cannot converge almost everywhere.

The sequence $\{v_\varepsilon\}$ is obviously bounded independently of ε in $L^2(a, b)$. We

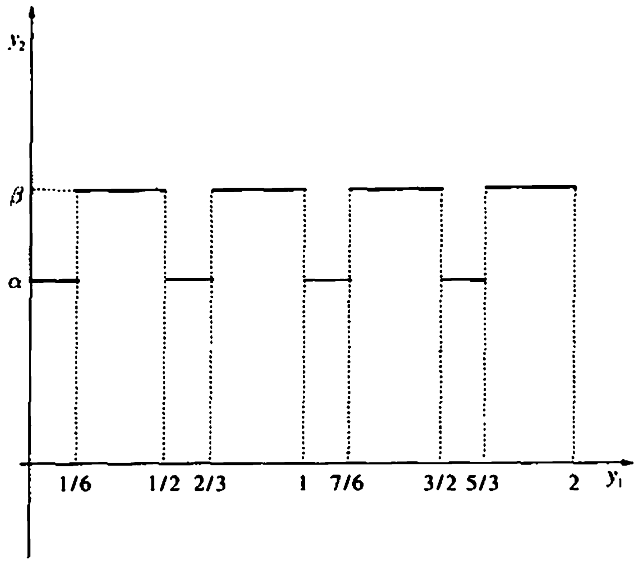


Fig. 2.7 $(n = 2)$

would like to apply Proposition 1.46 to this sequence. To do so, we need to verify assertion (ii) from this proposition. Let $I =]a_1, b_1[$ be an arbitrary interval in $]a, b[$ and let us compute

$$\mathcal{I}_\varepsilon = \int_{a_1}^{b_1} v_\varepsilon(x)\, dx.$$

For any positive ε, there exist k and θ such that

$$b_1 = a_1 + 2\,k\,\varepsilon + \theta\,\varepsilon, \quad k \in \mathbb{N}, \quad 0 \le \theta < 2.$$

Therefore,

$$\mathcal{I}_\varepsilon = \varepsilon \int_{\frac{a_1}{\varepsilon}}^{\frac{b_1}{\varepsilon}} v(y)\, dy = \varepsilon \int_{\frac{a_1}{\varepsilon}}^{\frac{a_1}{\varepsilon}+2k} v(y)\, dy + \varepsilon \int_{\frac{a_1}{\varepsilon}+2k}^{\frac{a_1}{\varepsilon}+2k+\theta} v(y)\, dy. \tag{2.4}$$

From Lemma 2.3 we have

$$\begin{aligned} \varepsilon \int_{\frac{a_1}{\varepsilon}}^{\frac{a_1}{\varepsilon}+2k} v(y)\, dy &= \varepsilon \sum_{h=1}^{k} \int_{\frac{a_1}{\varepsilon}+2(h-1)}^{\frac{a_1}{\varepsilon}+2h} v(y)\, dy \\ &= k\,\varepsilon \int_0^2 v(y)\, dy = \frac{b_1 - a_1 - \theta\,\varepsilon}{2} \int_0^2 v(y)\, dy. \end{aligned}$$

On the other hand, again by Lemma 2.3,

$$\left| \int_{\frac{a_1}{\varepsilon}+2k}^{\frac{a_1}{\varepsilon}+2k+\theta} v(y)\, dy \right| \le \int_{\frac{a_1}{\varepsilon}+2k}^{\frac{a_1}{\varepsilon}+2k+2} |v(y)|\, dy = \int_0^2 |v(y)|\, dy.$$

Consequently, passing to the limit as $\varepsilon \to 0$ in (2.4), we get

$$\mathcal{I}_\varepsilon \longrightarrow \frac{b_1 - a_1}{2} \int_0^2 v(y)\, dy = (b_1 - a_1)\frac{1}{2}\left(\frac{2}{3}\,\alpha + \frac{4}{3}\,\beta\right).$$

Then, Proposition 1.46 gives that

$$v_\varepsilon \rightharpoonup v_0 = \frac{1}{2}\left(\frac{2}{3}\,\alpha + \frac{4}{3}\,\beta\right) \quad \text{weakly in } L^2(a, b). \tag{2.5}$$

Here also, this convergence is not strong in $L^2(a, b)$. Indeed, if this convergence were strong, Proposition 1.17 would imply that

$$\|v_\varepsilon\|^2_{L^2(a,b)} \longrightarrow \|v_0\|^2_{L^2(a,b)}.$$

But a similar computation as that used to prove (2.5) gives

$$\|v_\varepsilon\|^2_{L^2(a,b)} = \int_a^b v_\varepsilon^2(x)\, dx \longrightarrow (b-a)\frac{1}{2}\left(\frac{2}{3}\,\alpha^2 + \frac{4}{3}\,\beta^2\right).$$

which is different from

$$\|v_0\|^2_{L^2(a,b)} = (b-a)\left[\frac{1}{2}\left(\frac{2}{3}\,\alpha + \frac{4}{3}\,\beta\right)\right]^2.$$

◊

Let us observe that in the two examples above, the weak limit given by (2.3) and (2.5) respectively, is equal to $\mathcal{M}_Y(v)$. This fact is contained in a general result concerning the weak limit of a sequence of rapidly oscillating functions. The aim of the next section is to give this result.

2.3 Weak limits of rapidly oscillating periodic functions

In this section we prove the following result:

Theorem 2.6. *Let $1 \leq p \leq +\infty$ and f be a Y-periodic function in $L^p(Y)$. Set*

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N. \tag{2.6}$$

Then, if $p < +\infty$, as $\varepsilon \to 0$

$$f_\varepsilon \rightharpoonup \mathcal{M}_Y(f) = \frac{1}{|Y|}\int_Y f(y)\, dy \quad \text{weakly in } L^p(\omega),$$

for any bounded open subset ω of $\mathbb{R}^N$.

If $p = +\infty$, one has

$$f_\varepsilon \rightharpoonup \mathcal{M}_Y(f) = \frac{1}{|Y|}\int_Y f(y)\, dy \quad \text{weakly* in } L^\infty(\mathbb{R}^N).$$

Proof. The proof is done in several steps.

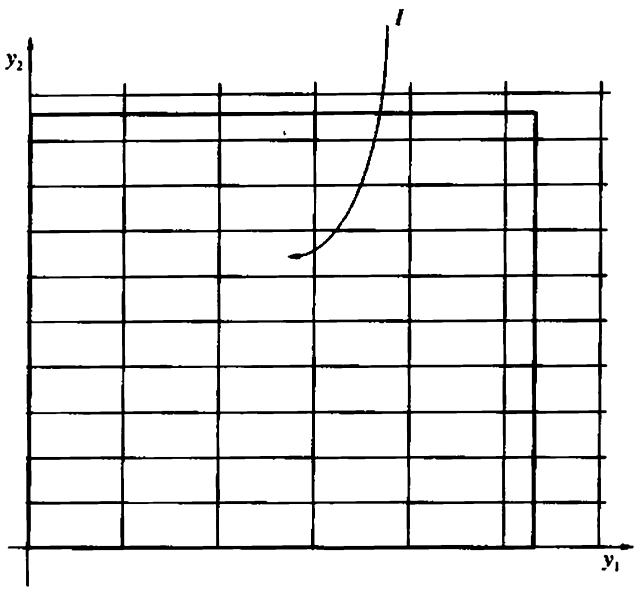


Fig. 2.8

2.3.1 A priori estimates

If $p = +\infty$, taking into account definition (2.6) of f_ε, one has

$$\|f_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = \|f\|_{L^\infty(Y)}.$$

Consequently, thanks to Theorem 1.26, one can extract a subsequence $\{f_{\varepsilon'}\}$ such that

$$f_{\varepsilon'} \rightharpoonup F \quad \text{weakly* in } L^\infty(\mathbb{R}^N). \tag{2.7}$$

Consider now the case $p < +\infty$ and let ω be a bounded open subset of $\mathbb{R}^N$. To obtain a priori estimates in $L^p(\omega)$, it is enough to show that for any open interval I of $\mathbb{R}^N$, there exists a constant C independent of ε such that

$$\|f_\varepsilon\|_{L^p(I)} \leq C.$$

It is not restrictive to suppose that I contains at least a translated set of Y.

Observe that one can find $N(\varepsilon)$ pairwise disjoint translated sets of Y denoted Y_k, $k = 1, \ldots, N(\varepsilon)$ with $\varepsilon\, Y_k \subset I$, and $N'(\varepsilon)$ pairwise disjoint translated sets of Y denoted Y'_j, $j = 1, \ldots, N'(\varepsilon)$ with $\varepsilon Y'_j \cap \partial I \neq \emptyset$, such that (see Fig 2.8)

$$I \subset \left(\bigcup_{k=1}^{N(\varepsilon)} \varepsilon \overline{Y}_k \right) \cup \left(\bigcup_{j=1}^{N'(\varepsilon)} \varepsilon \overline{Y}'_j \right). \tag{2.8}$$

Let $L_1, \dots, L_N$ be the lengths of the edges of I. Let us prove first that

$$\begin{cases} i) \quad \lim_{\varepsilon \to 0} \varepsilon^N N(\varepsilon) = \dfrac{|I|}{|Y|}, \\ ii) \quad \lim_{\varepsilon \to 0} \varepsilon^{N-1} N'(\varepsilon) \leq \displaystyle\sum_{i=1}^{N} \Big(\prod_{j \neq i} \frac{L_j}{\ell_j} \Big) \leq N \frac{|I|}{|Y|}. \end{cases} \tag{2.9}$$

Clearly, for any $\varepsilon > 0$, there exist $k_1^\varepsilon, \dots, k_N^\varepsilon \in \mathbb{N}$ such that, for any $i = 1, \dots, N$,

$$L_i = \varepsilon\, k_i^\varepsilon\, \ell_i + \gamma_i^\varepsilon \quad \text{with} \quad 0 \leq \gamma_i^\varepsilon < \varepsilon \ell_i,$$

and consequently,

$$\varepsilon k_i^\varepsilon \longrightarrow \frac{L_i}{\ell_i} \quad \text{when} \quad \varepsilon \longrightarrow 0. \tag{2.10}$$

On the other hand, observe that the number of translated periods of εY strictly included in I is

$$N_\varepsilon = k_1^\varepsilon \times \cdots \times k_N^\varepsilon$$

so that (2.10) implies

$$\varepsilon^N N_\varepsilon \longrightarrow \frac{L_1 \times \cdots \times L_N}{\ell_1 \times \cdots \times \ell_N} = \frac{|I|}{|Y|},$$

which is assertion (2.9)(i). To evaluate N'_ε, let us observe that the sets Y'_k and Y_j can be chosen such that the interval I is covered by the union of K^ε disjoint translated sets of $\varepsilon \overline{Y}$ with $K^\varepsilon = (k_1^\varepsilon + 1) \times \cdots \times (k_N^\varepsilon + 1)$. Then

$$N'_\varepsilon \leq K^\varepsilon - N^\varepsilon = A_\varepsilon + B_\varepsilon$$

where

$$A_\varepsilon = \sum_{i=1}^{N} \Big(\prod_{j \neq i} k_j^\varepsilon \Big).$$

From (2.10), we see that

$$\varepsilon^{N-1} A_\varepsilon \longrightarrow \sum_{i=1}^{N} \Big(\prod_{j \neq i} \frac{L_j}{\ell_j} \Big) \leq N \prod_{j=1}^{N} \frac{L_j}{\ell_j} = N \frac{|I|}{|Y|},$$

as for any $j = 1, \dots, N$, one has $\ell_j < L_j$. This implies (2.9)(ii) since, by construction

$$\varepsilon^{N-1} B_\varepsilon \longrightarrow 0.$$

Now, the periodicity of f, Lemma 2.3, and estimates (2.9) give

$$\begin{cases} \|f_\varepsilon\|^p_{L^p(I)} & \leq \displaystyle\sum_{k=1}^{N(\varepsilon)} \int_{\varepsilon Y_k} |f_\varepsilon|^p \, dx + \sum_{j=1}^{N'(\varepsilon)} \int_{\varepsilon Y'_j} |f_\varepsilon|^p \, dx \\ & = [N(\varepsilon) + N'(\varepsilon)] \displaystyle\int_{\varepsilon Y} |f_\varepsilon|^p \, dx \\ & = [N(\varepsilon) + N'(\varepsilon)] \varepsilon^N \displaystyle\int_Y |f(y)|^p \, dy \leq C \|f\|^p_{L^p(Y)}, \end{cases} \tag{2.11}$$

where C is a constant independent of ε.

This means that the sequence $\{f_\varepsilon\}$ is bounded in $L^p(\omega)$, for any bounded open subset ω of $\mathbb{R}^N$. In particular, for $1 < p < \infty$, we can apply Proposition 1.18 to get a subsequence $\{f_{\varepsilon'}\}$ such that

$$f_{\varepsilon'} \rightharpoonup G \quad \text{weakly in } L^p(\omega). \tag{2.12}$$

2.3.2 Identification of the limit

(a) Case $1 < p < +\infty$

Let ω be a bounded open subset of $\mathbb{R}^N$. From the first step and Proposition 1.46, to identify the limit in (2.12), it is sufficient to show that

$$\int_I f_\varepsilon(x)\,dx \longrightarrow \int_I \mathcal{M}_Y(f)\,dx = |I|\mathcal{M}_Y(f),$$

for any interval $I \subset \omega$. By using (2.8), one has

$$\int_I f_\varepsilon(x)\,dx = N(\varepsilon)\,\varepsilon^N \int_Y f(y)\,dy + \sum_{j=1}^{N'(\varepsilon)} \int_{\varepsilon Y_j' \cap I} f_\varepsilon(x)\,dx.$$

From (2.9) it follows that

$$N(\varepsilon)\,\varepsilon^N \int_Y f(y)\,dy \to \frac{|I|}{|Y|}\int_Y f(y)\,dy = |I|\mathcal{M}_Y(f)$$

and also

$$\left|\sum_{j=1}^{N'(\varepsilon)} \int_{\varepsilon Y_j' \cap I} f_\varepsilon(x)\,dx\right| \leq N'(\varepsilon)\left|\int_{\varepsilon Y} f_\varepsilon(y)\,dy\right| = \varepsilon^N\, N'(\varepsilon)\left|\int_Y f(y)\,dy\right| \to 0.$$

Consequently, in (2.12) one has $G = \mathcal{M}_Y(f)$. Moreover, (ii) of Theorem 1.18 implies that the whole sequence $\{f_\varepsilon\}$ converges to $\mathcal{M}_Y(f)$.

(b) Case $p = +\infty$

Let ω be a bounded open subset of $\mathbb{R}^N$ and χ_ω be its characteristic function (see Definition 1.40). Then, in particular for any $\varphi \in L^2(\omega)$, one has $\varphi\chi_\omega \in L^1(\mathbb{R}^N)$. Then, from (2.7) one gets

$$\int_\omega f_{\varepsilon'}\,\varphi\,dx \longrightarrow \int_\omega F\,\varphi\,dx.$$

Hence,

$$f_{\varepsilon'} \rightharpoonup F \quad \text{weakly in } L^2(\omega).$$

From step (a) and the uniqueness of the limit (see Remark 1.11), we know that

$$F = \mathcal{M}_Y(f),$$

a.e. on ω and therefore a.e. on $\mathbb{R}^N$, since ω is arbitrary. Again, Theorem 1.18 shows that the whole sequence $\{f_\varepsilon\}$ converges to $\mathcal{M}_Y(f)$.

(c) Case $p = 1$.

Since a bounded set in the space L^1 is not weakly compact (see Section 1.6), to prove the result in this case we will apply a density argument. Remark 1.43 implies that for any $\eta > 0$, there exists $g \in L^2(Y)$ such that

$$\|f - g\|_{L^1(Y)} \leq \eta. \tag{2.13}$$

Let us extend g by periodicity a.e. on $\mathbb{R}^N$ by setting

$$g(x + k\,\ell_i\, e_i) = g(x) \quad \text{a.e. on } Y, \quad \forall k \in \mathbb{Z}, \quad \forall i \in \{1, \ldots, N\},$$

where $\{e_1, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^N$. Define the function g_ε by

$$g_\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N.$$

Let ω be a bounded open subset of $\mathbb{R}^N$. Then, for any $\varphi \in L^\infty(\omega)$, one has

$$\begin{aligned}\int_\omega (f_\varepsilon - \mathcal{M}_Y(f))\, \varphi\, dx \;&=\; \int_\omega (f_\varepsilon - g_\varepsilon)\, \varphi\, dx + \int_\omega (g_\varepsilon - \mathcal{M}_Y(g))\, \varphi\, dx \\ &\quad + \int_\omega (\mathcal{M}_Y(g) - \mathcal{M}_Y(f))\, \varphi\, dx. \end{aligned} \tag{2.14}$$

If I is an interval in $\mathbb{R}^N$ such that $\omega \subset I$, from (2.11) and (2.13) it follows that there exists a constant C_1, independent of ε and η such that

$$\left| \int_\omega (f_\varepsilon - g_\varepsilon)\, \varphi\, dx \right| \leq \|\varphi\|_{L^\infty(\omega)} \|f_\varepsilon - g_\varepsilon\|_{L^1(I)} \leq C_1 \eta.$$

Obviously, from (2.13) one also has

$$\left| \int_\omega (\mathcal{M}_Y(g) - \mathcal{M}_Y(f))\, \varphi\, dx \right| \leq C_2 \eta,$$

where C_2 is a constant independent of ε and η. Finally, from step (a), as $\varepsilon \to 0$,

$$\int_\omega (g_\varepsilon - \mathcal{M}_Y(g))\, \varphi\, dx \to 0.$$

Consequently, from (2.14) we have

$$\int_\omega (f_\varepsilon - \mathcal{M}_Y(f))\, \varphi\, dx \to 0,$$

since η is arbitrary. This ends the proof of Theorem 2.6. □

Remark 2.7. Let E be a Banach space. Theorem 2.6 shows in particular, that if $\{u_\varepsilon\} \subset E$ and $\{v_\varepsilon\} \subset E'$ are two sequences such that, as $\varepsilon \to 0$,

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{weakly in } E, \\ v_\varepsilon &\rightharpoonup v \quad \text{weakly in } E', \end{aligned}$$

then, in general

$$\langle v_\varepsilon, u_\varepsilon \rangle_{E',E} \not\to \langle v, u \rangle_{E',E}.$$

Indeed, let f, g be two Y-periodic functions in $L^2(Y)$, and set

$$\begin{cases} u_\varepsilon(x) = f\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } \mathbb{R}^N, \\ v_\varepsilon(x) = g\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } \mathbb{R}^N. \end{cases} \tag{2.15}$$

Theorem 2.6 implies that, if ω is a bounded open subset of $\mathbb{R}^N$, then

$$u_\varepsilon v_\varepsilon = (fg)\left(\frac{\cdot}{\varepsilon}\right) \rightharpoonup \mathcal{M}_Y(fg) \quad \text{weakly in } L^1(\omega).$$

Hence using Remark 1.37 we have, in particular,

$$\langle v_\varepsilon, u_\varepsilon \rangle_{L^2(Y), L^2(Y)} = \int_\omega u_\varepsilon v_\varepsilon \, dx \longrightarrow |\omega| \mathcal{M}_Y(fg)$$

while, by using again Theorem 2.6

$$\langle v, u \rangle_{L^2(Y), L^2(Y)} = \int_\omega \mathcal{M}_Y(f) \mathcal{M}_Y(g) \, dx = |\omega| \mathcal{M}_Y(f) \mathcal{M}_Y(g).$$

In general, as it can be seen from Examples 2.4 and 2.5 above, one has

$$\mathcal{M}_Y(fg) \neq \mathcal{M}_Y(f) \mathcal{M}_Y(g).$$

◊

Remark 2.8. In Remark 2.7 we considered a particular case of two weakly converging sequences whose product is weakly converging in $L^1(\omega)$. This is a very special case, relying on the construction (2.15).

Let us consider now the general case of the product of two sequences $\{u_n\}$ and $\{v_n\}$, weakly converging in $L^2(\omega)$. Their product $\{u_n v_n\}$ is obviously bounded in $L^1(\omega)$ due to the Hölder inequality (Proposition 1.34). Hence, in view of Proposition 1.48, $\{u_n v_n\}$ is weakly* compact in $M(\omega)$. The question is: does the whole sequence $\{u_n v_n\}$ weakly* converge to some element of $M(\omega)$?

The following example shows that this is actually not true.

Let $Y =]0, 1[$ and f be a 1-periodic function in $L^2(0, 1)$ such that $\mathcal{M}_{(0,1)}(f) = 0$. Set

$$u_n = f(2^n x), \quad \text{a.e. on } \mathbb{R}$$

and

$$v_n = \begin{cases} u_n & \text{if } n \text{ is odd} \\ -u_n & \text{if } n \text{ is even.} \end{cases}$$

Since $\mathcal{M}_{(0,1)}(f) = 0$, both sequences $\{u_n\}$ and $\{v_n\}$ weakly converge to 0 in $L^2(\omega)$ for any interval ω in $\mathbb{R}$. On the other hand, by construction,

$$u_n\, v_n = \begin{cases} u_n^2 & \text{if } n \text{ is odd} \\ -u_n^2 & \text{if } n \text{ is even.} \end{cases}$$

Hence, thanks to Theorem 2.6 written for $\varepsilon = 1/2^n$, from $\{u_n\, v_n\}$ one can extract two weakly convergent sequences in $L^1(\omega)$, one converging to $\mathcal{M}_{(0,1)}(f^2)$ and the other one to $-\mathcal{M}_{(0,1)}(f^2)$. Since weak convergence in $L^1(\omega)$ obviously implies weak* convergence in $M(\omega)$ (see Section 1.6), this shows that the whole sequence $\{u_n\, v_n\}$ does not weakly* converge in $M(\omega)$. ◊

Remark 2.9. The weak convergences given by Theorem 2.6 are not strong, unless f is a constant and $|Y| = 1$. Indeed, strong convergence would imply that $\mathcal{M}_Y(f^p) = [\mathcal{M}_Y(f)]^p$. But it is easy to see that for any $p > 1$, one has

$$\mathcal{M}_Y(f^p) \neq [\mathcal{M}_Y(f)]^p.$$

◊

Remark 2.10. Let us notice one result, contained in the proof of Theorem 2.6, which is interesting by its own right. As in Theorem 2.6, let $1 \leq p \leq +\infty$ and f be a Y-periodic function in $L^p(Y)$. Set

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N.$$

Then, there exists a constant C depending on N only, such that for any open interval I containing at least a translated set of Y, one has

$$\|f_\varepsilon\|^p_{L^p(I)} \leq C\frac{|I|}{|Y|}\|f\|^p_{L^p(Y)}, \tag{2.16}$$

for ε small enough.

This inequality is a consequence of (2.9) and (2.11). Indeed, from the proof of (2.9), it is easily seen that

$$[N(\varepsilon) + N'(\varepsilon)]\varepsilon^N \leq C\frac{|I|}{|Y|}.$$

This, used into (2.11), gives (2.16) for ε small enough. ◊

3

Some classes of Sobolev spaces

In this chapter we introduce the functional setting, essentially based on the distribution theory and Sobolev spaces, which is the natural framework for the homogenization results we present in this book. Distributions and Sobolev spaces have been widely studied in the last fifty years. We refer the reader for instance to Schwartz (1951), Nečas (1967), Lions and Magenes (1968a,b), Adams (1975), and Mazya (1985). We will quote here just the main results which will be used later. We also present, in the same context, some specific spaces of periodic functions as well as their main properties.

Let us recall that, as in Chapter 2, $\mathcal{O}$ and Ω denote respectively, an open set and a bounded open set in $\mathbb{R}^N$.

3.1 Distributions

Let $\mathcal{D}(\mathcal{O})$ be the space introduced by Definition 1.28. We now give a notion of convergence for sequences in this space. To do so, we will make use of the following notations. If $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ (usually called a multi-index), we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_N,$$

and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}},$$

where for $|\alpha| = 0$, ∂^α is the identity.

Definition 3.1. Let $\{\varphi_n\}$ be a sequence in $\mathcal{D}(\mathcal{O})$. We say that φ_n converges to an element $\varphi \in \mathcal{D}(\mathcal{O})$, iff

i) there exists a compact set $K \subset \mathcal{O}$ such that, for any $n \in \mathbb{N}$, supp $\varphi_n \subset K$,

ii) for any $\alpha \in \mathbb{N}^N$, $\partial^\alpha \varphi_n$ converges uniformly to $\partial^\alpha \varphi$ on K.

Remark 3.2. This definition does not provide a topology on $\mathcal{D}(\mathcal{O})$. Nevertheless, one can define a suitable topology $\mathcal{T}$ on it, for which the convergence of sequences is exactly that given by Definition 3.1. This topology has a complicated structure, as can be seen in Schwartz (1951). In particular, $\mathcal{D}(\mathcal{O})$ is not a metric space. For our purpose, Definition 3.1 is enough so we do not give here more details. ◊

Definition 3.3. A map $T : \mathcal{D}(\mathcal{O}) \mapsto \mathbb{R}$ is called a *distribution* on $\mathcal{O}$, iff

i) T is linear, i.e.

$$\forall \lambda_1, \lambda_2 \in \mathbb{R},\ \varphi_1, \varphi_2 \in \mathcal{D}(\mathcal{O}), \quad T(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 T(\varphi_1) + \lambda_2 T(\varphi_2),$$

ii) T is continuous on sequences, i.e.

$$(\varphi_n \to \varphi \quad \text{in } \mathcal{D}(\mathcal{O})) \Longrightarrow (T(\varphi_n) \to T(\varphi)).$$

We denote by $\mathcal{D}'(\mathcal{O})$ the set of distributions on $\mathcal{O}$.

Remark 3.4. The notation $\mathcal{D}'(\mathcal{O})$ is motivated by the fact that one can prove that $\mathcal{D}'(\mathcal{O})$ is the dual of $\mathcal{D}(\mathcal{O})$ with respect to the topology $\mathcal{T}$ mentioned in Remark 3.2. This is why the usual notation for a distribution T is

$$T(\varphi) = \langle T, \varphi \rangle_{\mathcal{D}'(\mathcal{O}), \mathcal{D}(\mathcal{O})}, \quad \forall \varphi \in \mathcal{D}(\mathcal{O}).$$

◊

Example 3.5 (Dirac mass). Let $x_0 \in \mathbb{R}^N$ and define

$$\delta_{x_0} = \varphi(x_0), \quad \text{for any } \varphi \in \mathcal{D}(\mathbb{R}^N).$$

It is straightforward that $\delta_{x_0} \in \mathcal{D}'(\mathbb{R}^N)$. This distribution is called the Dirac function (or mass) in the point x_0. ◊

Example 3.6. Let $f \in L^1_{\text{loc}}(\mathcal{O})$ and set

$$T_f(\varphi) = \int_{\mathcal{O}} f\varphi\, dx, \quad \forall \varphi \in \mathcal{D}(\mathcal{O}). \tag{3.1}$$

This is a distribution on $\mathcal{O}$. Indeed, (i) from Definition 3.3 is obvious. Moreover, if $\{\varphi_n\}$ is a sequence converging to φ in $\mathcal{D}(\mathcal{O})$, one has

$$|T_f(\varphi_n) - T_f(\varphi)| = \left| \int_{\mathcal{O}} f(\varphi_n - \varphi)\, dx \right| \leq \|f\|_{L^1(K)} \max_K |\varphi_n - \varphi| \to 0,$$

where K is a compact set such that $\operatorname{supp} \varphi_n \subset K$. Hence (ii) from Definition 3.3 is satisfied. ◊

From Theorem 1.44, it follows that if T is defined by (3.1), then $T_f = 0$ iff $f = 0$. This observation leads to the following definition:

Definition 3.7. We say that a distribution T is in $L^1_{\text{loc}}(\mathcal{O})$ (respectively in $L^1(\mathcal{O})$), if there exists $f \in L^1_{\text{loc}}(\mathcal{O})$ (respectively in $L^1(\mathcal{O})$), such that $T = T_f$, where T_f is given by (3.1).

Remark 3.8. Suppose that $T \in \mathcal{D}'(\mathcal{O})$ is like in Definition 3.7. Then the function f is uniquely determined in view of Theorem 1.44. It is why usually one identifies T with f. ◊

We now give a notion of convergence for sequences in $\mathcal{D}'(\mathcal{O})$ which we will use in the next chapters. This is a weak* convergence but not in a Banach space context.

Definition 3.9. A sequence $\{T_n\}$ in $\mathcal{D}'(\mathcal{O})$ is said to converge (in the sense of distributions) to an element $T \in \mathcal{D}'(\mathcal{O})$ iff

$$\langle T_n, \varphi\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})} \longrightarrow \langle T, \varphi\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})}, \quad \forall\varphi \in \mathcal{D}(\mathcal{O}).$$

We denote this convergence by

$$T_n \longrightarrow T \quad \text{in } \mathcal{D}'(\mathcal{O}).$$

Example 3.10. Let $\{f_n\}$ be a sequence in $L^1(\mathcal{O})$ such that

$$f_n \rightharpoonup f \quad \text{weakly in } L^1(\mathcal{O}).$$

From Section 1.6 and Remark 3.8 it is obvious that

$$f_n \longrightarrow f \quad \text{in } \mathcal{D}'(\mathcal{O}).$$

◊

The following definition is essential in the theory of Sobolev spaces:

Definition 3.11 (derivative of a distribution). Let $T \in \mathcal{D}'(\mathcal{O})$. For any $i = 1, \dots, N$, the *derivative* $\partial T/\partial x_i$ of T with respect to x_i is defined by

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})} = -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})}, \quad \forall\varphi \in \mathcal{D}(\mathcal{O}).$$

Remark 3.12. It is easy to check that $\partial T/\partial x_i$, $i = 1, \dots, N$, is a distribution. Actually, it is linear and continuous on sequences, since $\partial\varphi/\partial x_i$ is in $\mathcal{D}(\mathcal{O})$ and T is a distribution.

Moreover, if T_n converges to T in the sense of distributions, then $\partial T_n/\partial x_i$ converges to $\partial T/\partial x_i$ in the sense of distributions for any $i = 1, \dots, N$. ◊

Example 3.13. For the distribution δ_{x_0} introduced in Example 3.5, one easily has that

$$\left\langle \frac{\partial \delta_{x_0}}{\partial x_i}, \varphi \right\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})} = -\frac{\partial \varphi}{\partial x_i}(x_0), \quad \forall\varphi \in \mathcal{D}(\mathbb{R}^N).$$

◊

Example 3.14. Let us consider the Heaviside function on $\mathbb{R}$, defined by

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Observe that its (usual) derivative is defined and equal to 0 in any point $x \neq 0$, so that this derivative is equal to 0 a.e. on $\mathbb{R}$.

On the other hand, since $H \in L^1_{\text{loc}}(\mathbb{R})$, by Remark 3.8 one can identify H with the distribution T_H, given by (3.1). Therefore,

$$\left\langle \frac{dT_H}{dx}, \varphi \right\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})} = -\int_{\mathbb{R}} H(x)\, \frac{d\varphi}{dx}\, dx = -\int_0^{\infty} \frac{d\varphi}{dx}\, dx = \varphi(0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Hence, the derivative of H in the sense of distributions is the Dirac mass in 0 (see Example 3.5). This shows that the usual derivative and the derivative in the sense of distributions are two different notions. ◊

Remark 3.15. Let $f \in L^1(\mathcal{O})$. Suppose that its derivative in the sense of distributions $\partial f/\partial x_i$ is in $L^1(\mathcal{O})$. From Remark 3.8 we have

$$\int_{\mathcal{O}} f\, \frac{\partial \varphi}{\partial x_i}\, dx = -\int_{\mathcal{O}} \frac{\partial f}{\partial x_i}\, \varphi\, dx, \quad \forall \varphi \in \mathcal{D}(\mathcal{O}). \tag{3.2}$$

This together with Theorem 1.44, shows in particular. that if a function is of class $C^1(\mathcal{O})$, its derivatives in the sense of distributions coincide with the usual partial derivatives since (3.2) is nothing else than the Green formula. ◊

3.2 The spaces $W^{1,p}$

In this section we define some classes of Sobolev spaces and recall their main properties. We refer to Nečas (1967), Lions and Magenes (1968a), and Adams (1975) for proofs and more details.

Definition 3.16. Let $1 \leq p \leq \infty$. The *Sobolev space* $W^{1,p}(\mathcal{O})$ is defined by

$$W^{1,p}(\mathcal{O}) = \left\{ u \,\middle|\, u \in L^p(\mathcal{O}),\ \frac{\partial u}{\partial x_i} \in L^p(\mathcal{O}),\ i = 1, \ldots, N \right\},$$

where the derivatives are taken in the sense of distributions of Definition 3.11.

For $p = 2$, one denotes $W^{1,2}(\mathcal{O}) = H^1(\mathcal{O})$, i.e.

$$H^1(\mathcal{O}) = \left\{ u \,\middle|\, u \in L^2(\mathcal{O}),\ \frac{\partial u}{\partial x_i} \in L^2(\mathcal{O}),\ i = 1, \ldots, N \right\}.$$

Proposition 3.17.

i) The space $W^{1,p}(\mathcal{O})$ is a Banach space for the norm

$$\|u\|_{W^{1,p}(\mathcal{O})} = \|u\|_{L^p(\mathcal{O})} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathcal{O})}.$$

For $1 \leq p < \infty$, this norm is equivalent to the following one:

$$\|u\|_{W^{1,p}(\mathcal{O})} = \left(\|u\|^p_{L^p(\mathcal{O})} + \|\nabla u\|^p_{L^p(\mathcal{O})} \right)^{\frac{1}{p}}, \tag{3.3}$$

where we have used the notations

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right).$$

and

$$\|\nabla u\|_{L^p(\mathcal{O})} = \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\mathcal{O})} \right)^{\frac{1}{p}}.$$

ii) The space $W^{1,p}(\mathcal{O})$ is separable for $1 \leq p < +\infty$ and reflexive for $1 < p < +\infty$.

iii) The space $H^1(\mathcal{O})$ is a Hilbert space for the scalar product

$$(v, w)_{H^1(\mathcal{O})} = (v, w)_{L^2(\mathcal{O})} + \sum_{i=1}^{N} \left(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right)_{L^2(\mathcal{O})}, \quad \forall v, w \in H^1(\mathcal{O}). \tag{3.4}$$

Proof. The equivalence between the two norms, for $1 \leq p < +\infty$, is a simple consequence of the following inequality:

$$\sum_{i=1}^{m} a_i^p \leq \left(\sum_{i=1}^{m} a_i \right)^p \leq m^{p-1} \sum_{i=1}^{m} a_i^p, \quad \forall m \in \mathbb{N} - \{0\},\ a_i \in \mathbb{R}_+,\ i = 1, \dots, m.$$

It is also clear that (3.3) and (3.4) define, respectively, a norm and a scalar product.

Let us prove that $W^{1,p}(\mathcal{O})$ is complete. Let $\{u_n\}$ be a Cauchy sequence in $W^{1,p}(\mathcal{O})$. Obviously, $\{u_n\}$ and $\{\partial u_n / \partial x_i\}$ for $i = 1, \dots, N$ are Cauchy sequences in $L^p(\mathcal{O})$ which is complete (see Proposition 1.31). Consequently, there exist $u \in L^p(\mathcal{O})$ and $v_i \in L^p(\mathcal{O})$ for $i = 1, \dots, N$, such that

$$u_n \longrightarrow u, \quad \frac{\partial u_n}{\partial x_i} \longrightarrow v_i \quad \text{strongly in } L^p(\mathcal{O}), \quad \forall i = 1, \dots, N.$$

Then, it is enough to prove that $v_i = \partial u / \partial x_i$ for $i = 1, \dots, N$. By Definition 3.11 and Remark 3.15, one has

$$\int_{\mathcal{O}} \frac{\partial u_n}{\partial x_i} \varphi \, dx = - \int_{\mathcal{O}} u_n \frac{\partial \varphi}{\partial x_i} \, dx, \quad \forall \varphi \in \mathcal{D}(\mathcal{O}).$$

We can now pass to the limit in this identity. Indeed, for the left-hand side integral, one has, by using the Hölder inequality (Proposition 1.34),

$$\left|\int_{\mathcal{O}} \frac{\partial u_n}{\partial x_i} \varphi\, dx - \int_{\mathcal{O}} v_i \varphi\, dx\right| \le \left\|\frac{\partial u_n}{\partial x_i} - v_i\right\|_{L^p(\mathcal{O})} \|\varphi\|_{L^{p'}(\mathcal{O})} \to 0,$$

where p' is the conjugate of p.

Similarly,

$$\left|\int_{\mathcal{O}} u_n \frac{\partial \varphi}{\partial x_i}\, dx - \int_{\mathcal{O}} u \frac{\partial \varphi}{\partial x_i}\, dx\right| \le \|u_n - u\|_{L^p(\mathcal{O})} \|\nabla\varphi\|_{L^{p'}(\mathcal{O})} \to 0.$$

Hence,

$$\int_{\mathcal{O}} v_i \varphi\, dx = -\int_{\mathcal{O}} u \frac{\partial \varphi}{\partial x_i}\, dx, \quad \forall \varphi \in \mathcal{D}(\mathcal{O}),$$

which, due to (3.2), proves that $v_i = \partial u / \partial x_i$ in the sense of distributions. This ends the proof of (i).

To show the other assertions of the theorem, let us consider the map

$$T : u \in W^{1,p}(\mathcal{O}) \longmapsto T(u) \in [L^p(\mathcal{O})]^{N+1}$$

defined by

$$T(u) = \left(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right),$$

which is clearly an isometry since, by definition,

$$\|T(u)\|_{[L^p(\mathcal{O})]^{N+1}} = \|u\|_{W^{1,p}(\mathcal{O})}.$$

Consequently, $T(W^{1,p}(\mathcal{O}))$ can be identified with $W^{1,p}(\mathcal{O})$, so it is a closed subspace of $[L^p(\mathcal{O})]^{N+1}$. From Proposition 1.32 and Remark 1.16 we know that the space $[L^p(\mathcal{O})]^{N+1}$ is separable for $1 \le p < +\infty$ and reflexive for $1 < p < +\infty$. This ends the proof of (ii) since any closed subspace of a separable and reflexive Banach space is separable and reflexive too (see, for instance Brezis, 1987).

Assertion (iii) is an immediate consequence of Definition 1.2. The proof of Theorem 3.17 is complete. □

In this book we will study the homogenization of boundary value problems posed in bounded open sets. It is why from now on, we consider Sobolev spaces only on a bounded open set Ω in $\mathbb{R}^N$. Many relevant properties of these spaces are true if $\partial\Omega$ is sufficiently smooth. There are many kind of regularity assumptions that can be made (see for instance. Nečas, 1967, Lions and Magenes, 1968a, Adams, 1975, and Mazya, 1985). The following one is due to J. Nečas (1967):

Definition 3.18. The boundary $\partial\Omega$ is *Lipschitz-continuous* iff there exist two constants $c_1 > 0$ and $c_2 > 0$ and a finite number M of local coordinate systems $(\tilde{x}^m, x_N^m)$ and local maps Φ_m $(m = 1, \dots, M)$ defined on the set

$$R_m = \{\tilde{x}^m \in \mathbb{R}^{N-1},\ \tilde{x}^m = (x_1^m, \dots, x_{N-1}^m),\ |x_i^m| \le c_1, \forall i = 1, \dots, N-1\},$$

which are Lipschitz continuous on their domain of definition, such that:

$$\partial\Omega = \bigcup_{m=1}^{M} \Gamma_m,$$

where, for any $m = 1, \ldots, M$,

$$\Gamma_m = \{(\widetilde{x}^m, x_N^m) |\ x_N^m = \Phi_m(\widetilde{x}^m),\ \widetilde{x}^m \in R_m\},$$

and

$$\begin{aligned} \Omega_m &= \Big\{(\widetilde{x}^m, x_N^m) |\ \Phi_m(\widetilde{x}^m) < x_N^m < \Phi_m(\widetilde{x}^m) + c_2,\ \widetilde{x}^m \in R_m\Big\} \subset \Omega, \\ C_m &= \{(\widetilde{x}^m, x_N^m) |\ \Phi_m(\widetilde{x}^m) - c_2 < x_N^m < \Phi_m(\widetilde{x}^m),\ \widetilde{x}^m \in R_m\} \subset \mathbb{R}^N \setminus \overline{\Omega}. \end{aligned}$$

Recall that Φ_m is Lipschitz continuous iff there exists a positive constant L_m such that

$$|\Phi_m(\widetilde{x}^m) - \Phi_m(\widetilde{y}^m)| \leq L_m |\widetilde{x}^m - \widetilde{y}^m|, \quad \forall\, \widetilde{x}^m, \widetilde{y}^m \in R_m.$$

The set $\partial\Omega$ is of class C^k, where k is a strictly positive integer, if for any $m = 1, \ldots, M$ the map Φ_m is in $C^k(R_m)$.

For $N = 2$, two possible configurations are drawn in Figs 3.1 and 3.2, where the dashed zones represent Ω_m.

Remark 3.19. The boundaries of open balls in $\mathbb{R}^N$ are of class C^∞. Polygons in $\mathbb{R}^2$, polyhedrons in R^3 and intervals in R^N have a Lipschitz continuous boundary. On the other hand, domains with cusps do not have a Lipschitz continuous boundary.

Let us recall that (see Nečas, 1967) if $\partial\Omega$ is Lipschitz continuous, then one can define a surface measure on $\partial\Omega$. In particular, $L^2(\partial\Omega)$ is well defined. ◊

Definition 3.20. Let $\mathcal{D}(\mathbb{R}^N)$ be given by Definition 1.28. We denote by $\mathcal{D}(\overline{\Omega})$ the set of restrictions to $\overline{\Omega}$ of functions in $\mathcal{D}(\mathbb{R}^N)$.

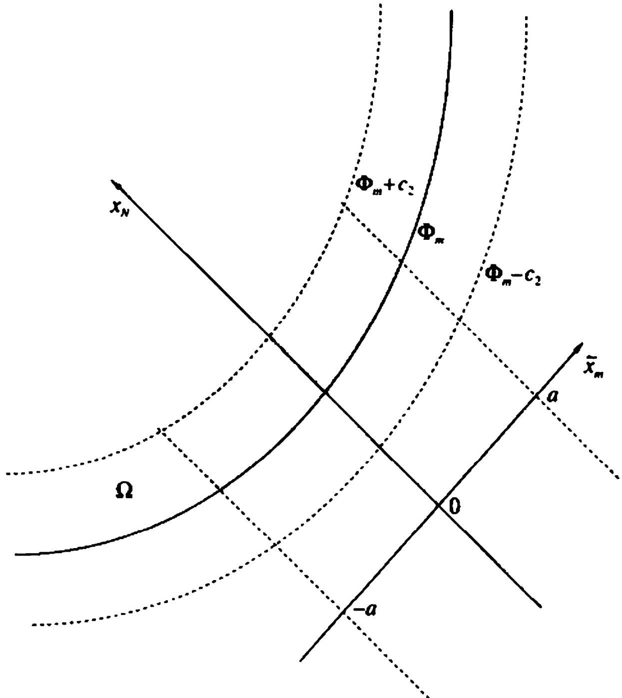
Remark 3.21. Let us point out that $\mathcal{D}(\Omega)$ is strictly contained in $\mathcal{D}(\overline{\Omega})$, since the functions of $\mathcal{D}(\overline{\Omega})$ are not required to vanish on the boundary $\partial\Omega$. ◊

The next three theorems are basic in the theory of Sobolev spaces. Their proofs are rather technical. We refer the reader to Nečas (1967), Adams (1975), and Brezis (1987) for them.

Theorem 3.22 (Density). *Let $1 \leq p < \infty$. Then $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$. Moreover, if $\partial\Omega$ is Lipschitz continuous, $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.*

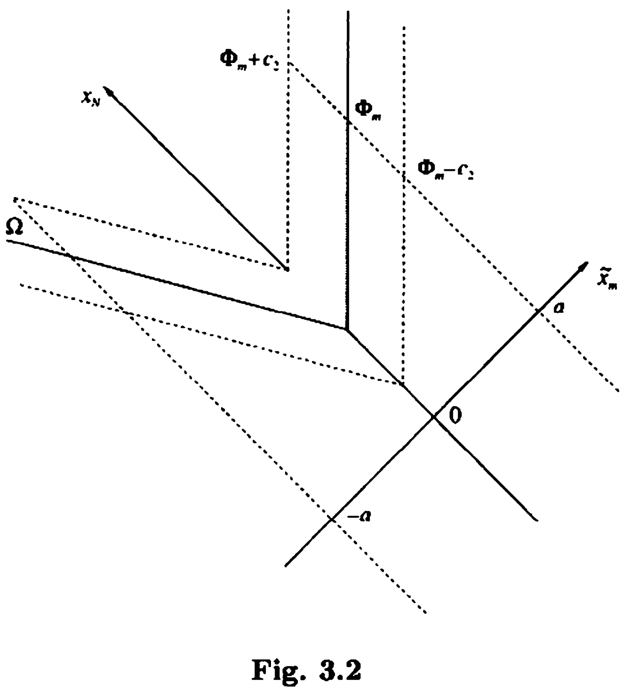
Recall that if E and E_1 are two Banach spaces, a map $h : E \mapsto E_1$ is compact iff the image $\{h(u_n)\}$ of a bounded sequence $\{u_n\}$ of E is relatively compact in E_1, i.e. iff there exists a subsequence $\{h(u_{n'})\}$ strongly convergent in E_1.

In the following, if $E \subset E_1$ the map $x \in E \mapsto x \in E_1$ is called an *injection.*



Φ_m+c_2
x_N
Φ_m
Φ_m-c_2
$\tilde{x}_m$
a
Ω
0
$-a$

Fig. 3.1



Φ_m+c_2
x_N
Φ_m
Φ_m-c_2
Ω
$\tilde{x}_m$
a
0
$-a$

Fig. 3.2

Theorem 3.23 (Sobolev embeddings). *Suppose that $\partial\Omega$ is Lipschitz continuous. Then*

i) if $1 \le p < N$, $W^{1,p}(\Omega) \subset L^q(\Omega)$ *with*

- compact injection for $q \in [1, p^\star[$. where $\dfrac{1}{p^\star} = \dfrac{1}{p} - \dfrac{1}{N}$,
- continuous injection for $q = p^\star$,

ii) if $p = N$, $W^{1,N}(\Omega) \subset L^q(\Omega)$ *with compact injection if* $q \in [1, +\infty[$,

iii) if $p > N$, $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ *with compact injection.*

Remark 3.24. One can prove that the inclusions given in this theorem are optimal. ◊

Theorem 3.23 is one of the main results in the theory of Sobolev spaces. Compact injections are an essential tool when studying partial differential equations. One has in particular that $H^1(\Omega) \subset L^2(\Omega)$ with compact injection, since $2 < 2^\star = 2N/(N-2)$ if $N > 2$ and the result is obvious when $N \le 2$. By definition, this means that any bounded sequence in $H^1(\Omega)$ contains at least a subsequence strongly convergent in $L^2(\Omega)$. This result will be widely used in the next chapters.

We end this section by another result, very important in applications, which allows us to extend functions in H^1.

Theorem 3.25 (Extension operator). *Suppose that $\partial\Omega$ is Lipschitz continuous. Then, there exists a linear continuous extension operator P from $H^1(\Omega)$ into $H^1(\mathbb{R}^N)$ satisfying*

$$\begin{cases} i) & Pu = u \quad \text{on } \Omega \\ ii) & \|Pu\|_{L^2(\mathbb{R}^N)} \le C\|u\|_{L^2(\Omega)} \\ iii) & \|Pu\|_{H^1(\mathbb{R}^N)} \le C\|u\|_{H^1(\Omega)}, \end{cases}$$

where C is a constant depending on Ω.

3.3 The space H_0^1 and the notion of trace

Theorem 3.23 shows that for $N = 1$, one has the inclusion $H^1(\Omega) \subset C^0(\overline{\Omega})$, so one can speak about the values on $\partial\Omega$ of a function $u \in H^1(\Omega)$. This inclusion is not true for higher dimensions. To give a sense to the restriction to $\partial\Omega$ of functions in $H^1(\Omega)$, we introduce in Theorem 3.28 below the notion of trace.

We are mainly interested in functions which vanish (in some sense) on the boundary. To do so, we introduce in this section a subspace of $H^1(\Omega)$, denoted $H_0^1(\Omega)$. We will see below that if $\partial\Omega$ is sufficiently smooth, a function in $H_0^1(\Omega)$ will vanish on the boundary in the sense of the trace. When we have no regularity on $\partial\Omega$, saying that a function u belongs to $H_0^1(\Omega)$ will replace the fact that u vanishes on the boundary.

Definition 3.26. For any $1 \leq p \leq \infty$, the *Sobolev space* $W_0^{1,p}(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$. We set

$$H_0^1(\Omega) = W_0^{1,p}(\Omega).$$

It is clear that $H_0^1(\Omega) \subset H^1(\Omega)$ so that $H_0^1(\Omega)$ is a Hilbert space for the scalar product (3.4). Moreover, it can be proved that Theorem 3.23 is still valid for $W_0^{1,p}(\Omega)$ without any regularity assumption on $\partial\Omega$, namely

Theorem 3.27. *One has the following inclusions:*

i) if $1 \leq p < N$, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ *with*

- compact injection for $q \in [1, p^*[$, where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$,
- continuous injection for $q = p^*$,

ii) if $p = N$, $W_0^{1,N}(\Omega) \subset L^q(\Omega)$, *with compact injection if* $q \in [1, +\infty[$,

iii) if $p > N$, $W_0^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ *with compact injection.*

In the sequel, we restrict our attention to the case of the space $H_0^1(\Omega)$. However, all we will say about it can be extended to the general case of $W_0^{1,p}(\Omega)$.

Theorem 3.28 (Trace theorem).

i) There exists a unique linear continuous map, called trace

$$\gamma : H^1(\mathbb{R}^{N-1} \times \mathbb{R}_+^*) \longmapsto L^2(\mathbb{R}^{N-1}),$$

such that for any $u \in H^1(\mathbb{R}^{N-1} \times \mathbb{R}_+^*) \cap C^0(\mathbb{R}^{N-1} \times \mathbb{R}_+)$, *one has* $\gamma(u) = u|_{\mathbb{R}^{N-1}}$.

ii) Assume now that Ω *is a bounded open set in* $\mathbb{R}^N$ *such that* $\partial\Omega$ *is Lipschitz continuous. Then, there exists a unique linear continuous map*

$$\gamma : H^1(\Omega) \longmapsto L^2(\partial\Omega),$$

such that for any $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ *one has* $\gamma(u) = u|_{\partial\Omega}$. *The function* $\gamma(u)$ *is called the trace of* u *on* $\partial\Omega$.

Proof. We only prove here the first statement of the theorem. Assertion (ii) follows from (i) by rather technical arguments, so we refer the reader to Nečas (1967) or Adams (1975) for details.

Let γ^0 be the linear map defined by

$$\gamma^0 : v \in \mathcal{D}(\mathbb{R}^{N-1} \times \mathbb{R}_+) \longmapsto v|_{\mathbb{R}^{N-1}}.$$

Let us first show that

$$\|\gamma^0(v)\|_{L^2(\mathbb{R}^{N-1})} \leq \|v\|_{H^1(\mathbb{R}^{N-1}\times\mathbb{R}_+^*)}, \quad \forall v \in \mathcal{D}(\mathbb{R}^{N-1} \times \mathbb{R}_+). \tag{3.5}$$

Since $v \in \mathcal{D}(\mathbb{R}^{N-1} \times \mathbb{R}_+)$ has a compact support, we have

$$\begin{aligned} |v(x',0)|^2 &= -\int_0^\infty \frac{\partial}{\partial x_N}\left(|v(x',x_N)|^2\right) dx_N \\ &= -\int_0^\infty 2v(x',x_N)\frac{\partial v}{\partial x_N}(x',x_N)\, dx_N. \end{aligned}$$

Therefore, by the Young inequality

$$|v(x',0)|^2 \leq \int_0^\infty |v(x',x_N)|^2\, dx_N + \int_0^\infty \left|\frac{\partial v}{\partial x_N}(x',x_N)\right|^2 dx_N.$$

Integrating over $\mathbb{R}^{N-1}$ in x' and using Fubini's theorem, one obtains

$$\int_{\mathbb{R}^{N-1}} |v(x',0)|^2\, dx' \leq \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+^*} |v(x)|^2\, dx + \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+^*} \left|\frac{\partial v}{\partial x_N}(x)\right|^2 dx,$$

which gives (3.5).

Suppose now that $u \in H^1(\mathbb{R}^{N-1} \times \mathbb{R}_+^*)$. From Theorem 3.22, there exists a sequence $\{u_n\} \in \mathcal{D}(\mathbb{R}^{N-1} \times \mathbb{R}_+)$ converging to u in $H^1(\mathbb{R}^{N-1} \times \mathbb{R}_+^*)$. By inequality (3.5) and the linearity of γ^0, we have

$$\|\gamma^0(u_n) - \gamma^0(u_m)\|_{L^2(\mathbb{R}^{N-1})} \leq \|u_n - u_m\|_{H^1(\mathbb{R}^{N-1}\times\mathbb{R}_+^*)}, \quad \forall m, n \in \mathbb{N}.$$

Consequently, $\{\gamma^0(u_n)\}$ is a Cauchy sequence in the complete space $L^2(\mathbb{R}^{N-1})$, so it has a limit $u_0 \in L^2(\mathbb{R}^{N-1})$. Define $\gamma(u) = u_0$. Obviously,

$$\gamma(v) = \gamma^0(v), \quad \text{for } v \in \mathcal{D}(\mathbb{R}^{N-1} \times \mathbb{R}_+),$$

so that γ is a linear extension of γ^0 to $H^1(\mathbb{R}^{N-1} \times \mathbb{R}_+^*)$. By construction, γ is uniquely determined and linear and continuous from $H^1(\mathbb{R}^{N-1} \times \mathbb{R}_+^*)$ to $L^2(\mathbb{R}^{N-1})$.

To conclude the proof, suppose now that u is in $H^1(\mathbb{R}^{N-1}\times\mathbb{R}_+^*)\cap C^0(\mathbb{R}^{N-1}\times \mathbb{R}_+)$. One can check (see for instance Brezis, 1987) that the approximating sequence $\{u_n\}$ can be chosen such that the norm $\|u_n - u\|_{C^0(\mathbb{R}^{N-1}\times\mathbb{R}_+)}$ converges to zero. Therefore, $\gamma^0(u_n) = u_n|_{\mathbb{R}^{N-1}}$ converges to $u|_{\mathbb{R}^{N-1}}$ in $C^0(\mathbb{R}^{N-1})$ and then in $L^2(\mathbb{R}^{N-1})$. This shows that $\gamma(u) = u|_{\mathbb{R}^{N-1}}$ and ends the proof of (i). □

One can prove that γ is not onto $L^2(\partial\Omega)$, i.e. that there exist functions in $L^2(\partial\Omega)$ which are not traces of any element of $H^1(\Omega)$. This leads to the following definition:

Definition 3.29. Suppose that $\partial\Omega$ is Lipschitz continuous. Define the set $H^{\frac{1}{2}}(\partial\Omega)$ as the range of the map γ given by Theorem 3.28, i.e. $H^{\frac{1}{2}}(\partial\Omega) = \gamma(H^1(\Omega))$.

The following theorem provides a structure of a Banach space for this set:

Theorem 3.30. *Suppose that $\partial\Omega$ is Lipschitz continuous. Then $H^{\frac{1}{2}}(\partial\Omega)$ is a Banach space for the norm defined by*

$$\|u\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = \int_{\partial\Omega} |u(x)|^2 \, ds_x + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+1}} \, ds_x \, ds_y.$$

The introduction of this norm is motivated by the fact that the continuity of the trace map γ is preserved when $L^2(\partial\Omega)$ is equipped with this norm. Indeed,

Proposition 3.31. *Suppose that $\partial\Omega$ is Lipschitz continuous. Then there exists a constant $C_\gamma(\Omega)$ such that*

$$\|\gamma(u)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_\gamma(\Omega)\|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Other properties of the space $H^{\frac{1}{2}}(\partial\Omega)$ are given by the following result:

Proposition 3.32. *Suppose that $\partial\Omega$ is Lipschitz continuous. Then, the space $H^{\frac{1}{2}}(\partial\Omega)$ has the following properties:*

i) The set $\{u|_{\partial\Omega}, \ u \in C^\infty(\mathbb{R}^N)\}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)$.

ii) The injection $H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$ is compact.

iii) There exists a linear continuous map

$$g \in H^{\frac{1}{2}}(\partial\Omega) \longmapsto u_g \in H^1(\Omega),$$

with $\gamma(u_g) = g$, and there exists a constant $C_1(\Omega)$ depending only on Ω, such that

$$\|u_g\|_{H^1(\Omega)} \leq C_1(\Omega)\,\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}, \quad \forall g \in H^{\frac{1}{2}}(\partial\Omega).$$

Let us mention now that if $\partial\Omega$ is Lipschitz continuous (see Nečas, 1967), then the unit outward normal vector to Ω is well defined almost everywhere. Then the following theorem extends to Sobolev spaces the well-known Green formula for smooth functions:

Theorem 3.33 (Green formula). *Suppose that $\partial\Omega$ is Lipschitz continuous. Let $u, v \in H^1(\Omega)$. Then,*

$$\int_\Omega u \frac{\partial v}{\partial x_i} \, dx = -\int_\Omega v \frac{\partial u}{\partial x_i} \, dx + \int_{\partial\Omega} \gamma(u)\,\gamma(v)\, n_i \, ds,$$

for $1 \leq i \leq N$ and where $n = (n_1, \dots, n_N)$ denotes the unit outward normal vector to Ω.

The next result gives the meaning of the trace for functions in $H_0^1(\Omega)$.

Proposition 3.34. *Suppose that $\partial\Omega$ is Lipschitz continuous. Then*

$$H_0^1(\Omega) = \{u \mid u \in H^1(\Omega), \gamma(u) = 0\}.$$

Recall now that by definition, the space $H_0^1(\Omega)$ is equipped with the H^1-norm. The following inequality allows us to introduce an equivalent norm on $H_0^1(\Omega)$ (see Remark 3.37 below):

Proposition 3.35 (Poincaré inequality). *There exists a constant C_Ω such that*

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \, \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega),$$

where the constant C_Ω is a constant depending on the diameter of Ω.

Proof. Let I be an interval of $\mathbb{R}^N$ containing $\overline{\Omega}$. Let $u \in H_0^1(\Omega)$ and denote by $\tilde{u}$ the extension by zero of u to the whole of I. From Definition 3.26 of $H_0^1(\Omega)$, it follows that $\tilde{u} \in H_0^1(I)$. Obviously,

$$\|u\|_{L^2(\Omega)} = \|\tilde{u}\|_{L^2(I)}, \quad \|\nabla u\|_{L^2(\Omega)} = \|\nabla \tilde{u}\|_{L^2(I)}, \quad \forall u \in H_0^1(\Omega).$$

Hence, it is enough to prove the result for the case where Ω is an interval I of the form $\Omega =]0, a[^N$. We have, for any $u \in \mathcal{D}(\Omega)$

$$u(x) = u(x', x_N) = \int_0^{x_N} \frac{\partial u}{\partial x_N}(x', t)\, dt.$$

Applying Cauchy–Schwarz inequality one has

$$\begin{aligned} |u(x)|^2 = \left| \int_0^{x_N} \frac{\partial u}{\partial x_N}(x', t)\, dt \right|^2 &\leq |x_N| \int_0^{x_N} \left| \frac{\partial u}{\partial x_N}(x', t) \right|^2 dt \\ &\leq a \int_0^{a} \left| \frac{\partial u}{\partial x_N}(x', t) \right|^2 dt. \end{aligned}$$

By integrating this inequality over Ω, we obtain

$$\int_\Omega u^2\, dx \leq a^2 \int_\Omega \left| \frac{\partial u}{\partial x_N} \right|^2 dx \leq a^2 \int_\Omega |\nabla u|^2\, dx.$$

Therefore, we have

$$\|u\|_{L^2(\Omega)} \leq a \|\nabla u\|_{L^2(\Omega)},$$

for all $u \in \mathcal{D}(\Omega)$, and by density for all $u \in H_0^1(\Omega)$.

Observe now that if Ω is an arbitrary bounded open set, one can always find an interval I with sides depending on the diameter of Ω such that $\Omega \subset I$. This ends the proof. □

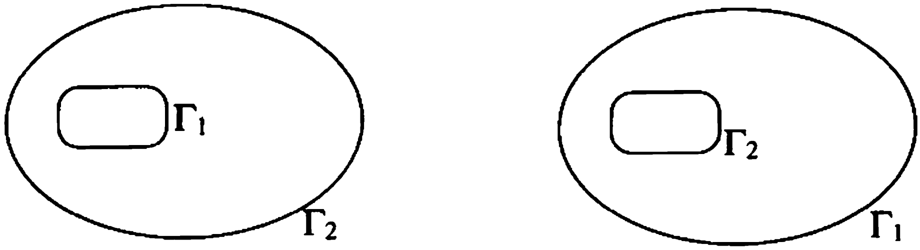


Fig. 3.3

A variant of the previous result is the following one:

Proposition 3.36. *Let Ω be connected. Suppose that $\partial\Omega$ is Lipschitz continuous and such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are two disjoint closed sets and Γ_1 is of positive measure. Then, there exists a constant C_Ω such that*

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \, \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega) \text{ such that } \gamma(u) = 0 \text{ on } \Gamma_1,$$

where the constant C_Ω depends on the diameter of Ω and on Γ_1.

Remark 3.37. Observe that by construction. Γ_1 can be regarded as the boundary (Lipschitz continuous) of some bounded open set in $\mathbb{R}^N$. Two possible configurations are shown in Fig. 3.3. Hence, the trace $\gamma(u)$ on Γ_1 makes sense. ◊

The Poincaré inequality implies that

$$\|u\| = \|\nabla u\|_{L^2(\Omega)}$$

is a norm on $H_0^1(\Omega)$, equivalent to the norm of $H^1(\Omega)$ defined in Proposition 3.17 (i). Of course, this equivalence does not hold in $H^1(\Omega)$ since for constant functions, the above quantity vanishes. As can be seen from the Proposition 3.38 below, such an equivalence holds for the subspace of functions with zero mean value. Moreover, Proposition 3.40 shows that this equivalence also holds on the quotient space $H^1(\Omega)/\mathbb{R}$ defined in Definition 3.39 below.

Proposition 3.38 (Poincaré–Wirtinger inequality). *Suppose that Ω is connected. Then. there exists a constant $C(\Omega)$ such that*

$$\|u - \mathcal{M}_\Omega(u)\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega),$$

where $\mathcal{M}_\Omega(u)$ denotes the mean value of u on Ω introduced in Definition 2.2.

Definition 3.39. Suppose that Ω is connected. The quotient space

$$W(\Omega) = H^1(\Omega)/\mathbb{R}$$

is defined as the space of classes of equivalence with respect to the relation

$$u \simeq v \Longleftrightarrow u - v \text{ is a constant}, \quad \forall u, v \in H^1(\Omega).$$

We denote by $\dot{u}$ the class of equivalence represented by u.

Proposition 3.40. *Suppose that Ω is connected. The following quantity:*

$$\|\dot{u}\|_{W(\Omega)} = \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in \dot{u}, \ \dot{u} \in W(\Omega),$$

defines a norm on $W(\Omega)$ for which $W(\Omega)$ is a Banach space.

Moreover, $W(\Omega)$ is a Hilbert space for the scalar product

$$(v, w)_{W(\Omega)} = \sum_{i=1}^{N} \left(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right)_{L^2(\Omega)}, \quad \forall v, w \in W(\Omega).$$

Proof. It is sufficient to observe that

$$\|\nabla u\|_{L^2(\Omega)} = 0,$$

implies that

$$u = \text{constant}, \text{ i.e. } \dot{u} \simeq 0,$$

which means that $u \in \dot{0}$.

The completeness of $W(\Omega)$ is straightforward from that of $H^1(\Omega)$. □

Another important space in the study of elliptic problems is the dual space of $H_0^1(\Omega)$. By making use of Proposition 1.4, we can give the following definition:

Definition 3.41. We denote by $H^{-1}(\Omega)$ the Banach space defined by

$$H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$$

equipped with the norm

$$\|F\|_{H^{-1}(\Omega)} = \sup_{H_0^1(\Omega)\setminus\{0\}} \frac{|\langle F, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}|}{\|u\|_{H_0^1(\Omega)}}.$$

The next proposition provides an important characterization of $H^{-1}(\Omega)$:

Proposition 3.42. *Let F be in $H^{-1}(\Omega)$. Then, there exists $N + 1$ functions $f_0, f_1, \ldots, f_N$ in $L^2(\Omega)$ such that*

$$F = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \tag{3.6}$$

in the sense of distributions. Moreover

$$\|F\|^2_{H^{-1}(\Omega)} = \inf \sum_{i=0}^{N} \|f_i\|^2_{L^2(\Omega)},$$

where the infimum is taken over all the vectors $(f_0, f_1, \ldots, f_N)$ in $[L^2(\Omega)]^{N+1}$ such that (3.6) holds.

Conversely, if $(f_0, f_1, \ldots, f_N)$ is a vector in $[L^2(\Omega)]^{N+1}$, then (3.6) defines an element F of $H^{-1}(\Omega)$ which satisfies

$$\|F\|^2_{H^{-1}(\Omega)} \leq \sum_{i=0}^{N} \|f_i\|^2_{L^2(\Omega)}.$$

Proposition 3.43. *Suppose that $\partial\Omega$ is Lipschitz continuous. One has $L^2(\Omega) \subset H^{-1}(\Omega)$ with compact injection.*

Remark 3.44. Putting together Remark 1.37, Theorem 1.38, Definitions 3.26 and 3.39, and taking into account Proposition 3.43 and Sobolev embeddings (Theorem 3.27), we have that the following inclusions are compact:

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega).$$

Notice also that, if $u \in H_0^1(\Omega)$ and $v \in L^2(\Omega)$, then from Theorem 1.36 one has

$$\langle v, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega u\, v\, dx.$$

Similarly, if $u \in H^{\frac{1}{2}}(\partial\Omega)$ and $v \in L^2(\partial\Omega)$, one also has

$$\langle v, u \rangle_{(H^{\frac{1}{2}}(\partial\Omega))', H^{\frac{1}{2}}(\partial\Omega)} = \int_{\partial\Omega} u\, v\, ds.$$

◇

Remark 3.45. Obviously, the restriction of any element of $(H^1(\Omega))'$ to $H_0^1(\Omega)$ is in $H^{-1}(\Omega)$. Let us notice that $(H^1(\Omega))'$ is not contained in $H^{-1}(\Omega)$ since it can be proved that the space $(H^1(\Omega))'$ can be identified with the direct sum $H^{-1}(\Omega) \oplus H^{-\frac{1}{2}}(\partial\Omega)$ where $H^{-\frac{1}{2}}(\partial\Omega)$ is defined below. ◇

Definition 3.46. Suppose that $\partial\Omega$ is Lipschitz continuous. We denote by $H^{-\frac{1}{2}}(\partial\Omega)$ the Banach space defined by

$$H^{-\frac{1}{2}}(\partial\Omega) = \left(H^{\frac{1}{2}}(\partial\Omega)\right)'$$

equipped with the norm

$$\|F\|_{H^{-\frac{1}{2}}(\partial\Omega)} = \sup_{H^{\frac{1}{2}}(\partial\Omega)\setminus\{0\}} \frac{|\langle F, u \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}|}{\|u\|_{H^{\frac{1}{2}}(\partial\Omega)}}.$$

Proposition 3.47. *The space $H^{-\frac{1}{2}}(\partial\Omega)$ has the following properties:*

i) Suppose that $\partial\Omega$ is Lipschitz continuous. Then, one has $L^2(\partial\Omega) \subset H^{-\frac{1}{2}}(\partial\Omega)$ with compact injection.

ii) Suppose that $\partial\Omega$ *is Lipschitz continuous and introduce the space*

$$H(\Omega, \mathrm{div}) = \{v \mid v \in (L^2(\Omega))^N,\ \mathrm{div}\ v \in L^2(\Omega)\}.$$

Then, $v \cdot n \in H^{-\frac{1}{2}}(\partial\Omega)$ *and the map*

$$v \in H(\Omega, \mathrm{div}) \longmapsto v \cdot n \in H^{-\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Moreover, if $v \in H(\Omega, \mathrm{div})$ *and* $w \in H^1(\Omega)$*, then*

$$-\int_\Omega (\mathrm{div}\ v)\ w\, dx = \int_\Omega v\ \nabla w\, dx\ + \langle v \cdot n,\ w\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}.$$

Observe that the first assertion of this proposition is dual with respect to (ii) of Proposition 3.32. The second assertion is an important result due to Lions et Magenes (1968a). It plays an essential role in many questions arising in the study of partial differential equations, as can be seen in Sections 4.5 and 6.4 (for further examples see also Duvaut and Lions, 1972). Let us point out that *a priori*, as already observed, a function in $L^2(\Omega)$ does not have a trace on the boundary. The fact that its divergence is also in $L^2(\Omega)$ allows us, nevertheless, to give a sense to $v \cdot n$.

3.4 The space H^1_{per}

In this section, we introduce a notion of periodicity for functions in the Sobolev space H^1. Let us recall that in Chapter 2 this notion was treated for function in L^1.

Let Y be the reference cell defined by (2.1). namely $Y =]0, \ell_1[\times \cdots \times]0, \ell_N[$, where $\ell_1, \ldots, \ell_N$ are given positive numbers.

Definition 3.48. Let $C^\infty_{\mathrm{per}}(Y)$ be the subset of $C^\infty(\mathbb{R}^N)$ of Y-periodic functions. We denote by $H^1_{\mathrm{per}}(Y)$ the closure of $C^\infty_{\mathrm{per}}(Y)$ for the H^1-norm.

From this definition and the proof of Theorem 3.28, it is obvious that the space $H^1_{\mathrm{per}}(Y)$ has the following properties:

Proposition 3.49. *Let* $u \in H^1_{\mathrm{per}}(Y)$*. Then.* u *has the same trace on the opposite faces of* Y.

Let g be a function defined a.e. on Y and denote by $g^\#$ its extension by periodicity to the whole of $\mathbb{R}^N$, defined by

$$g^\#(x + k\,\ell_i\, e_i) = g(x) \quad \text{a.e. on } Y, \quad \forall k \in \mathbb{Z}, \quad \forall i \in \{1, \ldots, N\}, \qquad (3.7)$$

where $\{e_1, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^N$.

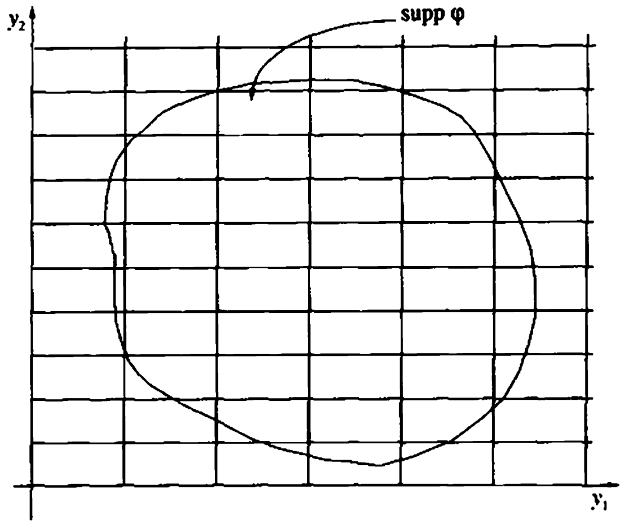


Fig. 3.4

Proposition 3.50. *Let $u \in H^1_{\text{per}}(Y)$ and $u^\#$ be its extension defined by (3.7). Then $u^\#$ is in $H^1(\omega)$ for any bounded open subset ω of $\mathbb{R}^N$.*

Proof. It is obvious that $u^\# \in L^2(\omega)$ and $(\partial u/\partial x_i)^\# \in L^2(\omega)$. Let us show that

$$\frac{\partial u^\#}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^\#, \quad \forall i \in \{1, \dots, N\},$$

which will prove the result.

To show this identity, let $\varphi \in \mathcal{D}(\omega)$. Remark that supp φ can be covered by a finite union of translated sets of $\overline{Y}$, as follows (see Fig. 3.4):

$$\text{supp } \varphi \subset \bigcup_{k \in K(\omega)} \overline{Y_k} = I_\omega, \tag{3.8}$$

where $K(\omega)$ is a finite subset of $\mathbb{Z}^N$ and the intervals Y_k are pairwise disjoint and

$$Y_k = Y + z(k), \quad \text{for some } z(k) = (k_1\ell_1, \dots, k_N\ell_N) \in \mathbb{R}^N, \ k \in \mathbb{Z}^N.$$

Then, by Definition 3.11

$$\left\langle \frac{\partial u^\#}{\partial x_i}, \varphi \right\rangle_{\mathcal{D}'(\omega),\mathcal{D}(\omega)} = -\int_\omega u^\# \frac{\partial \varphi}{\partial x_i}\, dx = -\sum_{k \in K(\omega)} \int_{Y_k} u^\# \frac{\partial \varphi}{\partial x_i}\, dx. \tag{3.9}$$

Hence, by a change of variables and using (3.7),

$$\int_{Y_k} u^\# \frac{\partial \varphi}{\partial x_i}\, dx = \int_Y u(y) \frac{\partial \varphi}{\partial x_i}(y - z(k))\, dy, \tag{3.10}$$

where we can use the Green formula from Theorem 3.33 to obtain

$$\left\{\begin{aligned}&\int_Y u(y)\frac{\partial\varphi}{\partial x_i}(y-z(k))\,dy\\&\quad=-\int_Y\frac{\partial u}{\partial y_i}(y)\varphi(y-z(k))\,dy+\int_{\partial Y}u(y)\varphi(y-z(k))\,n_i\,ds_y\\&\quad=-\int_Y\frac{\partial u}{\partial y_i}(y)\varphi(y-z(k))\,dy+\int_{F_i^+\cup F_i^-}u(y)\,\varphi(y-z(k))\,n_i\,ds_y,\end{aligned}\right.\tag{3.11}$$

where $F_i^{\pm}$ are the faces of ∂Y normal to the direction x_i. But

$$\begin{aligned}&\int_{F_i^+\cup F_i^-}u(y)\,n_i\,\varphi(y-z(k))\,ds_y\\&\quad=\int_{\Pi_{j\neq i}]0,\ell_j[}\big[u(y)\varphi(y-z(k))\big]\big|_{y_i=\ell_i}\,dy_1\cdots dy_{i-1}dy_{i+1}\cdots dy_N\\&\qquad-\int_{\Pi_{j\neq i}]0,\ell_j[}\big[u(y)\varphi(y-z(k))\big]\big|_{y_i=0}\,dy_1\cdots dy_{i-1}dy_{i+1}\cdots dy_N.\end{aligned}$$

Consider now the cell $Y_{k'}$, adjacent to Y_k in the x_i-direction, i.e.

$$Y_{k'}=Y+z(k'),$$

where $z(k')=z(k)+\ell_ie_i=(k_1\ell_1,\dots,k_{i-1}\ell_{i-1},(k_i+1)\ell_i,k_{i+1}\ell_{i+1},\dots,k_N\ell_N)$.
Performing the same computation, we have

$$\begin{aligned}&\int_{F_i^+\cup F_i^-}u(y)\,n_i\,\varphi(y-z(k'))\,ds_y\\&\quad=\int_{\Pi_{j\neq i}]0,\ell_j[}\big[u(y)\varphi(y-z(k'))\big]\big|_{y_i=\ell_i}\,dy_1\cdots dy_{i-1}dy_{i+1}\cdots dy_N\\&\qquad-\int_{\Pi_{j\neq i}]0,\ell_j[}\big[u(y)\varphi(y-z(k'))\big]\big|_{y_i=0}\,dy_1\cdots dy_{i-1}dy_{i+1}\cdots dy_N.\end{aligned}$$

Notice that

$$\begin{aligned}&\varphi(y-z(k'))|_{y_i=0}\\&\quad=\varphi(y_1-k_1\ell_1,\dots,y_{i-1}-k_{i-1}\ell_{i-1},-(k_i+1)\ell_i,y_{i+1}\\&\qquad-k_{i+1}\ell_{i+1},\dots,y_N-k_N\ell_N)\\&\quad=\varphi(y-z(k))|_{y_i=\ell_i},\end{aligned}$$

hence, since u is Y-periodic,

$$\left\{\begin{aligned}&-\int_{\Pi_{j\neq i}]0,\ell_j[}\big[u(y)\varphi(y-z(k'))\big]\big|_{y_i=0}\,dy_1\cdots dy_{i-1}dy_{i+1}\cdots dy_N\\&\quad=-\int_{\Pi_{j\neq i}]0,\ell_j[}\big[u(y)\,\varphi(y-z(k))\big]\big|_{y_i=\ell_i}\,dy_1\cdots dy_{i-1}dy_{i+1}\cdots dy_N.\end{aligned}\right.\tag{3.12}$$

Use now (3.10) and (3.11) in (3.9). When summing the boundary terms on the different faces, most of them cancel two by two due to (3.12). The only boundary terms for which we cannot use (3.12) are those corresponding to the boundary of the set I_ω defined by (3.8). Since φ vanishes on ∂I_ω, these terms also vanish. Hence, using again (3.7), (3.9) becomes

$$\left\langle \frac{\partial u^\#}{\partial x_i}, \varphi \right\rangle_{\mathcal{D}'(\omega),\mathcal{D}(\omega)} = \sum_{k\in K(\omega)} \int_Y \frac{\partial u}{\partial y_i}(y)\varphi(y - z(k))\, dy = \int_\omega \left(\frac{\partial u}{\partial x_i}\right)^\# \varphi\, dx,$$

which ends the proof. □

In the sequel, we will make use of the space $H^1_{\text{per}}(Y)/\mathbb{R}$ defined in the spirit of Definition 3.39 as follows:

Definition 3.51. The *quotient space*

$$\mathcal{W}_{\text{per}}(Y) = H^1_{\text{per}}(Y)/\mathbb{R}$$

is defined as the space of equivalence classes with respect to the relation

$$u \simeq v \iff u - v \text{ is a constant}, \quad \forall u, v \in H^1_{\text{per}}(Y).$$

We denote by $\dot{u}$ the equivalence class represented by u.

Thanks to Proposition 3.40 one has

Proposition 3.52. *The following quantity:*

$$\|\dot{u}\|_{\mathcal{W}_{\text{per}}(Y)} = \|\nabla u\|_{L^2(Y)}, \quad \forall u \in \dot{u},\ \dot{u} \in \mathcal{W}_{\text{per}}(Y),$$

defines a norm on $\mathcal{W}_{\text{per}}(Y)$.

Moreover, the dual space $(\mathcal{W}_{\text{per}}(Y))'$ can be identified with the set

$$\{F \in (H^1_{\text{per}}(Y))' \mid F(c) = 0,\ \forall c \in \mathbb{R}\},$$

with

$$\langle F, \dot{u}\rangle_{(\mathcal{W}_{\text{per}}(Y))', \mathcal{W}_{\text{per}}(Y)} = \langle F, u\rangle_{(H^1_{\text{per}}(Y))', H^1_{\text{per}}(Y)} \quad \forall u \in \dot{u},\ \forall \dot{u} \in \mathcal{W}_{\text{per}}(Y).$$

3.5 Vector-valued spaces of the type $L^p(a,b;X)$

The notion of distribution can be generalized to vector-valued functions as follows:

Definition 3.53. Let X be a Banach space and $\Omega \subset \mathbb{R}^N$. A map $T : \mathcal{D}(\Omega) \mapsto X$ is called a *distribution* on Ω with values in X, iff

i) T is linear, i.e.

$$\forall \lambda_1, \lambda_2 \in \mathbb{R},\ \varphi_1, \varphi_2 \in \mathcal{D}(\Omega), \qquad T(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 T(\varphi_1) + \lambda_2 T(\varphi_2).$$

ii) T is continuous on the sequences, i.e.

$$\big(\varphi_n \to \varphi \text{ in } \mathcal{D}(\Omega)\big) \Longrightarrow \big(T(\varphi_n) \to T(\varphi) \text{ in } X\big).$$

We denote by $\mathcal{D}'(\Omega;\, X)$ the set of distributions on Ω with values in X.

If $\Omega =]a, b[$, $\mathcal{D}'(a, b;\, X)$ denotes the set of distributions on $]a, b[$ with values in X.

Similarly, one can also define the L^p-spaces for vector-valued functions.

Definition 3.54. Let X be a Banach space, $\Omega \subset \mathbb{R}^N$ and p such that $1 \leq p \leq \infty$. We denote by $L^p(\Omega;\, X)$, the set of measurable functions $u : x \in \Omega \to u(x) \in X$ such that $\|u(x)\|_X \in L^p(\Omega)$.

Let us mention that the measurability of real functions may be defined in several equivalent ways (see for instance Dinculeanu, 1967). In the vector-valued case, these definitions are no longer equivalent. Here, a measurable function is an a.e. limit of step functions (see Definition 1.41).

Proposition 3.55. *The following quantity*

$$\|u\|_{L^p(\Omega;\, X)} = \left(\int_\Omega \|u(x)\|_X^p \, dx\right)^{\frac{1}{p}},$$

defines a norm on $L^p(\Omega;\, X)$, which is a Banach space.

If X is reflexive and $1 < p < \infty$, the space $L^p(\Omega;\, X)$ is reflexive too. Moreover, if X is separable and $1 \leq p < \infty$, then $L^p(\Omega;\, X)$ is separable.

To end this chapter, we investigate the properties of a class of vector-valued functions which is well adapted to the study of problems where one of the variables plays a special role, namely the space $L^p(a, b;\, X)$ (corresponding in Definition 3.54 to $\Omega =]a, b[$). This occurs, for example, for the variable 'time' in time-dependent problems. Another situation is that of layered materials where the periodicity concerns only one direction of the space, and this direction has therefore to be distinguished. For various results on vector-valued functions we refer to Schwartz (1951), Dinculeanu (1967), Lions and Magenes (1968a,b), Lions (1969), and Diestel and Uhl (1977).

It is obvious from Definition 3.54 that if B_0 and B are two Banach spaces such that $B_0 \subset B$ with continuous injection, then one has $L^p(a, b;\, B_0) \subset L^p(a, b;\, B)$ with continuous injection also.

Suppose now that the injection $B_0 \subset B$ is compact. A natural question is whether the injection $L^p(a, b;\, B_0) \subset L^p(a, b;\, B)$ is also compact. Actually, one can prove that this is not true, a trivial example being the case $B_0 = B = \mathbb{R}$. As Proposition 3.57 below shows, if one has some information about the derivative (in the sense of distributions) of u with respect to t, then one can give a positive answer to the above question. To do that, let us first make precise what we mean by the derivative in the sense of distributions of a vector-valued function on $]a,b[$.

Definition 3.56. Let u be in $L^p(a,b;X)$. The *derivative* $\partial u/\partial t$ is the distribution in $\mathcal{D}'(a,b;X)$ defined by

$$\frac{\partial u}{\partial t}(\varphi) = -\int_a^b u\,\frac{\partial \varphi}{\partial t}\,dt, \quad \forall \varphi \in \mathcal{D}(a,b).$$

The following result is due to J. L. Lions (1988, Chapitre 1, Theorem 5.1). We also refer to Aubin (1963) and Simon (1987) for some generalizations.

Proposition 3.57. *Let $B_0 \subset B \subset B_1$, three Banach spaces such that B_0 and B_1 are reflexive. Suppose also that the injection $B_0 \subset B$ is compact. Define*

$$W = \left\{ v \mid v \in L^{p_0}(a,b;\ B_0),\ \frac{\partial v}{\partial t} \in L^{p_1}(a,b;\ B_1) \right\},$$

with $1 < p_0, p_1 < +\infty$. Then

i) W is a Banach space with respect to the norm of the graph defined by

$$\|u\|_W = \|u\|_{L^{p_0}(a,b;\ B_0)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^{p_1}(a,b;\ B_1)},$$

ii) the injection $W \subset L^{p_0}(a,b;\ B)$ is compact.

The following theorem plays an important role in the study of partial differential equations:

Theorem 3.58. *Let us define the Banach spaces*

$$\mathcal{W} = \left\{ v \mid v \in L^2(a,b;\ H_0^1(\Omega)),\ \frac{\partial v}{\partial t} \in L^2(a,b;\ H^{-1}(\Omega)) \right\},$$

$$\mathcal{W}_1 = \left\{ v \mid v \in L^2(a,b;\ L^2(\Omega)),\ \frac{\partial v}{\partial t} \in L^2(a,b;\ H^{-1}(\Omega)) \right\},$$

equipped with the norm of the graph. Then, the following properties hold true:

i) the injections

$$\mathcal{W} \subset L^2(a,b;\ L^2(\Omega)), \quad \mathcal{W}_1 \subset L^2(a,b;\ H^{-1}(\Omega))$$

are compact,

ii) one has the inclusions

$$\begin{aligned} \mathcal{W} &\subset C([a,b];\, L^2(\Omega)), \\ \mathcal{W}_1 &\subset C([a,b];\, H^{-1}(\Omega)), \end{aligned}$$

where, for $X = L^2(\Omega)$ *or* $X = H^{-1}(\Omega)$, *one denotes by* $C([a,b];\, X)$ *the space of measurable functions on* $\Omega \times [a,b]$ *such that* $u(\cdot,t) \in X$ *for any* $t \in [a,b]$ *and such that the map* $t \in [a,b] \mapsto u(\cdot,t) \in X$ *is continuous,*

iii) for any $u, v \in \mathcal{W}$ *one has*

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(x,t)\, v(x,t)\, dx &= \langle u'(\cdot,t),\, v(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\quad + \langle v'(\cdot,t),\, u(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \end{aligned}$$

Proof. Statement (i) is an easy consequence of Theorem 3.27 and Proposition 3.43. On the contrary, the proof of the second statement is rather complicated, we skip it and refer the reader to Lions and Magenes (1968a, Chapter 1, Theorem 3.1).

Let us prove the third statement. Recall that $\mathcal{D}(]a,b[\times\Omega)$ is dense in $\mathcal{W}$ (see, for instance, Lions and Magenes, 1968a). Let $\{u_m\}$ and $\{v_m\}$ be two sequences in $\mathcal{D}(]a,b[\times\Omega)$, strongly converging in $\mathcal{W}$ respectively, to u and v. Let $\varphi \in \mathcal{D}(a,b)$. Due to Definition 3.11 of the derivatives in the sense of distributions, we have

$$\begin{aligned} &\left\langle \frac{d}{dt}\int_\Omega u(x,t)\, v(x,t)\, dx,\ \varphi \right\rangle_{\mathcal{D}'(a,b),\mathcal{D}(a,b)} \\ &= -\int_a^b\int_\Omega u(x,t)\, v(x,t)\, \varphi'(t)\, dx\, dt \\ &= -\lim_{m\to\infty}\int_a^b\int_\Omega u_m(x,t)\, v_m(x,t)\, \varphi'(t)\, dx\, dt \\ &= \lim_{m\to\infty}\int_a^b\int_\Omega \left[u_m'(x,t)\, v_m(x,t) + u_m(x,t)\, v_m'(x,t)\right] \varphi(t)\, dx\, dt \\ &= \lim_{m\to\infty}\int_a^b \big[\langle u_m'(\cdot,t), v_m(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\qquad + \langle v_m'(\cdot,t), u_m(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\big]\varphi(t)\, dt \\ &= \int_a^b \left[\langle u'(\cdot,t),\, v(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle v'(\cdot,t),\, u(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right] \varphi(t)\, dt \\ &= \langle [\langle u'(\cdot,t),\, v(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\qquad + \langle v'(\cdot,t),\, u(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}],\ \varphi\rangle_{\mathcal{D}'(a,b),\mathcal{D}(a,b)}. \end{aligned}$$

This implies the required equality in $\mathcal{D}'(a,b)$, and due to Remark 3.8, in $L^1(a,b)$.

□

The following proposition characterizes the dual of $L^p(a,b;\ H_0^1(\Omega))$ (for more details see Diestel and Uhl, 1977):

Proposition 3.59. *Let H be a Hilbert space and $1 \le p < \infty$. One has the following identification:*

$$[L^p(a,b;\ H)]' = L^{p'}(a,b;\ H'),$$

where p' is the conjugate of p.

In particular, $[L^2(a,b;\ H_0^1(\Omega))]' = L^2(a,b;\ H^{-1}(\Omega))$, and if $f \in L^2(a,b;\ H^{-1}(\Omega))$, one has

$$\langle f,u\rangle_{L^2(a,b;\ H^{-1}(\Omega)),L^2(a,b;\ H_0^1(\Omega))} = \int_a^b \langle f(t),u(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt, \tag{3.13}$$

for any $u \in L^2(a,b;\ H_0^1(\Omega))$.

We will now formulate a density result, which is also very important in the study of time-dependent problems.

Proposition 3.60. *Let $u \in L^2(a,b;\ H_0^1(\Omega)) \cap C([a,b];\ L^2(\Omega))$. Then, for any $\delta > 0$, there exists $\Phi \in C^\infty([a,b];\ \mathcal{D}(\Omega))$, such that*

$$\begin{cases} i) & \|u-\Phi\|_{C([a,b];\ L^2(\Omega))} \le \delta \\ ii) & \|\nabla u - \nabla\Phi\|_{L^2(\Omega\times]a,b[)} \le \delta, \end{cases}$$

where $C^\infty([a,b];\ \mathcal{D}(\Omega))$ is the space of measurable functions on $\Omega\times[a,b]$ such that $u(\cdot,t) \in \mathcal{D}(\Omega)$ for any $t \in [a,b]$, and such that the map $t \in [a,b] \mapsto u(\cdot,t) \in \mathcal{D}(\Omega)$ is indefinitely differentiable.

If further $u' \in C([a,b];\ L^2(\Omega))$, then for any $\delta > 0$, there exists $\Phi \in C^\infty([a,b];\ \mathcal{D}(\Omega))$, such that

$$\begin{cases} i) & \|u'-\Phi'\|_{C([a,b];\ L^2(\Omega))} \le \delta \\ ii) & \|\nabla u - \nabla\Phi\|_{L^2(\Omega\times]a,b[)} \le \delta. \end{cases}$$

We end this section by recalling some properties, useful in the sequel, concerning the space $L^2(\Omega;C_{\mathrm{per}}(Y))$ where $C_{\mathrm{per}}(Y)$ denotes the subset of $C(\overline{Y})$ of Y-periodic functions.

Proposition 3.61. *The following properties hold:*

i) The space $L^2(\Omega;C_{\mathrm{per}}(Y))$ is separable.

ii) The space $L^2(\Omega;C_{\mathrm{per}}(Y))$ is dense in $L^2(\Omega;\ L^2(Y)) = L^2(\Omega\times Y)$.

Proof. The first statement follows immediately from Proposition 3.55 since $C_{\mathrm{per}}(Y)$ is separable (see for instance Rudin, 1966). As concerning the second statement, observe that a consequence of Theorem 1.38 is the density of the space $L^2(\Omega;\ \mathcal{D}(Y))$ in $L^2(\Omega;\ L^2(Y))$. Then, property (ii) follows from the obvious inclusion $L^2(\Omega;\ \mathcal{D}(Y)) \subset L^2(\Omega;\ C_{\mathrm{per}}(Y))$. □

4

Some variational elliptic problems

We study in this chapter some classical elliptic partial differential equations in the framework of weak solutions. The problems we deal with are linear elliptic partial differential equations with different boundary conditions: Dirichlet, Neumann, Robin and periodic conditions. In all these cases, the existence and uniqueness of the solution are obtained by applying the Lax–Milgram theorem. This important theorem is proved in Section 4.2 below.

4.1 Bilinear forms on Banach spaces

Let us recall here some basic properties of bilinear maps on Banach spaces. In all this section V denotes a real Banach space.

Definition 4.1. Let a be a map from $V \times V$ to $\mathbb{R}$. It is called a *bilinear form* on V iff, for any fixed $u \in V$, the following maps:

$$\begin{aligned} a(u,\cdot) &: v \in V \longmapsto a(u,v) \in \mathbb{R}, \\ a(\cdot,u) &: v \in V \longmapsto a(v,u) \in \mathbb{R}, \end{aligned}$$

are linear.

Definition 4.2. A map a from $V \times V$ to $\mathbb{R}$ is *bounded* on V iff there exists $C > 0$ such that

$$|a(u,v)| \leq C \|u\|_V \|v\|_V. \tag{4.1}$$

Proposition 4.3. *Let $a : V \times V \longmapsto \mathbb{R}$ be a bilinear form. Then a is bounded if and only if a is continuous on $V \times V$.*

Proof. Suppose that (4.1) holds. Then, for (u,v), $(u_0, v_0) \in V \times V$, one has

$$\begin{aligned} |a(u,v) - a(u_0,v_0)| &\leq |a(u, v - v_0)| + |a(u - u_0, v_0)| \\ &\leq C \|u\|_V \|v - v_0\|_V + C \|u - u_0\|_V \|v_0\|_V, \end{aligned}$$

which gives the continuity of a on $V \times V$.

Suppose now that a is continuous, so in particular, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|w\|_V < \delta,\ \|z\|_V < \delta \Longrightarrow |a(w,z)| < \varepsilon. \tag{4.2}$$

By linearity, for any $u, v \in V \setminus \{0\}$, one has

$$\frac{|a(u,v)|}{\|u\|_V \|v\|_V} = \left| a\left(\frac{u}{\|u\|_V}, \frac{v}{\|v\|_V} \right) \right|. \tag{4.3}$$

Introduce w and z as follows:

$$w = \frac{\delta}{2} \frac{u}{\|u\|_V}, \qquad z = \frac{\delta}{2} \frac{v}{\|v\|_V}. \tag{4.4}$$

Obviously,

$$\|w\|_V = \frac{\delta}{2} < \delta, \qquad \|z\|_V = \frac{\delta}{2} < \delta.$$

Let us write (4.2) for w and z defined in (4.4). One has

$$\left| a\left(\frac{u}{\|u\|_V}, \frac{v}{\|v\|_V} \right) \right| = \frac{4}{\delta^2} |a(w,z)| \leq \frac{4}{\delta^2} \varepsilon,$$

which, together with (4.3) gives

$$|a(u,v)| \leq \frac{4}{\delta^2} \varepsilon \|u\|_V \|v\|_V.$$

This gives (4.1) with $C = \frac{4}{\delta^2} \varepsilon$. □

In the sequel we need the following definition (see Nečas, 1967):

Definition 4.4. A bilinear form a on V is called *symmetric* iff

$$a(u,v) = a(v,u), \quad \forall\, u, v \in V.$$

It is called *positive* iff

$$a(u,u) \geq 0, \quad \forall\, u \in V.$$

The form a is called *V-elliptic* (or coercive on V) with constant α_0, iff there exists $\alpha_0 > 0$ such that

$$a(u,u) \geq \alpha_0 \|u\|_V^2, \quad \forall\, u \in V.$$

4.2 The Lax–Milgram theorem

Let a be a bilinear form on a Hilbert space H (see Definition 1.2) and $F \in H'$. Let us consider the problem

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u,v) = \langle F, v \rangle_{H',H}, \quad \forall v \in H. \end{cases} \tag{4.5}$$

This abstract equation is called a variational equation and $v \in H$ is usually called a test function. Theorem 4.6 below gives, under suitable assumptions on a, the existence and uniqueness of a solution of (4.5). This theorem is based on the well-known Riesz representation theorem for the dual of a Hilbert space. Let us recall it.

Theorem 4.5 (Riesz theorem). *Let H be a Hilbert space and $F \in H'$. Then there exists a unique $\tau F \in H$ such that*

$$\langle F, v\rangle_{H',H} = (\tau F, v)_H. \quad \forall v \in H. \tag{4.6}$$

Moreover, the one-to-one application

$$\tau : F \in H' \longmapsto \tau F \in H$$

is an isometry (called the Riesz isometry), i.e. it satisfies

$$\|\tau F\|_H = \|F\|_{H'}. \tag{4.7}$$

We are now able to prove the following general result:

Theorem 4.6 (Lax–Milgram theorem). *Let a be a continuous bilinear form on a Hilbert space H and $F \in H'$. Assume that a is H-elliptic with constant α_0. Then the variational equation (4.5) has a unique solution $u \in H$.*

Moreover,

$$\|u\|_H \leq \frac{1}{\alpha_0}\|F\|_{H'}. \tag{4.8}$$

Proof. For $u \in H$ denote by Au the map

$$Au : v \in H \longmapsto a(u, v) \in \mathbb{R}. \tag{4.9}$$

From Proposition 4.3, we have

$$|\langle Au, v\rangle_{H',H}| = |a(u, v)| \leq C\|u\|_H\|v\|_H.$$

Hence $Au \in H'$ with

$$\|Au\|_{H'} \leq C\|u\|_H. \tag{4.10}$$

Then, from Theorem 4.5, there exists $\tau Au \in H$ such that

$$\langle Au, v\rangle_{H',H} = (\tau Au, v)_H. \quad \forall v \in H. \tag{4.11}$$

Similarly, since $F \in H'$, there exists $\tau F \in H$ such that (4.6) holds.

From (4.6), (4.9) and (4.11), it follows that problem (4.5) is equivalent to the following one: find $u \in H$ such that

$$(\tau Au - \tau F, v)_H = 0, \quad \forall v \in H,$$

i.e. such that

$$\tau A u = \tau F. \tag{4.12}$$

Let us observe now that in order to prove that (4.12) has a unique solution, it is enough to show that there exists $\rho > 0$, such that the map

$$\Phi : v \in H \longmapsto v - \rho(\tau A u - \tau F) \in H,$$

is a contraction, that is to say

$$\exists c < 1, \quad \|\Phi(w_1) - \Phi(w_2)\|_H \leq c\|w_1 - w_2\|_H, \quad \forall w_1, w_2 \in H. \tag{4.13}$$

Indeed, if Φ is a contraction, then by the Banach fixed point theorem (see, for instance, Dunford and Schwartz, 1958) it has a unique fixed point u such that

$$\Phi(u) = u, \quad \text{i.e.} \quad u - \rho(\tau A u - \tau F) = u,$$

which is equivalent to (4.12) if ρ is strictly positive.

To prove (4.13), remark that

$$\Phi(w_1) - \Phi(w_2) = w_1 - w_2 - \rho\tau A(w_1 - w_2).$$

Therefore, it is sufficient to show that there exists $c < 1$, such that

$$\|v - \rho\tau A v\|_H \leq c\|v\|_H, \quad \forall v \in H.$$

We have, by using (4.7), (4.9), (4.10), (4.11) and the H-ellipticity of the form a

$$\begin{aligned} \|v - \rho\tau A v\|_H^2 &= (v - \rho\tau A v, v - \rho\tau A v) = \|v\|_H^2 - 2\rho(\tau A v, v) + \rho^2\|\tau A v\|_H^2 \\ &= \|v\|_H^2 - 2\rho\, a(v, v) + \rho^2\|Av\|_{H'}^2 \leq (1 - 2\rho\alpha_0 + \rho^2 C^2)\|v\|_H^2. \end{aligned}$$

Choosing here $\rho \in]0, 2\alpha_0/C^2[$ one has (4.13). since then $(1 - 2\rho\alpha_0 + \rho^2 C^2) < 1$.

It remains to prove estimate (4.8). This is an obvious consequence of the H-ellipticity with constant α_0 of the bilinear form a and of inequality (1.1) applied to F. Indeed, one has

$$\alpha_0\|u\|_H^2 \leq a(u, u) = |\langle F, u\rangle_{H',H}| \leq \|F\|_{H'}\|u\|_H,$$

from which (4.8) is straightforward. The proof of Theorem 4.6 is complete. □

Remark 4.7. If the form a is symmetric, the proof of Theorem 4.6 is much simpler, since in this case $a(u, v)$ is a scalar product equivalent to $(\cdot, \cdot)_H$. Then the result is an easy consequence of the Riesz theorem (Theorem 4.5). ◊

As matter of fact, in the symmetric case the solution of (4.5) can be characterized as the minimum point of a suitable functional. Indeed, the following result holds true:

Theorem 4.8. *Let a be a continuous bilinear form on a Hilbert space H and $F \in H'$. Assume that a is positive and symmetric. Let J be the functional on H defined by*

$$J(v) = \tfrac{1}{2}a(v,v) - \langle F, v\rangle_{H',H}, \quad \forall v \in H. \tag{4.14}$$

Then u is solution of the variational equation (4.5) if and only if u is solution of the following problem:

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ J(u) = \inf_{v\in H} J(v). \end{cases} \tag{4.15}$$

Proof. Suppose first that u is solution of (4.5). Then the hypotheses on the form a and the linearity of F imply that

$$J(u+w) - J(u) = \{a(u,w) - F(w)\} + \tfrac{1}{2}a(w,w) = \tfrac{1}{2}a(w,w), \quad \forall w \in H.$$

Since a is positive, one has

$$J(u+w) - J(u) \geq 0, \quad \forall w \in H.$$

Hence, u satisfies (4.15) since any element $v \in H$ can be written as $v = u + w$ for some $w \in H$.

Suppose now that u is solution of (4.15) and let $v \in H$ and $t \in \mathbb{R}$. Then

$$0 \leq J(u+tv) - J(u) = t\{a(u,v) - F(v)\} + \frac{t^2}{2}a(v,v),$$

where we have used the assumptions on a and F. Since t is arbitrary in $\mathbb{R}$, one has that

$$a(u,v) - F(v) = 0,$$

hence u is solution of (4.5). □

The following corollary is an easy consequence of this result and Theorem 4.6:

Corollary 4.9. *Assume that the form a satisfies the hypotheses of Theorem 4.6 and that it is symmetric. Let $f \in H'$ and J be defined by (4.14). Then problem (4.15) admits a unique solution $u \in H$.*

Remark 4.10. This result is a particular case of some general results concerning the minimization of functionals on Banach spaces or on convex sets. There is a wide theory that solves this kind of problems, namely in the framework of calculus of variations and in optimization theory. We refer for instance to Kinderlehrer and Stampacchia (1980), Ciarlet (1982), Buttazzo (1989), and Dacorogna (1989). ◇

4.3 Setting of the variational formulation

The aim of this section is to introduce the reader to some classical boundary value problems in elliptic partial differential equations. The problems we present here will be formulated in the weak sense, that is to say the derivatives are taken in the sense of distributions and the solutions have to belong to some Sobolev space. Moreover, the equations are formulated in the variational sense, in the same spirit as the abstract formulation (4.5) above, in order to apply Lax–Milgram theorem.

To do so, we will have to write down, for any given boundary problem, a variational equation of the form

$$a(u,v) = \langle F, v\rangle_{H',H}$$

and introduce a suitable space H where this identity makes sense. In general, the variational equation is obtained by multiplying the partial differential equation by appropriate smooth test functions (i.e. taking into account the boundary conditions) and integrating by parts. This computation suggests the space H in which the problem has to be solved. This procedure is justified by the fact that if the data are sufficiently regular, the weak solution is also regular and is a solution in the classical sense (see, for instance Proposition 4.14 below).

As before, $\mathcal{O}$ and Ω denote respectively, an open set and a bounded open set in $\mathbb{R}^N$.

Definition 4.11. Let $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$. We denote by $M(\alpha,\beta,\mathcal{O})$ the set of the $N\times N$ matrices $A = (a_{ij})_{1\le i,j\le N} \in (L^\infty(\mathcal{O}))^{N\times N}$ such that

$$\begin{cases} i) & (A(x)\lambda,\lambda) \ge \alpha|\lambda|^2 \\ ii) & |A(x)\lambda| \le \beta|\lambda|, \end{cases} \tag{4.16}$$

for any $\lambda \in \mathbb{R}^N$ and a.e. on $\mathcal{O}$.

In the following, we will treat several examples of partial differential equations with an operator of the form

$$\mathcal{A} = -\text{div}\,(A(x)\,\nabla) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right). \tag{4.17}$$

Recall that if the matrix A is the identity, the operator in (4.17) is the classical Laplacian

$$-\Delta = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.$$

Remark 4.12. Notice first that condition (4.16)(i) is equivalent to the classical uniform ellipticity condition for the operator $\mathcal{A}$:

$$\exists\alpha > 0 \text{ such that } \sum_{i,j=1}^{N} a_{ij}(x)\lambda_i\lambda_j \ge \alpha\sum_{i=1}^{N}\lambda_i^2, \quad \text{a.e. on } \mathcal{O},\ \forall\lambda = (\lambda_1,\dots,\lambda_N)\in\mathbb{R}^N.$$

In particular, this inequality implies the invertibility of $A(x)$ a.e. on $\mathcal{O}$ (see for more details Lang, 1993, and Ciarlet, 1982). In general, if a matrix A satisfies this inequality, one says that A is elliptic.

On the other hand, condition (4.16)(ii). implies that

$$\|A(x)\|_2 \le \beta, \quad \text{a.e. on } \mathcal{O}.$$

where a.e. on $\mathcal{O}$ the following quantity

$$\|A(x)\|_2 = \sup_{\lambda \neq 0} \frac{|A(x)\lambda|}{|\lambda|},$$

is the norm of $A(x)$ as an element of $\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$, $\mathbb{R}^N$ being endowed with the Euclidean norm.

We recall now the notion of well-posed problem introduced by Hadamard. Let $\mathcal{P}$ be a boundary value problem and $\mathcal{U}, \mathcal{F}$ two Banach spaces.

Definition 4.13 (Well-posedness). We say that $\mathcal{P}$ is *well-posed* (with respect to $\mathcal{U}$ and $\mathcal{F}$) if

(1) for any element $f \in \mathcal{F}$ there exists a solution $u \in \mathcal{U}$ of $\mathcal{P}$,

(2) the solution is unique,

(3) the map $f \in \mathcal{F} \longmapsto u \in \mathcal{U}$ is continuous.

Obviously, the well-posedness of a problem depends on the choice of spaces $\mathcal{U}$ and $\mathcal{F}$. As a matter of fact, the examples we treat in the sequel have all this property. They are all related to an equation of the form

$$\mathcal{A}u = -\text{div}\,(A\,\nabla u) = f,$$

where the operator $\mathcal{A}$ is given by (4.17) and the matrix $A \in M(\alpha, \beta, \Omega)$. A boundary value problem is formulated by supplementing this equation with some boundary conditions.

Let us introduce the following notation:

$$\frac{\partial}{\partial \nu_A} = \sum_{i,j=1}^{N} a_{ij}(x)\, n_i\, \frac{\partial}{\partial x_j}. \tag{4.18}$$

where $n = (n_1, \ldots, n_N)$ denotes the unit outward normal to Ω.

We will treat the following boundary conditions:

$u = 0 \quad \text{on } \partial\Omega$ **Dirichlet condition**

$u = g \quad \text{on } \partial\Omega$ **Nonhomogeneous Dirichlet condition**

$\dfrac{\partial u}{\partial \nu_A} = 0 \quad \text{on } \partial\Omega$ **Neumann condition**

$$\frac{\partial u}{\partial \nu_A} = g \quad \text{on } \partial\Omega \qquad \textbf{Nonhomogeneous Neumann condition}$$

$$\frac{\partial u}{\partial \nu_A} + du = 0 \quad \text{on } \partial\Omega \qquad \textbf{Robin condition}$$

$$\frac{\partial u}{\partial \nu_A} + du = g \quad \text{on } \partial\Omega \qquad \textbf{Nonhomogeneous Robin condition}$$

In the last section of this chapter we also study, when $\Omega = Y$, Y being given by (2.1), a particular boundary condition which plays an essential role in the homogenization of periodic media, namely

$$u \ Y\text{-periodic} \qquad \textbf{Periodic condition}$$

4.4 The Dirichlet problem

Let $f \in H^{-1}(\Omega)$ and consider the problem

$$\begin{cases} -\text{div}\,(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.19}$$

The corresponding variational formulation is

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ a(u,v) = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega), \end{cases} \tag{4.20}$$

where

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx = \int_\Omega A\nabla u \nabla v \, dx, \quad \forall u,\, v \in H_0^1(\Omega). \tag{4.21}$$

The following proposition justifies, as we already mentioned in Section 4.3, the fact that a solution of system (4.20) is called a weak solution of system (4.19):

Proposition 4.14. *Suppose that $\partial\Omega$ is of class C^1. Let $A \in (C^1(\overline{\Omega}))^{N\times N}$, $f \in C^0(\overline{\Omega})$ and $u \in C^2(\overline{\Omega})$. Then u is solution of*

$$\begin{cases} -\text{div}\,\big(A(x)\nabla u(x)\big) = f(x) & \text{for any } x \in \Omega \\ u(x) = 0 & \text{for any } x \in \partial\Omega. \end{cases} \tag{4.22}$$

iff u is solution of (4.20).

Proof. Suppose that u is solution of (4.22). Notice that from Propositions 3.28 and 3.34, one has that $u \in H_0^1(\Omega)$. Let us multiply the equation in (4.22) by an arbitrary function $v \in \mathcal{D}(\Omega)$. By integrating by parts, we get

$$a(u,v) = \int_\Omega f\, v \, dx, \quad \forall v \in \mathcal{D}(\Omega).$$

where a is defined by (4.21). Recalling Definition 3.26 of the space $H_0^1(\Omega)$, we get by an obvious density argument. that

$$a(u,v) = \int_\Omega f\,v\,dx, \quad \forall v \in H_0^1(\Omega).$$

which, together with Remark 3.44, shows that u is a solution of (4.20).

To prove the converse implication, notice that if $u \in C^2(\overline{\Omega})$ is a solution of (4.20), then

$$\int_\Omega A\nabla u\,\nabla v\,dx = \int_\Omega f\,v\,dx. \quad \forall v \in \mathcal{D}(\Omega).$$

Integrating by parts, one has

$$\int_\Omega \left[-\operatorname{div}(A\nabla u) - f \right] v\,dx = 0, \quad \forall v \in \mathcal{D}(\Omega).$$

This, together with Theorem 1.44, implies that u is solution of (4.19), the fact that u satisfies the Dirichlet boundary being a simple consequence of Theorem 3.28 and Proposition 3.34. □

Remark 4.15. For a complete exposition concerning the properties of classical solutions, we refer the reader to Ladyzhenskaya and Uraltseva (1968), Gilbarg and Trudinger (1977), Troianiello (1987). Let us just point out here that some counterexamples (see Gilbarg and Trudinger, 1977) show that the assumptions on the data from Proposition 4.14 are not sufficient to insure the existence of a classical solution, i.e. a function in $C^2(\overline{\Omega})$ satisfying (4.22). For the existence of such a solution. more regularity on the data and on Ω are necessary. This justifies the introduction of the notion of weak solution. ◊

The first application of the Lax-Milgram theorem concerns the Dirichlet boundary value problem (4.20).

Theorem 4.16 (Homogeneous Dirichlet problem). *Suppose that the matrix A belongs to $M(\alpha,\beta,\Omega)$. Then, for any $f \in H^{-1}(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega)$ of problem (4.20). Moreover,*

$$\|u\|_{H_0^1(\Omega)} \le \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{4.23}$$

where $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$.

If $f \in L^2(\Omega)$, the solution satisfies the estimate

$$\|u\|_{H_0^1(\Omega)} \le \frac{C_\Omega}{\alpha}\|f\|_{L^2(\Omega)}. \tag{4.24}$$

where C_Ω is the Poincaré constant given by Theorem 3.35.

Proof. The proof is straightforward by applying the Lax–Milgram theorem. Indeed, from (4.21) and Remark 4.12, it follows that

$$a(v,v) \geq \alpha \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)} = \alpha \|\nabla v\|^2_{L^2(\Omega)}, \quad \forall v \in H^1_0(\Omega). \tag{4.25}$$

Due to the Poincaré inequality, the space $H^1_0(\Omega)$ can be equipped by the equivalent norm $\|\nabla v\|_{L^2(\Omega)}$, so that

$$a(v,v) \geq \alpha \|v\|^2_{H^1_0(\Omega)}, \tag{4.26}$$

which means that a is $H^1_0(\Omega)$-coercive. On the other hand, from the assumptions on the matrix A and the Cauchy–Schwarz inequality (Proposition 1.34), we get

$$|a(w,v)| \leq \beta \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} = \beta \|w\|^2_{H^1_0(\Omega)} \|v\|^2_{H^1_0(\Omega)}, \tag{4.27}$$

which gives the continuity of the form a on $H^1_0(\Omega) \times H^1_0(\Omega)$.

Consequently, we can apply the Lax–Milgram theorem for $H = H^1_0(\Omega)$, $F = f$ and a defined by (4.21) to obtain the existence and uniqueness of the solution of (4.20) as well as estimate (4.23).

Suppose now that $f \in L^2(\Omega)$ and let $u \in H^1_0(\Omega)$ be the solution of (4.20). One can choose u as test function to get

$$a(u,u) = \langle f, u \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

From Remark 3.44 and Proposition 3.35, by using again the Cauchy–Schwarz inequality, one has

$$\left| \langle f, u \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \right| = \left| \int_\Omega f\, u \, dx \right| \leq C_\Omega \|f\|_{L^2(\Omega)} \|u\|_{H^1_0(\Omega)}.$$

Then, (4.24) follows from (4.25). □

Remark 4.17. If the matrix A is symmetric then, by Corollary 4.9 it follows that the solution u given by Theorem 4.16, is the unique minimum point of the functional J defined by

$$J(v) = \frac{1}{2} \int_\Omega A \nabla v \, \nabla v \, dx - \langle f, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}, \quad \forall v \in H^1_0(\Omega).$$

◊

Remark 4.18. Theorem 4.16 shows that the Dirichlet problem (4.20) is well-posed (in the sense of Definition 4.13) for the choices

$$\mathcal{U} = H^1_0(\Omega), \qquad \mathcal{F} = H^{-1}(\Omega)$$
$$\mathcal{U} = H^1_0(\Omega), \qquad \mathcal{F} = L^2(\Omega).$$

◊

Assume now that $\partial\Omega$ is Lipschitz continuous. Suppose we are given f in $H^{-1}(\Omega)$ and g in $H^{\frac{1}{2}}(\partial\Omega)$. Consider the nonhomogeneous Dirichlet problem

$$\begin{cases} -\text{div}\,(A\nabla u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The trace notion (see Theorem 3.28 and Proposition 3.31) allows to give a weak sense to this equation. We say that u is a weak solution of this problem iff

$$\begin{cases} -\text{div}\,(A\nabla u) = f & \text{in } \mathcal{D}'(\Omega) \\ \gamma(u) = g & \text{in } H^{\frac{1}{2}}(\partial\Omega). \end{cases} \tag{4.28}$$

Then, the following result holds:

Theorem 4.19 (Nonhomogeneous Dirichlet problem). *Suppose that $\partial\Omega$ is Lipschitz continuous and that the matrix A belongs to $M(\alpha,\beta,\Omega)$. Let f in $H^{-1}(\Omega)$ and g in $H^{\frac{1}{2}}(\partial\Omega)$. Then problem (4.28) has a unique solution u in $H^1(\Omega)$. Moreover,*

$$\|u\|_{H^1(\Omega)} \leq C_1\|f\|_{H^{-1}(\Omega)} + C_2\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{4.29}$$

where C_1 and C_2 are two positive constants depending on Ω, α and β.

Proof. Since $g \in H^{\frac{1}{2}}(\partial\Omega)$, from Proposition 3.32(iii) there exists $G \in H^1(\Omega)$ such that $\gamma(G) = g$ and

$$\|G\|_{H^1(\Omega)} \leq C_1(\Omega)\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Observe that by Proposition 3.42, $f + \text{div}\,(A\nabla G) \in H^{-1}(\Omega)$. Hence, by Theorem 4.16 the following (homogeneous) Dirichlet problem with a defined by (4.21)

$$\begin{cases} \text{Find } z \in H_0^1(\Omega) \text{ such that} \\ a(z,v) = \langle f + \text{div}\,(A\nabla G), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega), \end{cases} \tag{4.30}$$

admits a unique solution $z \in H_0^1(\Omega)$. Moreover.

$$\|z\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\|f + \text{div}\,(A\nabla G)\|_{H^{-1}(\Omega)}. \tag{4.31}$$

Set $u = z + G$. From Proposition 3.34 and the linearity of γ, one has $\gamma(u) = g$ in $H^{\frac{1}{2}}(\partial\Omega)$. Further, choosing $v \in \mathcal{D}(\Omega)$ as test function in (4.30), one obtains

$$\begin{aligned} \langle -\text{div}\,(A\nabla u), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \int_\Omega A\nabla u\,\nabla v\,dx = a(u,v) = a(z,v) + a(G,v) \\ &= \langle f + \text{div}\,(A\nabla G), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\quad + \int_\Omega A\nabla G\,\nabla v\,dx \\ &= \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \qquad \forall v \in \mathcal{D}(\Omega), \end{aligned}$$

which means that $-\text{div}\,(A\nabla u) = f$ in $\mathcal{D}'(\Omega)$ and hence u satisfies (4.28).

We now make use of estimate (4.31) to derive (4.29). One has successively, by using Proposition 3.32(iii) and Proposition 3.35,

$$\begin{aligned}\|u\|_{H^1(\Omega)} &\leq \|u-G\|_{H^1(\Omega)} + \|G\|_{H^1(\Omega)} \leq \|z\|_{L^2(\Omega)} + \|\nabla z\|_{L^2(\Omega)} + \|G\|_{H^1(\Omega)} \\ &\leq (1+C_\Omega)\|z\|_{H_0^1(\Omega)} + C_1(\Omega)\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq \frac{1+C_\Omega}{\alpha}\left(\|f\|_{H^{-1}(\Omega)} + \|\text{div}\,(A\nabla G)\|_{H^{-1}(\Omega)}\right) + C_1(\Omega)\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}.\end{aligned} \tag{4.32}$$

On the other hand, by the Cauchy–Schwarz inequality (Proposition 1.34) and again by Proposition 3.32(iii),

$$\left|\langle \text{div}\,(A\nabla G), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right| = \left|\int_\Omega A\nabla G\,\nabla v\,dx\right| \leq \beta C_1(\Omega)\|g\|_{H^{\frac{1}{2}}(\Omega)}\|\nabla v\|_{L^2(\Omega)},$$

for any $v \in H_0^1(\Omega)$. This gives (due to Definition 3.41)

$$\|\text{div}\,(A\nabla G)\|_{H^{-1}(\Omega)} \leq \beta C_1(\Omega)\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

This, together with (4.32), implies that

$$\|u\|_{H^1(\Omega)} \leq \frac{1+C_\Omega}{\alpha}\|f\|_{H^{-1}(\Omega)} + \frac{1+C_\Omega}{\alpha}\beta C_1(\Omega)\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + C_1(\Omega)\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Hence, estimate (4.29) holds with

$$C_1 = \frac{1+C_\Omega}{\alpha}, \quad C_2 = 2\frac{1+C_\Omega}{\alpha}\beta C_1(\Omega),$$

since $[(1+C_\Omega)/\alpha]\beta > 1$. □

4.5 The Neumann problem

Let $f \in (H^1(\Omega))'$ and consider the homogeneous Neumann problem

$$\begin{cases} -\text{div}\,(A\nabla u) + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_A} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.33}$$

where $\dfrac{\partial}{\partial \nu_A}$ is defined by (4.18).

The corresponding variational formulation is

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ a(u,v) = \langle f, v\rangle_{(H^1(\Omega))',H^1(\Omega)}, \quad \forall v \in H^1(\Omega), \end{cases} \tag{4.34}$$

where

$$a(u,v) = \int_\Omega A\nabla u\,\nabla v\,dx + \int_\Omega u\,v\,dx, \quad \forall u, v \in H^1(\Omega). \tag{4.35}$$

Theorem 4.20 (Homogeneous Neumann problem). *Suppose that the matrix $A \in M(\alpha, \beta, \Omega)$. Then, for any $f \in (H^1(\Omega))'$, there exists a unique solution $u \in H^1(\Omega)$ of problem (4.34). Moreover,*

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{\alpha_0}\|f\|_{(H^1(\Omega))'}, \tag{4.36}$$

where $\alpha_0 = \min\{1, \alpha\}$. If $f \in L^2(\Omega)$, the solution satisfies the estimate

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{\alpha_0}\|f\|_{L^2(\Omega)}. \tag{4.37}$$

Proof. By the definition of a, one has

$$a(v,v) \geq \alpha\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \geq \alpha_0\|v\|^2_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega),$$

where $\alpha_0 = \min\{1, \alpha\}$. Hence, the form a is $H^1(\Omega)$-elliptic with constant α_0. Therefore, Theorem 4.6 applies and estimates (4.36) and (4.37) are straightforward. □

Assume now that $\partial\Omega$ is Lipschitz continuous. Let $f \in L^2(\Omega)$, $g \in H^{-\frac{1}{2}}(\partial\Omega)$ and consider the following nonhomogeneous Neumann problem:

$$\begin{cases} -\text{div}\,(A\nabla u) + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_A} = g & \text{on } \partial\Omega. \end{cases} \tag{4.38}$$

The corresponding variational formulation is

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ a(u,v) = \displaystyle\int_\Omega f\,v\,dx + \langle g, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}, \\ \forall v \in H^1(\Omega). \end{cases} \tag{4.39}$$

where now a is defined by (4.35).

Let us observe that if u is a solution of problem (4.39), the equation in (4.38) holds in $\mathcal{D}'(\Omega)$. Then, due to Proposition 3.47(ii), $A\nabla u$ belongs to $H(\Omega, \text{div})$ and therefore $\partial u/\partial\nu_A$ is well-defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$. This is the sense to be given to the boundary condition in (4.38).

We have the following existence and uniqueness result:

Theorem 4.21 (Nonhomogeneous Neumann problem). *Suppose that $\partial\Omega$ is Lipschitz continuous and that the matrix $A \in M(\alpha, \beta, \Omega)$. Then, for any $f \in L^2(\Omega)$ and for any $g \in H^{-\frac{1}{2}}(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of problem (4.39). Moreover,*

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{\alpha_0}\left(\|f\|_{L^2(\Omega)} + C_\gamma(\Omega)\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right), \tag{4.40}$$

where $\alpha_0 = \min\{1, \alpha\}$ and $C_\gamma(\Omega)$ is the trace constant defined by Proposition 3.31.

Proof. Set for any $v \in H^1(\Omega)$,

$$F(v) = \int_\Omega f\, v\, dx + \langle g, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}. \tag{4.41}$$

One has,

$$\begin{aligned} |F(v)| &\leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}\| + C_\gamma(\Omega)\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|v\|_{H^1(\Omega)}, \end{aligned}$$

where we made use of Proposition 3.31. Hence $F \in (H^1(\Omega))'$ with

$$\|F\|_{(H^1(\Omega))'} \leq \|f\|_{L^2(\Omega)} + C_\gamma(\Omega)\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{4.42}$$

Again we apply the Lax–Milgram theorem 4.6 with F defined by (4.41) to get a unique solution $u \in H^1(\Omega)$. Estimate (4.40) is a direct consequence of (4.42). □

Suppose that Ω is connected and consider now instead of (4.38), the following nonhomogeneous Neumann problem:

$$\begin{cases} -\text{div}\,(A\nabla u) = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_A} = g & \text{on } \partial\Omega, \end{cases} \tag{4.43}$$

under the same hypotheses on g and f as in Theorem 4.21. The corresponding bilinear form is

$$a(u, v) = \int_\Omega A\nabla u\, \nabla v\, dx, \quad \forall u,\, v \in H^1(\Omega). \tag{4.44}$$

One notices immediately that now, this form is no longer coercive on $H^1(\Omega)$ but, due to Proposition 3.40, it is coercive on the Hilbert space

$$W(\Omega) = H^1(\Omega)/\mathbb{R}$$

given by Definition 3.39. Consequently, the natural variational formulation of (4.43) is

$$\begin{cases} \text{Find } \dot{u} \in W(\Omega) \text{ such that} \\ \dot{a}(\dot{u}, \dot{v}) = \displaystyle\int_\Omega f\, v\, dx + \langle g, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}, \\ \forall v \in \dot{v}, \quad \forall \dot{v} \in W(\Omega), \end{cases} \tag{4.45}$$

where $\dot{a}$ is defined by

$$\dot{a}(\dot{u}, \dot{v}) = \int_\Omega A\nabla u\, \nabla v\, dx, \quad \forall u \in \dot{u},\, v \in \dot{v},\, \forall \dot{u}, \dot{v} \in W(\Omega). \tag{4.46}$$

This problem makes sense if the right-hand side term is independent of $v \in \dot{v}$. This is expressed by the compatibility condition (4.47) written below. Indeed, we have the following result:

Theorem 4.22. (Variant of the nonhomogeneous Neumann problem) *Assume that Ω is connected and $\partial\Omega$ is Lipschitz continuous. Let A be a matrix in $M(\alpha,\beta,\Omega)$. Suppose that $f \in L^2(\Omega)$ and $g \in H^{-\frac{1}{2}}(\partial\Omega)$ satisfy the following compatibility condition:*

$$\int_\Omega f\, dx + \langle g, 1\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = 0. \tag{4.47}$$

Then, there exists a unique solution $u \in W(\Omega)$ of problem (4.45). Moreover,

$$\|u\|_{W(\Omega)} \le \frac{1}{\alpha}\left(\|f\|_{L^2(\Omega)} + C_\gamma(\Omega)\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right), \tag{4.48}$$

where $C_\gamma(\Omega)$ is the trace constant defined by Proposition 3.31.

Proof. We will again apply the Lax-Milgram theorem (Theorem 4.6) to problem (4.45) with $a = \dot{a}$ defined by (4.46), $H = W(\Omega)$ and F defined by

$$F(\dot{v}) = \int_\Omega f\, v\, dx + \langle g, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}, \quad \forall v \in \dot{v}, \forall \dot{v} \in W(\Omega).$$

We have only to check that F is well defined on $W(\Omega)$, i.e. that

$$F(w) = F(v) \quad \text{iff } w, v \in \dot{v}.$$

This is a consequence of the compatibility condition (4.47). Indeed, if $w \simeq v$ then, there exists a real constant C such that $w - v = C$. By linearity, the condition $F(w) = F(v)$ reads $F(w - v) = 0$, that is to say

$$\int_\Omega f\, C\, dx + \langle g, C\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = 0,$$

which again by linearity is equivalent to (4.47). □

Remark 4.23. Observe that if, in particular, $g \in L^2(\partial\Omega)$, then the compatibility condition (4.46) becomes

$$\int_\Omega f\, dx + \int_{\partial\Omega} g\, ds = 0. \tag{4.49}$$

◊

4.6 The Robin problem

In this section we consider two other examples of boundary conditions, namely the nonhomogeneous Robin condition and the case where one has a Dirichlet condition on a part of the boundary and a homogeneous Robin one on the rest of the boundary. Suppose that $\partial\Omega$ is Lipschitz continuous and let $f \in L^2(\Omega)$, $g \in H^{-\frac{1}{2}}(\partial\Omega)$. Consider the problem

$$\begin{cases} -\text{div}\,(A\nabla u) + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu_A} + du = g & \text{on } \partial\Omega, \end{cases} \tag{4.50}$$

where $d \in \mathbb{R}$ is such that $d \geq 0$. The variational formulation of problem (4.50) is then

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ a(u,v) = \displaystyle\int_\Omega f v\, dx + \langle g, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}, \\ \forall v \in H^1(\Omega), \end{cases} \tag{4.51}$$

where

$$a(u,v) = \int_\Omega A\nabla u\, \nabla v\, dx + \int_\Omega u v\, dx + d\int_{\partial\Omega} u v\, ds, \quad \forall u, v \in H^1(\Omega). \tag{4.52}$$

Theorem 4.24 (Nonhomogeneous Robin problem). *Suppose that $\partial\Omega$ is Lipschitz continuous and that the matrix $A \in M(\alpha, \beta, \Omega)$. Then, for any $f \in L^2(\Omega)$ and for any $g \in H^{-\frac{1}{2}}(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of problem (4.50). Moreover,*

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{\alpha_0}\left(\|f\|_{L^2(\Omega)} + C_\gamma(\Omega)\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right). \tag{4.53}$$

where $\alpha_0 = \min\{1, \alpha\}$ and $C_\gamma(\Omega)$ is the trace constant defined by Proposition 3.31.

Proof. As in the proof of Theorem 4.21, let $F \in (H^1(\Omega))'$ be defined by

$$F(v) = \int_\Omega f v\, dx + \langle g, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}. \tag{4.54}$$

We will again apply the Lax Milgram theorem (Theorem 4.6) with $H = H^1(\Omega)$. Observe that as a consequence of Proposition 3.31, the bilinear form $a(u,v)$ given by (4.52) is continuous on $H^1(\Omega) \times H^1(\Omega)$ and coercive. since d is positive. Observe also that the functional F in (4.54) is the same as that in (4.41) introduced for the nonhomogeneous Neumann problem. Therefore, estimate (4.53) follows from estimate (4.42). □

Other boundary conditions can be studied, always as applications of the Lax–Milgram theorem. Let us finish this section by the following example. Let Ω be connected. Suppose that $\partial\Omega$ is Lipschitz continuous and such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two disjoint closed sets and Γ_1 is of positive measure. Consider the problem

$$\begin{cases} -\text{div}\,(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial \nu_A} + du = 0 & \text{on } \Gamma_2, \end{cases} \tag{4.55}$$

where $d \geq 0$. Let us introduce the space

$$V = \{v \mid v \in H^1(\Omega),\ \gamma(v) = 0 \text{ on } \Gamma_1\},$$

which is well defined due to Remark 3.37. Observe that, thanks to Proposition 3.36, V can be equipped with the norm

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)}. \tag{4.56}$$

Let $f \in L^2(\Omega)$. The variational formulation of problem (4.55) is

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = \displaystyle\int_\Omega f\, v\, dx \\ \forall v \in V, \end{cases} \tag{4.57}$$

where

$$a(u,v) = \int_\Omega A\nabla u\, \nabla v\, dx + d\int_{\partial\Omega} u\, v\, ds, \quad \forall u, v \in V. \tag{4.58}$$

Theorem 4.25 (Mixed Dirichlet–Robin condition). *Let Ω be connected. Suppose that $\partial\Omega$ is Lipschitz continuous and such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are two disjoint closed sets and Γ_1 is of positive measure. Let A be a matrix in $M(\alpha, \beta, \Omega)$, $f \in L^2(\Omega)$ and $d \geq 0$. Then, there exists a unique solution $u \in V$ of problem (4.57). Moreover,*

$$\|u\|_V \leq \frac{C_\Omega}{\alpha}\|f\|_{L^2(\Omega)}, \tag{4.59}$$

where C_Ω is the Poincaré constant given by Proposition 3.36.

Proof. We apply the Lax–Milgram theorem (Theorem 4.6) with $H = V$, a defined by (4.58) and F given by

$$F(v) = \int_\Omega f\, v\, dx. \tag{4.60}$$

Due to (4.56) and since d is positive, the bilinear form a is coercive on V. The continuity of a is again a consequence of Proposition 3.38. Obviously, (4.60) defines a continuous form on V and due to Proposition 3.36, one has

$$\|F\|_{V'} \leq C_\Omega \|f\|_{L^2(\Omega)}.$$

Then, estimate (4.59) is straightforward. □

4.7 Periodic boundary conditions

We consider now the case of periodic boundary conditions.

Let Y be the interval of $\mathbb{R}^n$ defined by (2.1), i.e. $Y =]0, \ell_1[\times \cdots \times]0, \ell_N[$, where $\ell_1, \ldots, \ell_N$ are given positive numbers.

Suppose that the coefficients a_{ij} are Y-periodic (in the sense of Definition 2.1). Let f be Y-periodic and consider the problem

$$\begin{cases} -\text{div}\,(A\nabla u) = f & \text{in } Y \\ u \quad Y\text{-periodic}. \end{cases} \tag{4.61}$$

A natural space for the solutions is $\mathcal{W}_{\text{per}}(Y)$, introduced by Definition 3.51. Hence, for f given in $(\mathcal{W}_{\text{per}}(Y))'$, the variational formulation of problem (4.61) is

$$\begin{cases} \text{Find } \dot{u} \in \mathcal{W}_{\text{per}}(Y) \text{ such that} \\ \dot{a}_Y(\dot{u}, \dot{v}) = \langle f, \dot{v} \rangle_{(\mathcal{W}_{\text{per}}(Y))', \mathcal{W}_{\text{per}}(Y)} \\ \forall \dot{v} \in \mathcal{W}_{\text{per}}(Y), \end{cases} \tag{4.62}$$

where

$$\dot{a}_Y(\dot{u}, \dot{v}) = \int_Y A\nabla u\, \nabla v\, dy, \quad \forall u \in \dot{u}, \forall v \in \dot{v}.$$

Theorem 4.26 (Periodic boundary condition). *Let A be a matrix in $M(\alpha, \beta, Y)$ with Y-periodic coefficients and $f \in (\mathcal{W}_{\text{per}}(Y))'$. Then problem (4.62) has a unique solution. Moreover,*

$$\|\dot{u}\|_{\mathcal{W}_{\text{per}}(Y)} \leq \frac{1}{\alpha} \|f\|_{(\mathcal{W}_{\text{per}}(Y))'}. \tag{4.63}$$

Proof. The claimed result is a simple application of Lax–Milgram theorem (Theorem 4.6) with $H = \mathcal{W}_{\text{per}}(Y)$ and

$$a(u, v) = \dot{a}_Y(\dot{u}, \dot{v}), \quad \forall \dot{u}, \dot{v} \in \mathcal{W}_{\text{per}}(Y),$$

since, due to Proposition 3.52, the bilinear form $\dot{a}_Y$ is coercive on $\mathcal{W}_{\text{per}}(Y)$. □

Let us recall that an element of $\mathcal{W}_{\text{per}}(Y)$ is a class of $H^1_{\text{per}}(Y)$-functions, equivalent in the sense of Definition 3.51. Hence Theorem 4.26 shows that problem (4.61) admits a solution in $H^1_{\text{per}}(Y)$, defined up to an additive constant.

We can choose a representative element of the class of equivalence of $\dot{u}$ by fixing this constant. In particular, we can ask for the solution of the initial problem (4.61) to have a zero mean value, i.e. to solve the problem

$$\begin{cases} -\text{div}\,(A\nabla u) = f & \text{in } Y \\ u \quad Y\text{-periodic} \\ \mathcal{M}_Y(u) = 0, \end{cases} \tag{4.64}$$

where f is still in $(\mathcal{W}_{\mathrm{per}}(Y))'$.

The corresponding variational formulation is

$$\begin{cases} \text{Find } u \in W_{\mathrm{per}}(Y) \text{ such that} \\ \displaystyle\int_Y A\nabla u \nabla v \, dy = \langle f, v\rangle_{(W_{\mathrm{per}}(Y))', W_{\mathrm{per}}(Y)} \\ \forall v \in W_{\mathrm{per}}(Y), \end{cases} \tag{4.65}$$

where

$$W_{\mathrm{per}}(Y) = \left\{ v \mid v \in H^1_{\mathrm{per}}(Y),\ \mathcal{M}_Y(v) = 0 \right\}. \tag{4.66}$$

and the bracket $\langle f, v\rangle_{(W_{\mathrm{per}}(Y))', W_{\mathrm{per}}(Y)}$ is well-defined by Proposition 3.52.

Due to the Poincaré–Wirtinger inequality (Proposition 3.38), $W_{\mathrm{per}}(Y)$ is a Banach space for the norm

$$\|u\|_{W_{\mathrm{per}}(Y)} = \|\nabla u\|_{L^2(Y)}, \qquad \text{for any } u \in W_{\mathrm{per}}(Y).$$

In this setting, Theorem 4.26 reads as follows:

Theorem 4.27 (Variant of periodic boundary condition). *Let A be a matrix in $M(\alpha,\beta,Y)$ with Y-periodic coefficients and $f \in (\mathcal{W}_{\mathrm{per}}(Y))'$. Then problem (4.65) has a unique solution. Moreover,*

$$\|u\|_{W_{\mathrm{per}}(Y)} \leq \frac{1}{\alpha}\|f\|_{(\mathcal{W}_{\mathrm{per}}(Y))'}. \tag{4.67}$$

Let us recall that, from Proposition 3.50, the extension by periodicity given by (3.7) for an element of $H^1_{\mathrm{per}}(Y)$ is in $H^1(\omega)$ for any bounded open set ω of $\mathbb{R}^N$. A natural question arises now: does the extension $u^\#$ of the solution u of problem (4.65) satisfy some equation (at least locally) in $\mathbb{R}^N$? If $f = -\mathrm{div}\, h$, with $h \in (L^2(Y))^N$ and Y-periodic, we can give a positive answer to this question. In this case, the variational formulation (4.65) becomes

$$\begin{cases} \text{Find } u \in W_{\mathrm{per}}(Y) \text{ such that} \\ \displaystyle\int_Y A\nabla u \nabla v \, dy = \int_Y h \nabla v \, dx \\ \forall v \in W_{\mathrm{per}}(Y). \end{cases} \tag{4.68}$$

Observe that the following relation:

$$\langle -\mathrm{div}\, h, v\rangle_{(\mathcal{W}_{\mathrm{per}}(Y))'} = \int_Y h \nabla v \, dy, \quad \forall v \in W_{\mathrm{per}}(Y),$$

identifies $-\mathrm{div}\, h$ as an element in $(W_{\mathrm{per}}(Y))'$. Moreover, it is obvious that $-\mathrm{div}\, h \in (\mathcal{W}_{\mathrm{per}}(Y))'$ in the sense of Proposition 3.52. Therefore, Theorem 4.27 applies and the following result holds:

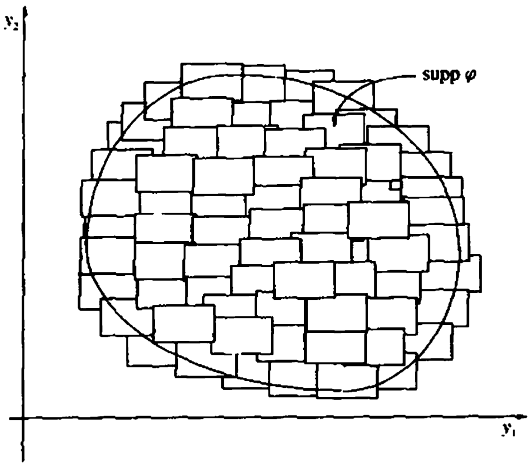


Fig. 4.1

Theorem 4.28. *Let A be a matrix in $M(\alpha, \beta, Y)$ with Y-periodic coefficients and h in $(L^2(Y))^N$ and Y-periodic. Let $u \in W_{\text{per}}(Y)$ be the solution of problem (4.68) and $u^\#$ its extension by periodicity given by (3.7). Then $u^\#$ is the unique solution of the problem*

$$\begin{cases} -\text{div } (A \nabla u^\#) = -\text{div } h & \text{in } \mathcal{D}'(\mathbb{R}^N) \\ u^\# \quad Y\text{-periodic} \\ \mathcal{M}_Y(u^\#) = 0. \end{cases} \tag{4.69}$$

Proof. It is easy to check by using Green's formula (Theorem 3.33) that $u^\#|_Y$ solves (4.68). Then, the uniqueness of the solution of (4.69) follows from that of problem (4.68). To prove (4.69) we have to check that

$$\int_{\mathbb{R}^N} A(\nabla u^\#) \nabla \varphi \, dx = \int_{\mathbb{R}^N} h \nabla \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N). \tag{4.70}$$

To begin with, observe that if $\psi \in C^\infty_{\text{per}}(Y)$ (see Definition 3.48), we can choose $v = \psi - \mathcal{M}_Y(\psi)$ as test function in (4.68) to obtain

$$\int_Y A \nabla u \nabla \psi \, dy = \int_Y h \nabla \psi \, dy. \tag{4.71}$$

Let now $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and $K = \text{supp } \varphi$. Let $(Y_i)_{i=1,\dots,m}$ be a finite set of translated cells of Y, recovering K (see Fig. 4.1), i.e.

$$K \subset \bigcup_{i=1}^m Y_i.$$

Let $(\theta_i)_{i=0,\dots,m}$ be a partition of the unity associated to this covering, i.e. a family of functions such that

$$\begin{cases} \theta_i \in \mathcal{D}(\mathbb{R}^N), \quad 0 \le \theta_i \le 1, \quad \forall i \in \{0,\dots,m\}, \quad \displaystyle\sum_{i=0}^{m} \theta_i = 1 \text{ in } \mathbb{R}^N, \\ \text{supp } \theta_i \subset Y_i, \quad \forall i \in \{1,\dots,m\}, \quad \text{supp } \theta_0 \subset \mathbb{R}^N \setminus K. \end{cases}$$

For the existence of this partition we refer for instance to Yosida (1964).

Since $\varphi = 0$ on the support of θ_0 one has

$$\varphi = \varphi \sum_{i=1}^{m} \theta_i = \sum_{i=1}^{m} (\varphi\theta_i) \quad \text{in } \mathbb{R}^N. \tag{4.72}$$

Denote by $(\varphi\theta_i)^\#$ the extension by periodicity of $\varphi\theta_i$ for any $i = 1,\dots,m$. Since $\theta_i = 0$ in a neighbourhood of ∂Y_i, the function $(\varphi\theta_i)^\#$ is in $C^\infty_{\text{per}}(Y)$, hence also in $H^1_{\text{per}}(Y)$.

Using Lemma 2.3, the properties of θ_i and taking into account (4.72), one has

$$\begin{aligned} \int_{\mathbb{R}^N} A(\nabla u^\#)\nabla\varphi \, dx &= \sum_{i=1}^{m} \int_{\mathbb{R}^N} A(\nabla u^\#)\nabla(\varphi\theta_i) \, dx \\ &= \sum_{i=1}^{m} \int_{Y_i} A(\nabla u^\#)\nabla(\varphi\theta_i) \, dx \\ &= \sum_{i=1}^{m} \int_{Y} A\nabla u \, \nabla((\varphi\theta_i)^\#) \, dx. \end{aligned} \tag{4.73}$$

Observe now that the properties of θ_i and φ allow to choose $\psi = (\varphi\theta_i)^\#$ as test function in (4.71). Consequently,

$$\sum_{i=1}^{m} \int_{Y} A\nabla u \, \nabla((\varphi\theta_i)^\#) \, dx = \sum_{i=1}^{m} \int_{Y} h \, \nabla((\varphi\theta_i)^\#) \, dx. \tag{4.74}$$

Using again Lemma 2.3 and the properties of θ_i, it follows that

$$\begin{aligned} \sum_{i=1}^{m} \int_{Y} h \, \nabla((\varphi\theta_i)^\#) \, dx &= \sum_{i=1}^{m} \int_{Y_i} h \, \nabla(\varphi\theta_i) \, dx \\ &= \sum_{i=1}^{m} \int_{\mathbb{R}^N} h \, \nabla(\varphi\theta_i) \, dx = \int_{\mathbb{R}^N} h \, \nabla\varphi \, dx, \end{aligned}$$

which, together with (4.73) and (4.74), gives (4.70) and ends the proof. □

5

Examples of periodic composite materials

In this chapter we introduce the periodic framework in which we will work throughout this book. We give in Section 5.2 some examples of physical problems for composite materials which are modelled by partial differential equations.

In Sections 5.3 and 5.4 we focus our attention on two particular situations, the one-dimensional case and the layered materials. The first example is due to S. Spagnolo and can be found in Spagnolo (1967, 1968) in the context of the G-convergence (cf. Chapter 13 below). The case of layered materials was studied by L. Tartar and by F. Murat (see Murat, 1978a, Murat and Tartar 1997a, and Tartar 1977a) in the context of the H-convergence (cf. again Chapter 13).

For general references in periodic homogenization, we refer the reader to Spagnolo (1968), De Giorgi and Spagnolo (1973), Babuška (1976), Bensoussan, Lions, and Papanicolaou (1978), Sanchez-Palencia (1980), Ene and Paşa (1987), Bakhvalov and Panasenko (1989), Jikov, Kozlov, and Oleinik (1994) and references herein. For further developments concerning perforated domains and periodic structures we refer to Lions (1981), Cioranescu and Saint Jean Paulin (1999) as well as to references therein.

5.1 Setting of the problem

In this chapter, Ω denotes as previously, a bounded open set in $\mathbb{R}^N$ and $\varepsilon > 0$ is a parameter taking its values in a sequence which tends to zero.

Let

$$A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1\le i,j\le N} \quad \text{a.e. on } \Omega, \tag{5.1}$$

be a sequence of non-constant matrices such that

$$A^\varepsilon \in M(\alpha, \beta, \Omega). \tag{5.2}$$

This means (see (4.16) in Definition 4.11) that A^ε satisfies the following inequalities:

$$\begin{cases} (A^\varepsilon(x)\lambda, \lambda) \ge \alpha|\lambda|^2 \\ |A^\varepsilon(x)\lambda| \le \beta|\lambda|, \end{cases} \tag{5.3}$$

for any $\lambda \in \mathbb{R}^N$ and a.e. on Ω.

Introduce the operator

$$\mathcal{A}_\varepsilon = -\text{div } (A^\varepsilon \nabla) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}\frac{\partial}{\partial x_j}\right). \tag{5.4}$$

As we shall see in this book, the homogenization theory allows to describe the asymptotic behaviour as $\varepsilon \to 0$ of partial differential equations of many types. To begin with, we study the equation

$$\mathcal{A}_\varepsilon u^\varepsilon = f \tag{5.5}$$

with a Dirichlet boundary condition on $\partial\Omega$.

This equation is a model case, particularly relevant both from the mathematical point of view and for applications. As a matter of fact, the main mathematical difficulties occurring in homogenization theory, are already present in this model problem. On the other hand, as we will see in Section 5.2, the equations of type (5.5) model thermal as well as electrical or linear elastic properties of materials. When treating such problems for composite materials, the parameter ε describes the heterogeneities of the material.

A classical problem of type (5.5) is the Dirichlet problem

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

where f is given in $H^{-1}(\Omega)$. From Theorem 4.16 it follows that for any fixed ε, there exists a unique solution $u^\varepsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla v \, dx \;=\; \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{5.7}$$

Moreover, estimate (4.23) holds, i.e.

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \le \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{5.8}$$

Consequently, from Theorem 1.18(i) and Proposition 3.17, it follows that there exist a subsequence, denoted by $\{u^{\varepsilon'}\}$, and an element $u^0 \in H_0^1(\Omega)$ such that

$$u^{\varepsilon'} \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega). \tag{5.9}$$

Observe that *a priori* the limit u^0 depends on the subsequence for which (5.9) holds.

At this point two natural questions arise:

- does u^0 satisfy some boundary value problem in Ω?
- and if so, is u^0 uniquely determined?

In order to investigate these questions, let us introduce the vector

$$\xi^\varepsilon = (\xi_1^\varepsilon, \ldots, \xi_N^\varepsilon) = \left(\sum_{j=1}^N a_{1j}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}, \ldots, \sum_{j=1}^N a_{Nj}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}\right) = A^\varepsilon \nabla u^\varepsilon, \tag{5.10}$$

which (see (5.7)) satisfies

$$\int_\Omega \xi^\varepsilon \nabla v \, dx = \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{5.11}$$

Obviously, from (5.2) and (5.8) it follows that

$$\|\xi^\varepsilon\|_{L^2(\Omega)} \leq \frac{\beta}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{5.12}$$

Then, again from Theorem 1.18(i) there exist a subsequence, still denoted by $\{\xi^{\varepsilon'}\}$, and an element $\xi^0 \in L^2(\Omega)$, such that

$$\xi^{\varepsilon'} \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^N. \tag{5.13}$$

Hence, we can pass to the limit in (5.11) written for the subsequence ε', to get

$$\int_\Omega \xi^0 \nabla v \, dx = \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega), \tag{5.14}$$

i.e.

$$-\text{div } \xi^0 = f \quad \text{in } \Omega. \tag{5.15}$$

Consequently, the first above question has a positive answer if one can describe ξ^0 in terms of u^0.

Remark 5.1. If A^ε is such that

$$A^\varepsilon \to \hat{A} \quad \text{strongly in } [L^\infty(\Omega)]^{N\times N},$$

one can easily give the relation between u^0 and ξ^0. Indeed, in view of (5.9) and Proposition 1.19, one has

$$\begin{aligned}
&\lim_{\varepsilon'\to 0} \int_\Omega A_{\varepsilon'} \nabla u^{\varepsilon'} \varphi \, dx \\
&= \lim_{\varepsilon'\to 0} \langle {}^t A_{\varepsilon'} \varphi, \nabla u^{\varepsilon'}\rangle_{[L^2(\Omega)]^N, [L^2(\Omega)]^N} \\
&= \langle {}^t \hat{A}\varphi, \nabla u^0\rangle_{[L^2(\Omega)]^N, [L^2(\Omega)]^N} = \int_\Omega \hat{A}\nabla u^0 \varphi \, dx. \qquad \forall \varphi \in [L^2(\Omega)]^N,
\end{aligned}$$

where, for any matrix B, ${}^tB$ denotes its transposed. Therefore,

$$\xi^0 = \hat{A}\nabla u^0.$$

Hence, from (5.9) and (5.15) one deduces that u^0 is the unique solution of

$$\begin{cases} -\text{div } (\hat{A}\nabla u^0) = f & \text{in } \Omega \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

From the Lax–Milgram theorem, this problem has a unique solution u^0 since obviously, $\hat{A} \in M(\alpha, \beta, \Omega)$. Thus, in this case one have also that the whole sequence u^ε converges. ◊

Let us point out that the case considered in Remark 5.1 is a very peculiar one and not relevant for the study of composite materials. Indeed, as we will see in Section 5.3, for composite materials a strong convergence of the matrix A^ε can never occur. For a sequence of matrices satisfying (5.2) (periodic or not), one can only deduce a weakly* compactness in $L^\infty(\Omega)$ (see Remark 1.54) to some matrix A^*.

On the other hand, as we will see in Chapter 13, the fact that A^ε satisfies (5.2) for any ε, implies the existence of a matrix A_0 (depending on the subsequence ε'), such that $\xi^0 = A_0 \nabla u^0$ so that u^0 is the unique solution of

$$\begin{cases} -\text{div}\,(A_0 \nabla u^0) = f & \text{in } \Omega \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.16}$$

with

$$A_0 \in M(\alpha, \beta', \Omega), \tag{5.17}$$

for some $\beta' \geq \beta$.

In general, A_0 is different from A^*. Moreover, one cannot uniquely identify the matrix A_0 so that one cannot say that the whole sequence u^ε converges to u^0.

In some situations, in particular in the periodic case, one can give explicit formulas for the matrix A_0 which show that A_0 is independent of the subsequence ε'. This implies that the limit u^0 is also independent of the subsequence ε'.

Consequently, from Theorem 1.18(ii) it follows that the whole sequence u^ε converges to u^0. In this case problem (5.16) is called the homogenized problem, A_0 the homogenized (or effective) matrix and u^0 the homogenized solution.

Let us now introduce the general periodic framework in which we will work from now on. As in Chapter 2, set

$$Y =]0, \ell_1[\times \cdots \times]0, \ell_N[,$$

where $\ell_1, \ldots, \ell_N$ are given positive numbers. It is called the reference period or reference cell.

Let $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$ and $A = (a_{ij})_{1 \leq i,j \leq N}$ be a $N \times N$ matrix such that

$$\begin{cases} a_{ij} \text{ is } Y\text{-periodic}, & \forall\, i, j = 1, \ldots, N \\ A \in M(\alpha, \beta, Y), \end{cases} \tag{5.18}$$

where the periodicity is taken in the sense of Definition 2.1 and the class $M(\alpha, \beta, Y)$ is given by Definition 4.11 for $\mathcal{O} = Y$, i.e.

$$\begin{cases} (A(y)\lambda, \lambda) \geq \alpha|\lambda|^2 \\ |A(y)\lambda| \leq \beta|\lambda|, \end{cases} \tag{5.19}$$

for any $\lambda \in \mathbb{R}^N$ and a.e. on Y.

Set

$$a_{ij}^\varepsilon(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall\, i, j = 1, \ldots, N \tag{5.20}$$

and

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) = (a^\varepsilon_{ij}(x))_{1\le i,j\le N} \quad \text{a.e. on } \mathbb{R}^N. \tag{5.21}$$

It is easy to check that A^ε satisfies (5.2) and (5.3) for any ε. Then, all the considerations above hold for problem (5.6) written for A^ε given by (5.21).

Observe that from Theorem 2.6 it follows that if $\varepsilon \to 0$

$$A^\varepsilon \rightharpoonup \mathcal{M}_Y(A). \quad \text{weakly* in } L^\infty(\Omega), \tag{5.22}$$

where the matrix $(\mathcal{M}_Y(A))_{ij}$ is defined by

$$(\mathcal{M}_Y(A))_{ij} = \frac{1}{|Y|}\int_Y a_{ij}(y)\,dy. \tag{5.23}$$

Moreover, from Remark 2.9 we know that in general. convergence (5.22) is not strong. Hence, in view of convergences (5.9) and (5.22), $A^\varepsilon\nabla u^\varepsilon$ is the product of two weakly convergent sequences. From Remark 2.7 we know that in general

$$\xi^0 \neq \mathcal{M}_Y(A)\nabla u^0,$$

with ξ^0 given by (5.13).

In Sections 5.3 and 5.4 we will see that for examples of the one-dimensional case and of layered materials, the homogenized matrix A^0 is obtained by taking weak limits of some algebraic formulas involving the coefficients of the matrix A defined by (5.18). As seen in Chapter 6, for the general N-dimensional case, the situation is completely different, since the coefficients of A^0 are no longer obtained as algebraic formulas from A. Indeed, they are defined by means of some periodic functions which are solutions in the reference cell Y of boundary value problems of the same type as that studied in Section 4.7.

5.2 Some physical models

In this section we show how some classical physical problems can be modelled by the Dirichlet problem (5.6), introduced in Section 5.1, as well as by other boundary value problems that we will consider in this book.

A composite is a material containing two or more finely mixed components. Composite materials are widely used nowadays in any kind of industries, since they have very interesting properties. It is known in practice that they exhibit in general 'better' behaviour (according to the performance one looks for), than the average behaviour of its components, classical examples being ceramics or supraconducting multifilamentary materials.

In a good composite, the heterogeneities are very small compared with the global dimension of the sample. The smaller are the heterogeneities, the better is the mixture, which appears then, at a first glance, as a 'homogeneous' material. It is for this reason that one can assume that the heterogeneities are evenly distributed.

From the mathematical point of view, one can model this distribution by supposing that it is a periodic one. This periodicity can be represented by a small parameter, ε.

In practice, one is interested to know the global behaviour of the composite material when the heterogeneities are very very small. This means that ε is very small, which mathematically signifies making ε tend to zero.

As showed in the examples below, in the mathematical model the main characteristics constants of a material are the coefficients of a partial differential equation. For a composite with a ε-periodic distribution, these coefficients will clearly depend on the parameter ε, so they jump between the different values of the characteristic of its components.

This makes the model very difficult to treat, in particular from the numerical point of view. Also, the pointwise knowledge of the characteristic of the material does not provide in a simple way any information on its global behaviour.

As we will see throughout all this book, when passing to the limit as $\varepsilon \to 0$, we obtain 'homogenized' limit problems with constant coefficients. This is very interesting in applications since, as is well known from engineers and physicians, these coefficients are good approximations of the global characteristics of the composite material, when regarded as an homogeneous one. Moreover, replacing the problem by the limit one, allows to make easy numerical computations.

Let us introduce the geometrical model of a periodic mixture corresponding to the problems we treat in this book. For the sake of simplicity, we describe here the case of a mixture of two components.

Let Ω be the domain occupied by the material, Y the reference cell, $Y_1 \subset Y$ and $Y_2 \subset Y$ such that

$$\overline{Y} = \overline{Y_1} \cup \overline{Y_2}, \quad Y_1 \cap Y_2 = \emptyset.$$

Let $\varepsilon > 0$ be a parameter which takes its values in a sequence which tends to zero and set

$$\Omega_1^\varepsilon = \left\{ x \,\Big|\, \chi_1\Big(\frac{x}{\varepsilon}\Big) = 1 \right\}, \qquad \Omega_2^\varepsilon = \Omega \setminus \overline{\Omega_1^\varepsilon} = \left\{ x \,\Big|\, \chi_2\Big(\frac{x}{\varepsilon}\Big) = 1 \right\},$$

where χ_i for $i = 1, 2$, is the characteristic function of the set Y_i (see Definition 1.40) extended by periodicity with period Y.

By this construction, the set Ω is covered by a pavement of cells of the form $\varepsilon Y = \varepsilon Y_1 \cup \varepsilon Y_2$ (see Fig. 5.1).

Remark, in particular, that if we have

$$Y_1 =]0, \frac{\ell_1}{2}[\times \cdots \times]0, \ell_N[, \qquad Y_2 =]\frac{\ell_1}{2}, \ell_1[\times \cdots \times]0, \ell_N[$$

we are in the case of a layered material (see Fig. 5.2).

When taking $\varepsilon \to 0$, the cells εY covering Ω are smaller and smaller and their number goes to ∞. This signifies that, in this procedure, we are mixing the two materials 'better and better' in the sense that the heterogeneities are

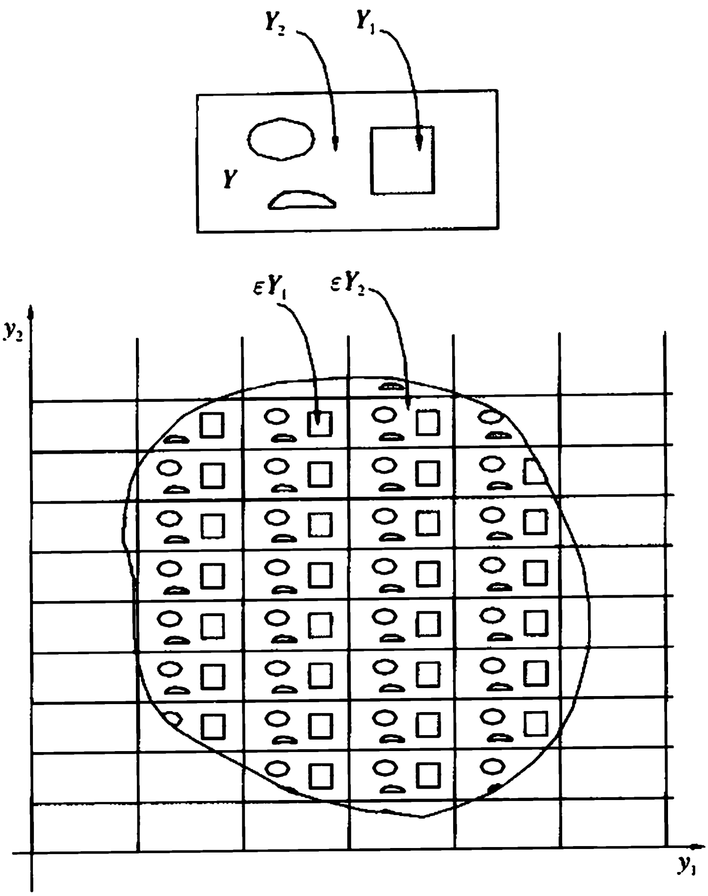


Fig. 5.1

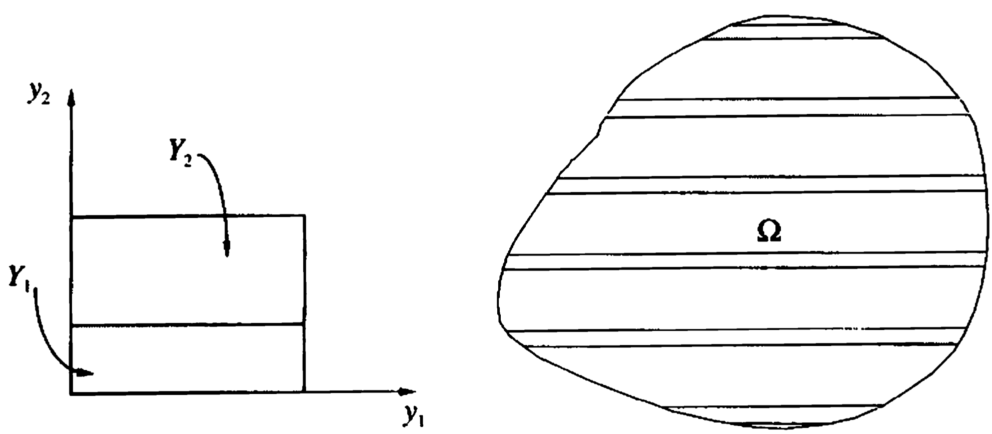


Fig. 5.2 (A layered material)

finer and finer (see Figures 5.1 and 5.2). This is why this procedure is called homogenization. Indeed, in this process of homogenization, the proportion of the materials is kept constant. Actually, the proportion θ_i^ε of the material occupying the set Ω_i^ε and given by

$$\theta_i^\varepsilon = \frac{|\Omega_i^\varepsilon|}{|\Omega|} = \frac{1}{|\Omega|}\int_\Omega \chi_i\left(\frac{x}{\varepsilon}\right)dx, \quad (\theta_1^\varepsilon + \theta_2^\varepsilon = 1),$$

is of the order of a constant C_i, independent of ε. Indeed, from Theorem 2.6 one has, for $\varepsilon \to 0$, that

$$\chi_i\left(\frac{\cdot}{\varepsilon}\right) \rightharpoonup \mathcal{M}_Y(\chi_i) = \frac{1}{|Y|}\int_Y \chi_i(y)\,dy = \frac{|Y_i|}{|Y|} \quad \text{weakly* in } L^\infty(\Omega), \quad i = 1,2.$$

Therefore,

$$\theta_1^\varepsilon \to C_1 = \frac{|Y_1|}{|Y|}, \quad \theta_2^\varepsilon \to C_2 = \frac{|Y_2|}{|Y|}.$$

In the examples below, we will place ourselves in this geometrical framework.

Example 5.2 (Dirichlet problems). Consider first a homogeneous body occupying Ω with thermal conductivity γ. For simplicity, we assume that the material is isotropic, which means that γ is a scalar. If f represents the heat source and g the temperature on the surface $\partial\Omega$ of the body, then the temperature $u = u(x)$ at the point $x \in \Omega$ satisfies the following Dirichlet problem:

$$\begin{cases} -\gamma\,\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\Delta u = \operatorname{div}(\nabla u)$. The flux of the temperature is defined by

$$q = \gamma\,\nabla u.$$

By linearity, we can always suppose that $g = 0$ on $\partial\Omega$ and consider the following Dirichlet homogeneous problem:

$$\begin{cases} -\gamma\,\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If now the body is composed of two different materials of thermal conductivity γ_1 and γ_2, occupying respectively the subdomains Ω_1 and Ω_2, the temperature and the flux of the temperature in a point $x \in \Omega$ of the composite take, respectively, the values

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega_1 \\ u_2(x) & \text{if } x \in \Omega_2. \end{cases}$$

and

$$q = \begin{cases} q_1 = \gamma_1\,\nabla u_1 & \text{in } \Omega_1 \\ q_2 = \gamma_2\,\nabla u_2 & \text{in } \Omega_2. \end{cases}$$

The usual physical assumptions are the continuity of the temperature u and of the flux q at the interface of the two materials, i.e.

$$\begin{cases} u_1 = u_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2 \\ q_1 \cdot n_1 = q_2 \cdot n_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2, \end{cases}$$

where n_i is the outward normal unit vector to $\partial\Omega_i$, $i = 1, 2$. Obviously, $n_1 = -n_2$ on $\partial\Omega_1 \cap \partial\Omega_2$.

Taking into account these continuity conditions, if we set

$$\gamma(x) = \begin{cases} \gamma_1 & \text{if } x \in \Omega_1 \\ \gamma_2 & \text{if } x \in \Omega_2, \end{cases}$$

the temperature u is solution of the stationary thermal problem

$$\begin{cases} -\mathrm{div}(\gamma\, \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, let us formulate the thermal problem in the above periodic setting. To do so, set

$$\gamma_\varepsilon(x) = \gamma_1 \chi_1\left(\frac{x}{\varepsilon}\right) + \gamma_2 \chi_2\left(\frac{x}{\varepsilon}\right),$$

which represents the conductivity of the periodic composite material since, obviously,

$$\gamma_\varepsilon(x) = \begin{cases} \gamma_1 & \text{if } x \in \Omega_1^\varepsilon \\ \gamma_2 & \text{if } x \in \Omega_2^\varepsilon. \end{cases}$$

Then, for any ε, the temperature u^ε satisfies the problem

$$\begin{cases} -\mathrm{div}(\gamma_\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

This is a homogeneous Dirichlet problem with the rapidly oscillating coefficient γ_ε. Observe that this problem is a particular case of problem (5.6). We will study it in details in Sections 5.3 and 5.4 and also in Chapters 5–9. Notice that if we are in the case of layered material of Figure 5.2, obviously we have that γ_ε depends only on the variable x_1, namely $\gamma_\varepsilon(x) = \gamma_\varepsilon(x_1)$, which is the example studied in Section 5.4.

It is clear from the equation that the temperature u^ε depends on two scales which are described by the two variables x and x/ε. The first one, called 'macroscopic' is slow, and it gives the position of the point in Ω. The second one, the 'microscopic' variable, oscillates rapidly with ε and determines whether the point is in Y_1 or in Y_2.

In the case of anisotropic materials, γ is no longer a scalar but a matrix A representing the thermal conductivity (in the axis directions) so that the problem to treat can be written as

$$\begin{cases} -\mathrm{div}\,(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

which is exactly problem (5.6).

Let us also mention that this Dirichlet problem models many other physical phenomena. Of course, the coefficient γ_ε will describe other characteristics of the material. For instance, in the study of the torsion of a cylindrical composite bar, one has $\gamma = 1/\mu$ where μ is the shear modulus of the material, f is the angle of the twist and u represents the stress function.

Also, in electricity the electrostatic potential u satisfies the same equation, where γ is the electrical conductivity and f stands for the distribution of the electric charges.

If the phenomenon depends on the time, we will have the following equation, called the 'heat equation' in thermicity:

$$\begin{cases} u_\varepsilon' - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega\times]0,T[\\ u_\varepsilon = 0 \quad \text{on } \partial\Omega\times]0,T[\\ u_\varepsilon(x,0) = u_\varepsilon^0 \quad \text{in } \Omega, \end{cases}$$

where the operators div and ∇ are taken with respect to the space variable $x \in \Omega$, the sign $'$ denotes the derivative with respect to the time variable $t \in]0,T[$ with $T > 0$, and the initial state u_ε^0 is given. We will study it in Chapter 11. ◊

Example 5.3 (The wave equation). One can also study the wave propagation in a composite material. Then under the same notations as in the previous example one has the system

$$\begin{cases} u_\varepsilon'' - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega\times]0,T[\\ u_\varepsilon = 0 \quad \text{on } \partial\Omega\times]0,T[\\ u_\varepsilon(x,0) = u_\varepsilon^0 \quad \text{in } \Omega \\ u'(x,0) = u_\varepsilon^1 \quad \text{in } \Omega, \end{cases}$$

where u_ε is the wave propagation, f_ε is a source term, and u_ε^0, u_ε^1 are the initial data. We will study it in Chapter 12. ◊

Example 5.4 (The linearized elasticity system). Suppose that we have an elastic homogeneous body occupying the domain $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is the union of two disjoint parts. Γ_1 and Γ_2. In linearized elasticity, the displacement $u = (u_1, u_2, u_3)$ is a solution of the system

$$\begin{cases} -\displaystyle\sum_{j,k,h=1}^{3} \frac{\partial}{\partial x_j}\left(a_{ijkh}\,\frac{\partial u_k}{\partial x_h}\right) = f_i & \text{in } \Omega \\ \displaystyle\sum_{j,k,h=1}^{3} a_{ijkh}\,\frac{\partial u_k}{\partial x_h}\, n_j = g_i & \text{on } \Gamma_1 \\ u = 0 \quad \text{on } \Gamma_2. \end{cases}$$

for $i = 1, 2, 3$, where the coefficients of elasticity a_{ijkh} satisfy the usual symmetries in elasticity, i.e.

$$a_{ijkh} = a_{jikh} = a_{khij}, \qquad \forall i, j, k, h \in \{1, 2, 3\}$$

and where $f = (f_1, f_2, f_3)$ denotes the volume density of applied body forces and $g = (g_1, g_2, g_3)$ the density of surface forces. The last boundary condition means that the Ω is clamped on Γ_2.

Consider now a composite of two materials with the above geometry, whose coefficients of elasticity are a^1_{ijkh} and a^2_{ijkh} respectively, and set

$$a^\varepsilon_{ijkh}(x) = a^1_{ijkh}\,\chi_1\left(\frac{x}{\varepsilon}\right) + a^2_{ijkh}\,\chi_2\left(\frac{x}{\varepsilon}\right).$$

Then the linearized elasticity system for the composite material is

$$\begin{cases} -\displaystyle\sum_{j,k,h=1}^{3} \frac{\partial}{\partial x_j}\left(a^\varepsilon_{ijkh}\,\frac{\partial u^\varepsilon_k}{\partial x_h}\right) = f_i & \text{in } \Omega \\ \displaystyle\sum_{j,k,h=1}^{3} a^\varepsilon_{ijkh}\,\frac{\partial u^\varepsilon_k}{\partial x_h}\,n_j = g & \text{on } \Gamma_1 \\ u^\varepsilon = 0 & \text{on } \Gamma_2, \end{cases}$$

for any $i = 1, 2, 3$. This system will be studied in Chapter 10. ◇

5.3 The one-dimensional case

In this section we present a one-dimensional problem which was studied by Spagnolo (1967).

Let $\Omega =]d_1, d_2[$ be an interval in $\mathbb{R}$ and consider the problem

$$\begin{cases} -\dfrac{d}{dx}\left(a^\varepsilon \dfrac{du^\varepsilon}{dx}\right) = f & \text{in }]d_1, d_2[\\ u^\varepsilon(d_1) = u^\varepsilon(d_2) = 0. \end{cases} \tag{5.24}$$

We assume here that a is a positive function in $L^\infty(0, \ell_1)$ such that

$$\begin{cases} a \text{ is } \ell_1\text{-periodic}, \\ 0 < \alpha \le a(x) \le \beta < +\infty, \end{cases} \tag{5.25}$$

where α and β are constants. The a^ε from (5.24) is the function defined by

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right). \tag{5.26}$$

We have the following result:

Theorem 5.5. *Let $f \in L^2(d_1, d_2)$ and a^ε be defined by (5.25) and (5.26). Let $u^\varepsilon \in H_0^1(d_1, d_2)$ be the solution of problem (5.24). Then,*

$$u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(d_1, d_2),$$

where u^0 is the unique solution in $H_0^1(d_1, d_2)$ of the problem

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\mathcal{M}_{(0,\ell_1)}\left(\dfrac{1}{a}\right)}\dfrac{du^0}{dx}\right) = f \quad \text{in }]d_1, d_2[\\ u^0(d_1) = u^0(d_2) = 0. \end{cases} \tag{5.27}$$

Proof. Observe first that estimate (4.24) holds true, that is

$$\|u^\varepsilon\|_{H_0^1(d_1,d_2)} \leq \frac{d_2 - d_1}{\alpha}\|f\|_{L^2(\Omega)}.$$

Indeed, from the proof of Proposition 3.35, it is immediate that for $\Omega =]d_1, d_2[$, the Poincaré constant C_Ω is equal to $d_2 - d_1$. Then, one still has convergences (5.9) for a subsequence still denoted by ε, which reads here

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } L^2(d_1, d_2) \\ \dfrac{du^\varepsilon}{dx} \rightharpoonup \dfrac{du^0}{dx} \quad \text{weakly in } L^2(d_1, d_2). \end{cases} \tag{5.28}$$

Define

$$\xi^\varepsilon = a^\varepsilon \frac{du^\varepsilon}{dx}$$

which satisfies

$$-\frac{d\xi^\varepsilon}{dx} = f \quad \text{in }]d_1, d_2[. \tag{5.29}$$

Moreover, from the estimate on u^ε and (5.25) one has

$$\|\xi^\varepsilon\|_{L^2(d_1,d_2)} \leq \frac{\beta(d_2 - d_1)}{\alpha}\|f\|_{L^2(d_1,d_2)}.$$

Hence, from Theorem 1.18 one has the convergence (up to a subsequence)

$$\xi^\varepsilon \rightharpoonup \xi^0 \quad \text{weakly in } L^2(d_1, d_2). \tag{5.30}$$

Moreover, the limit ξ^0 satisfies (see (5.15))

$$-\frac{d\xi^0}{dx} = f \quad \text{in }]d_1, d_2[. \tag{5.31}$$

Clearly, from the former estimate on ξ^ε and from equation (5.29), we have

$$\|\xi^\varepsilon\|_{L^2(d_1,d_2)} + \left\|\frac{d\xi^\varepsilon}{dx}\right\|_{L^2(d_1,d_2)} \leq \frac{\beta(d_2 - d_1)}{\alpha}\|f\|_{L^2(d_1,d_2)} + \|f\|_{L^2(d_1,d_2)}.$$

Hence, ξ^ε is bounded in $H^1(d_1, d_2)$, then compact in $L^2(d_1, d_2)$ thanks to Theorem 3.23. Consequently, there exists a subsequence, still denoted by ε, such that

$$\xi^\varepsilon \longrightarrow \xi^0 \quad \text{strongly in } L^2(d_1, d_2). \tag{5.32}$$

We now establish the relation between ξ^0 and u^0.

By definition

$$\frac{du^\varepsilon}{dx} = \frac{1}{a^\varepsilon}\xi^\varepsilon. \tag{5.33}$$

Assumption (5.25) implies that $\dfrac{1}{a^\varepsilon}$ is bounded in $L^\infty(d_1, d_2)$, since

$$0 < \frac{1}{\beta} \leq \frac{1}{a^\varepsilon} \leq \frac{1}{\alpha} < +\infty. \tag{5.34}$$

Therefore, Theorem 2.6 applies to $1/a^\varepsilon$ and gives

$$\frac{1}{a^\varepsilon} \rightharpoonup \mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right) = \frac{1}{\ell_1}\int_0^{\ell_1} \frac{1}{a(x)}\, dx \quad \text{weakly* in } L^\infty(d_1, d_2),$$

where, due to (5.34),

$$\mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right) \neq 0. \tag{5.35}$$

Hence, using (5.32) and in view of Proposition 1.19, we can pass to the limit in the 'weak–strong' product in the right-hand term in (5.33), to obtain

$$\frac{du^\varepsilon}{dx} \rightharpoonup \mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right)\xi^0 \quad \text{weakly in } L^2(d_1, d_2).$$

Consequently, from (5.28) we have

$$\frac{du^0}{dx} = \mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right)\xi^0.$$

Making now use of (5.31), it follows that u^0 is solution of the limit 'homogenized' equation (5.27). Due to (5.35), this problem has a unique solution. Hence, by Theorem 1.18(ii), the whole sequence $\{u^\varepsilon\}$ weakly converges in $H_0^1(d_1, d_2)$ to u^0. This ends the proof. □

Remark 5.6. In this particular case of the dimension one, since $\mathcal{M}_{(0,\ell_1)}(1/a)$ is a constant, one can compute explicitly the limit solution u^0. For example, if $]d_1, d_2[=]0, 1[$, one has

$$u^0(x) = -\mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right)\int_0^x dy \int_0^y f(t)\, dt + \mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right)\left(\int_0^1 dy \int_0^y f(t)\, dt\right) x.$$

◊

Remark 5.7. Let us observe that the coefficient of the limit equation (5.27) only depends on a. ◊

Remark 5.8. Since in general, the harmonic mean value $1/[\mathcal{M}_{(0,\ell_1)}(1/a)]$ of a is different from its arithmetic mean value $\mathcal{M}_{(0,\ell_1)}(a)$, clearly

$$\frac{1}{\mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right)}\frac{du^0}{dx} \neq \mathcal{M}_{(0,\ell_1)}(a)\frac{du^0}{dx},$$

so that

$$\lim_{\varepsilon\to 0}\left(a^\varepsilon \frac{du^\varepsilon}{dx}\right) \neq (\lim_{\varepsilon\to 0} a^\varepsilon)\left(\lim_{\varepsilon\to 0}\frac{du^\varepsilon}{dx}\right),$$

in the sense of the L^2-weak convergence. ◊

Remark 5.9. We considered here the periodic case. Suppose now that $\{a^\varepsilon\}$ is a sequence (not necessarily periodic) such that

$$0 < \alpha \leq a^\varepsilon(x) \leq \beta < +\infty.$$

Then, due to Theorem 1.26, there exists a subsequence ε' such that

$$\frac{1}{a^{\varepsilon'}} \rightharpoonup a^0 \quad \text{weakly* in } L^\infty(d_1, d_2).$$

Let u^ε be the solution of (5.24). Same arguments as in the proof of Theorem 5.5, show that

$$u^{\varepsilon'} \rightharpoonup u^0 \quad \text{weakly in } H_0^1(d_1, d_2),$$

where u^0 is solution of the equation

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{a_0}\dfrac{du^0}{dx}\right) = f & \text{in }]d_1, d_2[, \\ u^0(d_1) = u^0(d_2) = 0. \end{cases}$$

◊

5.4 Layered materials

The result obtained for the one-dimensional case could suggest that in the N-dimensional case the limit problem can be described in terms of the mean value of the inverse matrix A^{-1} of A. This is not true, as can already be seen in the case of the layered materials we treat in this section, where the homogenized coefficients are again the mean value of algebraic expressions of the components of A (but not only those of A^{-1}).

In the case of layered materials the coefficients in system (5.6) depend on one variable only. For the sake of simplicity, we restrict ourselves to the two-dimensional periodic case. So, in this section, Ω denotes a bounded open set in $\mathbb{R}^2$ and

$$Y =]0, \ell_1[\times]0, \ell_2[.$$

where ℓ_1, ℓ_2 are given positive numbers.

Assume then that $A = (a_{ij})_{1 \le i,j \le 2}$ is a 2×2 matrix such that

$$a_{ij}(y) = a_{ij}(y_1, y_2) = a_{ij}(y_1), \quad \forall i, j \in \{1, 2\}, \tag{5.36}$$

and satisfying (5.18), that is

$$\begin{cases} a_{ij} \text{ is } \ell_1\text{-periodic}, & \forall i, j \in \{1, 2\}, \\ A \in M(\alpha, \beta, Y). \end{cases}$$

Set, as before,

$$\begin{cases} a_{ij}^{\varepsilon}(x) = a_{ij}^{\varepsilon}(x_1) = a_{ij}\left(\dfrac{x_1}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^2, \quad \forall i, j \in \{1, 2\} \\ A^{\varepsilon}(x) = A^{\varepsilon}(x_1) = A\left(\dfrac{x_1}{\varepsilon}\right) = (a_{ij}^{\varepsilon}(x))_{1 \le i,j \le 2} \quad \text{a.e. on } \mathbb{R}^2. \end{cases} \tag{5.37}$$

Problem (5.6) reads

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}^{\varepsilon}(x_1)\frac{\partial u^{\varepsilon}}{\partial x_j}\right) = f \quad \text{in } \Omega \\ u^{\varepsilon} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.38}$$

Clearly, all the considerations in Section 5.1 concerning problem (5.6) still hold for this particular case. The following result is due to Murat and Tartar (Murat, 1978a, Murat and Tartar 1997a and Tartar 1977a):

Theorem 5.10. *Let $f \in L^2(\Omega)$ and a_{ij}^{ε} satisfying (5.18), (5.36) and (5.37). Let $u^{\varepsilon} \in H_0^1(\Omega)$ be the solution of problem (5.38). Then,*

$$u^{\varepsilon} \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega),$$

where u^0 is the unique solution in $H_0^1(\Omega)$ of the homogenized problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\breve{a}_{ij}\frac{\partial u^0}{\partial x_j}\right) = f \quad \text{in } \Omega \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.39}$$

The matrix $\breve{A} = (\breve{a}_{ij})_{1 \le i,j \le 2}$ is a constant positive definite matrix defined by

$$\begin{aligned} \breve{a}_{11} &= \frac{1}{\mathcal{M}_{(0,\ell_1)}\left(\dfrac{1}{a_{11}}\right)} \\ \breve{a}_{12} &= \breve{a}_{11}\, \mathcal{M}_{(0,\ell_1)}\left(\frac{a_{12}}{a_{11}}\right) \\ \breve{a}_{21} &= \breve{a}_{11}\, \mathcal{M}_{(0,\ell_1)}\left(\frac{a_{21}}{a_{11}}\right) \\ \breve{a}_{22} &= \breve{a}_{11}\mathcal{M}_{(0,\ell_1)}\left(\frac{a_{12}}{a_{11}}\right)\mathcal{M}_{(0,\ell_1)}\left(\frac{a_{21}}{a_{11}}\right) + \mathcal{M}_{(0,\ell_1)}\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right). \end{aligned}$$

Proof. Observe first that estimate (4.24) holds true from problem (5.38), i.e.

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \leq \frac{C_\Omega}{\alpha}\|f\|_{L^2(\Omega)},$$

where C_Ω is the Poincaré constant given by Proposition 3.35. Also,

$$\|\xi^\varepsilon\|_{L^2(\Omega)} \leq \frac{\beta C_\Omega}{\alpha}\|f\|_{L^2(\Omega)}. \tag{5.40}$$

where (see (5.10)) ξ^ε is defined by

$$\xi^\varepsilon = (\xi_1^\varepsilon, \xi_2^\varepsilon) = \left(\sum_{j=1}^{2} a_{1j}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}, \ \sum_{j=1}^{2} a_{2j}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \right) \tag{5.41}$$

and satisfies

$$-\frac{\partial \xi_1^\varepsilon}{\partial x_1} - \frac{\partial \xi_2^\varepsilon}{\partial x_2} = f \quad \text{in } \Omega. \tag{5.42}$$

As was proved in Section 5.1 (see (5.9) and (5.13)) one has the following convergences for a subsequence still denoted by ε,

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega) \\ \xi^\varepsilon \rightharpoonup \xi^0 & \text{weakly in } (L^2(\Omega))^2. \end{cases}$$

Moreover $\xi^0 = (\xi_1^0, \xi_2^0)$ satisfies

$$-\frac{\partial \xi_1^0}{\partial x_1} - \frac{\partial \xi_2^0}{\partial x_2} = f \quad \text{in } \Omega. \tag{5.43}$$

The question is how to relate ξ^0 to the limit function u^0. Of course, we cannot compute directly the limit of (5.41) since, as remarked before, in this formula we have products of only weakly convergent sequences. Neither can we make use of equation (5.42) to derive, as we did for the one-dimensional case, further information about $\partial\xi_1^\varepsilon/\partial x_1$ or $\partial\xi_2^\varepsilon/\partial x_2$ separately. The idea is then to make use of a compactness result in order to obtain a strong convergence in some functional space. It is at this point that the fact that the coefficients depend on only one variable is essential since, due to this property, we prove a strong convergence for the sequence ξ_1^0. The tools to do so are Proposition 3.57 and Theorem 3.58 dealing with vector-valued functions.

We will identify ξ^0 on any open interval I of Ω of the form

$$I =]a_1, b_1[\times]a_2, b_2[\subset \Omega.$$

Due to the particular form of I, the space $L^2(I)$ can be regarded as a vector-valued space (see Section 3.5) since Fubini's theorem implies

$$L^2(I) = L^2(a_1, b_1; \ L^2(a_2, b_2)). \tag{5.44}$$

Consequently, from (5.40) one has in particular

$$\|\xi_i^\varepsilon\|_{L^2(a_1,b_1;\, L^2(a_2,b_2))} \leq C, \quad i = 1, 2, \tag{5.45}$$

where C is a constant independent of ε.

We now prove that

$$\left\| \frac{\partial \xi_1^\varepsilon}{\partial x_1} \right\|_{L^2(a_1,b_1;\, H^{-1}(a_2,b_2))} \leq C. \tag{5.46}$$

From (5.42), one has that

$$-\frac{\partial \xi_1^\varepsilon}{\partial x_1} = f + \frac{\partial \xi_2^\varepsilon}{\partial x_2} \quad \text{in } \Omega, \tag{5.47}$$

and we will estimate the right-hand terms.

On one hand, since f belongs to $L^2(\Omega)$, from (5.44) it follows (see Section 3.5) that $f \in L^2(a_1, b_1;\ L^2(a_2, b_2)) \subset L^2(a_1, b_1;\ H^{-1}(a_2, b_2))$ with

$$\|f\|_{L^2(a_1,b_1;\, H^{-1}(a_2,b_2))} \leq \|f\|_{L^2(a_1,b_1;\, L^2(a_2,b_2))}, \tag{5.48}$$

which is a consequence of Proposition 3.42.

On the other hand, by Proposition 3.59, for any $v \in L^2(a_1, b_1;\ H_0^1(a_2, b_2))$, one has

$$\begin{cases} \left\langle \dfrac{\partial \xi_2^\varepsilon}{\partial x_2}, v \right\rangle_{L^2(a_1,b_1;\, H^{-1}(a_2,b_2)),\, L^2(a_1,b_1;\, H_0^1(a_2,b_2))} \\ \quad = \displaystyle\int_{a_1}^{b_1} \left\langle \frac{\partial \xi_2^\varepsilon}{\partial x_2}(x_1, \cdot), v(x_1, \cdot) \right\rangle_{H^{-1}(a_2,b_2),\, H_0^1(a_2,b_2)} dx_1. \end{cases}$$

Furthermore, Remark 3.44, Green's formula (Theorem 3.33) and Proposition 3.34 lead to

$$\begin{aligned} &\int_{a_1}^{b_1} \left\langle \frac{\partial \xi_2^\varepsilon}{\partial x_2}(x_1, \cdot), v(x_1, \cdot) \right\rangle_{H^{-1}(a_2,b_2),\, H_0^1(a_2,b_2)} dx_1 \\ &\quad = -\int_{a_1}^{b_1} \left\langle \xi_2^\varepsilon(x_1, \cdot), \frac{\partial v(x_1, \cdot)}{\partial x_2} \right\rangle_{L^2(a_2,b_2),\, L^2(a_2,b_2)} dx_1. \end{aligned} \tag{5.49}$$

Consequently, by using the Cauchy-Schwarz inequality and estimate (5.45), we have successively

$$\begin{aligned} &\left| \left\langle \frac{\partial \xi_2^\varepsilon}{\partial x_2}, v \right\rangle_{L^2(a_1,b_1;\, H^{-1}(a_2,b_2)),\, L^2(a_1,b_1;\, H_0^1(a_2,b_2))} \right| \\ &\quad \leq \int_{a_1}^{b_1} \|\xi_2^\varepsilon\|_{L^2(a_2,b_2)} \|v\|_{H_0^1(a_2,b_2)}\, dx_1 \\ &\quad \leq C \|v\|_{L^2(a_1,b_1;\, H_0^1(a_2,b_2))}. \end{aligned}$$

This, together with (5.47) and (5.48), gives (5.46). Estimates (5.45) and (5.46) imply that ξ_1^ε is bounded in the space $\mathcal{W}_1$ defined by

$$\mathcal{W}_1 = \left\{ v \mid v \in L^2(a_1, b_1;\ L^2(a_2, b_2)),\ \frac{\partial v}{\partial x_1} \in L^2(a_1, b_1;\ H^{-1}(a_2, b_2)) \right\}.$$

By Theorem 3.58 we know that the following injection is compact:

$$\mathcal{W}_1 \subset L^2(a_1, b_1;\ H^{-1}(a_2, b_2)).$$

Therefore, the sequence ξ_1^ε is compact in $L^2(a_1, b_1;\ H^{-1}(a_2, b_2))$. This, together with its weak convergence to ξ^0 in $L^2(I)$, gives

$$\xi_1^\varepsilon \to \xi_1^0 \quad \text{strongly in } L^2(a_1, b_1;\ H^{-1}(a_2, b_2)). \tag{5.50}$$

We now show that this convergence is sufficient to identify ξ_1^0 and ξ_2^0 in terms of u^0.

To begin with, observe that from the definition of ξ_1^ε (see (5.41)) and taking into account the fact that a_{ij}^ε are dependent on x_1 only, one has

$$\begin{aligned} \int_I \frac{\partial u^\varepsilon}{\partial x_1} \varphi\, dx &= \int_I \frac{1}{a_{11}^\varepsilon} \xi_1^\varepsilon \varphi\, dx - \int_I \frac{a_{12}^\varepsilon}{a_{11}^\varepsilon} \frac{\partial u^\varepsilon}{\partial x_2} \varphi\, dx \\ &= \int_I \frac{1}{a_{11}^\varepsilon} \xi_1^\varepsilon \varphi\, dx - \int_I \frac{\partial}{\partial x_2} \left(\frac{a_{12}^\varepsilon}{a_{11}^\varepsilon} u^\varepsilon \right) \varphi\, dx \\ &= \int_I \frac{1}{a_{11}^\varepsilon} \xi_1^\varepsilon \varphi\, dx + \int_I \frac{a_{12}^\varepsilon}{a_{11}^\varepsilon} u^\varepsilon \frac{\partial \varphi}{\partial x_2}\, dx, \end{aligned} \tag{5.51}$$

for any $\varphi \in \mathcal{D}(I)$. We will now pass to the limit in both sides of this identity. To do so, remark first that, by the same arguments as those used for proving (5.49), one has

$$\begin{aligned} \int_I \frac{1}{a_{11}^\varepsilon} \xi_1^\varepsilon \varphi\, dx &= \int_{a_1}^{b_1} \left\langle \xi_1^\varepsilon(x_1, \cdot),\ \frac{\varphi(x_1, \cdot)}{a_{11}^\varepsilon(x_1)} \right\rangle_{H^{-1}(a_2,b_2),\, H_0^1(a_2,b_2)} dx_1 \\ &= \left\langle \xi_1^\varepsilon,\ \frac{\varphi}{a_{11}^\varepsilon} \right\rangle_{L^2(a_1,b_1;\ H^{-1}(a_2,b_2)),\ L^2(a_1,b_1;\ H_0^1(a_2,b_2))}. \end{aligned} \tag{5.52}$$

Moreover, $\varphi / a_{11}^\varepsilon$ is bounded in $L^2(a_1, b_1;\ H_0^1(a_2, b_2))$. To see that, observe first that by choosing $\lambda = (0, 1)$ in the coerciveness condition from (5.3), one has

$$0 < \alpha \leq a_{11}^\varepsilon \quad \text{a.e. on }]0, \ell_1[,$$

whence

$$0 < \frac{1}{a_{11}^\varepsilon} \leq \frac{1}{\alpha}.$$

Consequently,

$$
\begin{aligned}
\int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} \left| \frac{\partial}{\partial x_2}\left(\frac{\varphi(x_1,x_2)}{a_{11}^{\varepsilon}(x_1)}\right)\right|^2 dx_2 \\
&= \int_{a_1}^{b_1} \frac{1}{(a_{11}^{\varepsilon}(x_1))^2}\, dx_1 \int_{a_2}^{b_2} \left|\frac{\partial \varphi(x_1,x_2)}{\partial x_2}\right|^2 dx_2 \\
&\leq \frac{1}{\alpha^2}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|\frac{\partial \varphi(x_1,x_2)}{\partial x_2}\right|^2 dx_2\, dx_1 \leq \frac{1}{\alpha^2}\|\varphi\|^2_{H^1(I)}. \qquad (5.53)
\end{aligned}
$$

On the other hand, if $\{h^\varepsilon\}$ is a sequence in $L^2(a_1,b_1)$ then

$$
\left(h^\varepsilon \rightharpoonup h^0 \quad \text{weakly in } L^2(a_1,b_1) \right) \Longrightarrow \left(h^\varepsilon\varphi \rightharpoonup h^0\varphi \quad \text{weakly in } L^2(I) \right), \qquad (5.54)
$$

for any $\varphi \in \mathcal{D}(I)$. This follows by Fubini's theorem, since

$$
\int_I h^\varepsilon\, \varphi\, \psi\, dx = \int_{a_1}^{b_1} h^\varepsilon(x_1)\left[\int_{a_2}^{b_2} \varphi(x_1,x_2)\,\psi(x_1,x_2)\, dx_2\right] dx_1
$$

for any $\psi \in L^2(I)$. Then, (5.54) applied to $h^\varepsilon = 1/a_{11}^\varepsilon$ together with Theorem 2.6, shows that

$$
\frac{\varphi}{a_{11}^{\varepsilon}} \rightharpoonup \mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a_{11}}\right)\varphi = \frac{1}{\breve{a}_{11}}\,\varphi \quad \text{weakly in } L^2(I),
$$

where we used the definition of $\breve{a}_{11}$ in the statement of Theorem 5.10. Then,

$$
\frac{\varphi}{a_{11}^{\varepsilon}} \rightharpoonup \frac{1}{\breve{a}_{11}}\,\varphi \quad \text{weakly in } L^2(a_1,b_1;\ H_0^1(a_2,b_2)),
$$

since, due to (5.53), it is bounded in this space.

Consequently, by using (5.50) and Proposition 1.19, we can pass to the limit in (5.52) to get

$$
\lim_{\varepsilon\to 0}\int_I \frac{1}{a_{11}^{\varepsilon}}\,\xi_1^\varepsilon\,\varphi\, dx = \frac{1}{\breve{a}_{11}}\int_I \xi_1^0\,\varphi\, dx. \qquad (5.55)
$$

Recall now that from (5.9) and Theorem 3.23 one has in particular

$$
u_1^\varepsilon \to u_1^0 \quad \text{strongly in } L^2(I). \qquad (5.56)
$$

This, together with (5.54) and Proposition 1.19, shows that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_I \frac{a_{12}^{\varepsilon}}{a_{11}^{\varepsilon}}\, u^\varepsilon\, \frac{\partial\varphi}{\partial x_2}\, dx &= \mathcal{M}_{(0,\ell_1)}\left(\frac{a_{12}}{a_{11}}\right)\int_I u^0\,\frac{\partial\varphi}{\partial x_2}\, dx \\
&= -\mathcal{M}_{(0,\ell_1)}\left(\frac{a_{12}}{a_{11}}\right)\int_I \frac{\partial u^0}{\partial x_2}\,\varphi\, dx, \qquad (5.57)
\end{aligned}
$$

where we have made use of Definition 3.11.

Hence, passing to the limit in (5.51), due to (5.55) and (5.57), one gets

$$\int_I \frac{\partial u^0}{\partial x_1}\varphi\,dx = \frac{1}{\breve{a}_{11}}\int_I \xi_1^0\,\varphi\,dx - \mathcal{M}_{(0,\ell_1)}\left(\frac{a_{12}}{a_{11}}\right)\int_I \frac{\partial u^0}{\partial x_2}\varphi\,dx, \quad \forall\varphi\in\mathcal{D}(I).$$

Consequently, Theorem 1.44 implies that

$$\frac{\partial u^0}{\partial x_1} = \frac{1}{\breve{a}_{11}}\xi_1^0 - \mathcal{M}_{(0,\ell_1)}\left(\frac{a_{12}}{a_{11}}\right)\frac{\partial u^0}{\partial x_2},$$

a.e. on I and therefore on Ω, since I is arbitrary.

This, together with the definition of a_{12}^0, identifies ξ_1^0 as

$$\xi_1^0 = \breve{a}_{11}\frac{\partial u^0}{\partial x_1} + \breve{a}_{11}\mathcal{M}_{(0,\ell_1)}\left(\frac{a_{12}}{a_{11}}\right)\frac{\partial u^0}{\partial x_2} = \breve{a}_{11}\frac{\partial u^0}{\partial x_1} + \breve{a}_{12}\frac{\partial u^0}{\partial x_2}. \tag{5.58}$$

Let us now identify ξ_2^0. By again using Definition (5.41) and the fact that a_{ij}^ε are dependent on x_1 only, one has

$$\xi_2^\varepsilon = a_{21}^\varepsilon\frac{\partial u^\varepsilon}{\partial x_1} + a_{22}^\varepsilon\frac{\partial u^\varepsilon}{\partial x_2} = \frac{a_{21}^\varepsilon}{a_{11}^\varepsilon}\xi_1^\varepsilon + \frac{\partial}{\partial x_2}\left[\left(a_{22}^\varepsilon - \frac{a_{12}^\varepsilon a_{21}^\varepsilon}{a_{11}^\varepsilon}\right)u^\varepsilon\right]. \tag{5.59}$$

The same arguments as those used to prove (5.55) and (5.57), give respectively

$$\lim_{\varepsilon\to 0}\int_I \frac{a_{21}^\varepsilon}{a_{11}^\varepsilon}\xi_1^\varepsilon\,\varphi\,dx = \mathcal{M}_{(0,\ell_1)}\left(\frac{a_{21}}{a_{11}}\right)\int_I \xi_1^0\,\varphi\,dx,$$

and

$$\lim_{\varepsilon\to 0}\int_I\left(a_{22}^\varepsilon - \frac{a_{12}^\varepsilon a_{21}^\varepsilon}{a_{11}^\varepsilon}\right)u^\varepsilon\,\frac{\partial\varphi}{\partial x_2}\,dx = -\mathcal{M}_{(0,\ell_1)}\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)\int_I \frac{\partial u^0}{\partial x_2}\varphi\,dx.$$

Hence, from (5.59) one derives

$$\xi_2^0 = \mathcal{M}_{(0,\ell_1)}\left(\frac{a_{21}}{a_{11}}\right)\xi_1^0 + \mathcal{M}_{(0,\ell_1)}\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)u^0,$$

which, together with (5.58) and the definition of $\breve{a}_{22}$, leads to

$$\begin{cases} \xi_2^0 = \breve{a}_{11}\mathcal{M}_{(0,\ell_1)}\left(\dfrac{a_{21}}{a_{11}}\right)\dfrac{\partial u^0}{\partial x_1} + \left[\breve{a}_{11}\mathcal{M}_{(0,\ell_1)}\left(\dfrac{a_{12}}{a_{11}}\right)\mathcal{M}_{(0,\ell_1)}\left(\dfrac{a_{21}}{a_{11}}\right)\right. \\ \left. +\mathcal{M}_{(0,\ell_1)}\left(a_{22} - \dfrac{a_{12}a_{21}}{a_{11}}\right)\right]\dfrac{\partial u^0}{\partial x_2} = \breve{a}_{21}\dfrac{\partial u^0}{\partial x_1} + \breve{a}_{22}\dfrac{\partial u^0}{\partial x_2}. \end{cases}$$

Replacing this equality and (5.58) into (5.43), one obtains equation (5.39).

To complete the proof, it remains to show that problem (5.39) has a unique solution. This will imply that the whole sequence u^ε converges to the limit u^0.

To do so, in view of Lax–Milgram theorem, it is sufficient to show that the constant matrix $\breve{A} = (\breve{a}_{ij})_{1\le i,j\le 2}$ satisfies an ellipticity condition of the form (4.16) (see also Remark 4.7). From (5.19) and the characterization of a positive definite matrix (see for instance Ciarlet, 1982, Lang, 1993) it follows that a_{11} and $(a_{11}a_{22} - a_{12}a_{21})$ are positive almost everywhere in Y. This implies that $\breve{a}_{11}$ and $\mathcal{M}_{(0,\ell_1)}(a_{22} - a_{12}a_{21}/a_{11})$ are positive constants too.

Then, an easy computation shows that the determinant of $\breve{A}$ is also positive. This, together with the positivity of $\breve{a}_{11}$, implies

$$\sum_{i,j=1}^{2} \breve{a}_{ij}(y)\xi_i\xi_j > 0, \quad \text{for any } \xi \in \mathbb{R}^2, \ \xi \neq 0.$$

To finish the proof of ellipticity, let h be the following function:

$$h(\zeta,\zeta) = \sum_{i,j=1}^{2} \breve{a}_{ij}(y)\zeta_i\zeta_j.$$

This function is continuous on the unit sphere S_1 which is a compact set of $\mathbb{R}^2$. Hence, h achieves its minimum on S_1 and, due to the previous result, this minimum is positive. So, there exists $\alpha_0 > 0$ such that

$$h(\zeta,\zeta) \ge \alpha_0, \quad \forall \zeta \in S_1.$$

Consequently,

$$\sum_{i,j=1}^{2} \breve{a}_{ij}\xi_i\xi_j = |\xi|^2 \sum_{i,j=1}^{2} \breve{a}_{ij}\frac{\xi_i}{|\xi|}\frac{\xi_j}{|\xi|} \ge \alpha_0, \quad \text{for any } \xi \in \mathbb{R}^2, \ \xi \neq 0,$$

since the vector $(\xi_1/|\xi|, \xi_2/|\xi|)$ belongs to S_1. This ends the proof of Theorem 5.10. □

Remark 5.11. As for the one-dimensional case, the coefficients of the limit problem only depend on the matrix A, and not on the other data f and Ω. ◊

It is only for simplicity that we treated the two-dimensional case of layered materials. A similar result holds for the N-dimensional case. Actually, suppose now that $A = (a_{ij})_{1\le i,j\le N}$ is an $N \times N$ matrix satisfying

$$a_{ij}(y) = a_{ij}(y_1, \dots, y_N) = a_{ij}(y_1), \quad i, j \in \{1, \dots, N\},$$

and (5.36). Then, the following result (see Murat and Tartar, 1997a) holds:

Theorem 5.12. *Let Ω be a bounded domain in $\mathbb{R}^N$. Under hypotheses of Theorem 5.10, if $u^\varepsilon \in H_0^1(\Omega)$ is the solution of problem (5.39), then*

$$u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega),$$

where u^0 is the unique solution in $H_0^1(\Omega)$ of the homogenized problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\breve{a}_{ij}\frac{\partial u^0}{\partial x_j}\right) = f & \text{in } \Omega \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The matrix $\breve{A} = (\breve{a}_{ij})_{1\leq i,j\leq 2}$ is a constant positive definite matrix and its coefficients are given by

$$\begin{aligned}
\breve{a}_{11} &= \frac{1}{\mathcal{M}_{(0,\ell_1)}\left(\dfrac{1}{a_{11}}\right)} \\
\breve{a}_{1j} &= \breve{a}_{11}\,\mathcal{M}_{(0,\ell_1)}\left(\frac{a_{1j}}{a_{11}}\right), \quad \text{for } 2\leq j\leq N \\
\breve{a}_{j1} &= \breve{a}_{11}\,\mathcal{M}_{(0,\ell_1)}\left(\frac{a_{j1}}{a_{11}}\right), \quad \text{for } 2\leq j\leq N \\
\breve{a}_{ij} &= \breve{a}_{11}\mathcal{M}_{(0,\ell_1)}\left(\frac{a_{1j}}{a_{11}}\right)\mathcal{M}_{(0,\ell_1)}\left(\frac{a_{i1}}{a_{11}}\right) + \mathcal{M}_{(0,\ell_1)}\left(a_{ij} - \frac{a_{1j}a_{i1}}{a_{11}}\right) \\
&\quad \text{for } 2\leq i,j\leq N.
\end{aligned}$$

The proof follows exactly that of Theorem 5.10 with obvious modifications.

6

Homogenization of elliptic equations: the convergence result

We place ourselves in the framework introduced in Section 5.1. The aim of this chapter is to describe the asymptotic behaviour as $\varepsilon \to 0$ of problem (5.6), namely

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where f is given in $H^{-1}(\Omega)$ and the matrix A^ε is the Y-periodic matrix defined by

$$a_{ij}^\varepsilon(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall i, j = 1, \dots, N \tag{6.2}$$

and

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) = (a_{ij}^\varepsilon(x))_{1\leq i,j\leq N} \quad \text{a.e. on } \mathbb{R}^N, \tag{6.3}$$

where

$$\begin{cases} a_{ij} \text{ is } Y\text{-periodic}, \quad \forall i, j = 1, \dots, N \\ A = (a_{ij})_{1\leq i,j\leq N} \in M(\alpha, \beta, Y), \end{cases} \tag{6.4}$$

with $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$ and $M(\alpha, \beta, Y)$ is given by Definition 4.11. Here, as before, Y denotes the reference cell defined by

$$Y =]0, \ell_1[\times \cdots \times]0, \ell_N[,$$

where $\ell_1, \dots, \ell_N$ are given positive numbers. The Y-periodicity is taken in the sense of Definition 2.1.

In Chapter 5 we have studied the particular cases of the dimension one and of layered materials. As we already mentioned, to study the general N-dimensional case we need to introduce some auxiliary functions which are solutions of periodic boundary value problems in the reference cell Y. This is done in Section 6.1 below. In Section 6.2 we state the main homogenization result for problem (6.1). This result will be proved in the next chapters by several methods. In Sections 6.3 and 6.4 we give the main properties of the homogenized matrix.

6.1 Auxiliary periodic problems

In this section, we introduce two families of auxiliary periodic boundary value problems defined on the reference cell Y. The first one involves the matrix A and consequently, the operator $\mathcal{A} = -\mathrm{div}(A\nabla)$. The second family involves the transposed matrix ${}^tA$ and so, the operator $\mathcal{A}^* = -\mathrm{div}({}^tA\nabla)$.

The functions $\widehat{\chi}_\lambda$ and $\widehat{w}_\lambda$

Consider, for any $\lambda \in \mathbb{R}^N$ the solution of the problem

$$\begin{cases} -\mathrm{div}(A(y)\nabla\widehat{\chi}_\lambda) = -\mathrm{div}(A(y)\lambda) & \text{in } Y \\ \widehat{\chi}_\lambda \quad Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{\chi}_\lambda) = 0. \end{cases} \tag{6.5}$$

This kind of problem has been studied in Section 4.4. The variational formulation of (6.5) is (see (4.68))

$$\begin{cases} \text{Find } \widehat{\chi}_\lambda \in W_{\mathrm{per}}(Y) \text{ such that} \\ a_Y(\widehat{\chi}_\lambda, v) = \displaystyle\int_Y A\lambda\nabla v\, dy, \\ \forall v \in W_{\mathrm{per}}(Y), \end{cases} \tag{6.6}$$

where

$$a_Y(u, v) = \int_Y A\nabla u\, \nabla v\, dy, \quad \forall u, v \in W_{\mathrm{per}}(Y) \tag{6.7}$$

and (see (4.66))

$$W_{\mathrm{per}}(Y) = \{v \in H^1_{\mathrm{per}}(Y);\ \mathcal{M}_Y(v) = 0\}.$$

with $H^1_{\mathrm{per}}(Y)$ given by Definition 3.48.

From Theorem 4.27 we know that (6.6) has a unique solution $\widehat{\chi}_\lambda \in W_{\mathrm{per}}(Y)$ since $\mathrm{div}(A\lambda) \in (W_{\mathrm{per}}(Y))'$.

Let us now extend by periodicity (see (3.7)) $\widehat{\chi}_\lambda$ to the whole of $\mathbb{R}^N$ and still denote by $\widehat{\chi}_\lambda$ this extension. Then, Theorem 4.28 shows that $\widehat{\chi}_\lambda$ is the unique solution of the following problem:

$$\begin{cases} -\mathrm{div}(A(y)\nabla\widehat{\chi}_\lambda) = -\mathrm{div}(A(y)\lambda) & \text{in } \mathcal{D}'(\mathbb{R}^N) \\ \widehat{\chi}_\lambda \quad Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{\chi}_\lambda) = 0. \end{cases} \tag{6.8}$$

Set now, for any $\lambda \in \mathbb{R}^N$,

$$\widehat{w}_\lambda = -\widehat{\chi}_\lambda + \lambda \cdot y, \tag{6.9}$$

which from (6.5) and (6.6), satisfies

$$a_Y(\widehat{w}_\lambda, v) = 0, \quad \forall v \in W_{\mathrm{per}}(Y), \tag{6.10}$$

and is the unique solution of

$$\begin{cases} -\mathrm{div}(A(y)\nabla\widehat{w}_\lambda) = 0 & \text{in } Y \\ \widehat{w}_\lambda - \lambda \cdot y & Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{w}_\lambda - \lambda \cdot y) = 0, \end{cases} \tag{6.11}$$

whose variational formulation is

$$\begin{cases} \text{Find } \widehat{w}_\lambda \text{ such that } \widehat{w}_\lambda - \lambda \cdot y \in W_{\mathrm{per}}(Y) \text{ and} \\ a_Y(\widehat{w}_\lambda, v) = 0 \\ \forall v \in W_{\mathrm{per}}(Y). \end{cases} \tag{6.12}$$

Let us remark that from (6.8) and (6.9) one also has that $\widehat{w}_\lambda$ satisfies

$$\begin{cases} -\mathrm{div}(A(y)\,\nabla\widehat{w}_\lambda) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^N) \\ \widehat{w}_\lambda - \lambda \cdot y \quad Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{w}_\lambda - \lambda \cdot y) = 0. \end{cases} \tag{6.13}$$

In the sequel, we will often use the functions $\widehat{\chi}_\lambda$ and $\widehat{w}_\lambda$ for the choice $\lambda = e_i$ for $i = 1, \dots, N$, where $(e_i)_{i=1}^N$ is the canonical basis of $\mathbb{R}^N$. We set, for simplicity,

$$\begin{cases} \widehat{\chi}_i = \widehat{\chi}_{e_i} \\ \widehat{w}_i = \widehat{w}_{e_i} = y_i - \widehat{\chi}_i, \end{cases} \tag{6.14}$$

for any $i = 1, \dots, N$. They obviously satisfy, respectively, the following problems:

$$\begin{cases} \text{Find } \widehat{\chi}_i \in W_{\mathrm{per}}(Y) \text{ such that} \\ a_Y(\widehat{\chi}_i, v) = \displaystyle\int_Y Ae_i \nabla v \, dy \\ \forall v \in W_{\mathrm{per}}(Y). \end{cases} \tag{6.15}$$

and

$$\begin{cases} \text{Find } \widehat{w}_i \text{ such that } \widehat{w}_i - y_i \in W_{\mathrm{per}}(Y) \text{ and} \\ a_Y(\widehat{w}_i, v) = 0 \\ \forall v \in W_{\mathrm{per}}(Y). \end{cases} \tag{6.16}$$

It is easily seen that, by linearity,

$$\widehat{\chi}_\lambda = \sum_{i=1}^N \lambda_i \, \widehat{\chi}_i, \quad \widehat{w}_\lambda = \sum_{i=1}^N \lambda_i \, \widehat{w}_i, \quad \forall \lambda \in \mathbb{R}^N.$$

The functions χ_λ and w_λ

Let now consider the transposed matrix ${}^tA$ of A, defined by

$$ {}^tA(y) = (a_{ji}(y))_{1\le i,j\le N} \quad \text{a.e. on } \mathbb{R}^N. $$

Obviously, ${}^tA$ satisfies (6.4) with the same constants α et β, i.e. ${}^tA \in M(\alpha, \beta, Y)$. Consequently, if in problem (6.5) A is replaced by ${}^tA$, we define another set of functions, namely χ_λ, solutions of

$$ \begin{cases} -\mathrm{div}({}^tA(y)\nabla\chi_\lambda) = -\mathrm{div}({}^tA(y)\lambda) & \text{in } Y \\ \chi_\lambda \quad Y\text{-periodic} \\ \mathcal{M}_Y(\chi_\lambda) = 0, \end{cases} \tag{6.17} $$

and all the considerations above hold true.

So, for any $\lambda \in \mathbb{R}^N$, χ_λ is the unique solution of the variational problem

$$ \begin{cases} \text{Find } \chi_\lambda \in W_{\mathrm{per}}(Y) \text{ such that} \\ a^*_Y(\chi_\lambda, v) = \displaystyle\int_Y {}^tA\lambda\nabla v\, dy \\ \forall v \in W_{\mathrm{per}}(Y), \end{cases} \tag{6.18} $$

where

$$ a^*_Y(u, v) = \int_Y {}^tA\nabla u\,\nabla v\, dy, \quad \forall u, v \in W_{\mathrm{per}}(Y). \tag{6.19} $$

Moreover, its extension by periodicity to the whole $\mathbb{R}^N$, still denoted by χ_λ, is the unique solution of the following problem:

$$ \begin{cases} -\mathrm{div}({}^tA(y)\,\nabla\chi_\lambda) = -\mathrm{div}({}^tA(y)\lambda) & \text{in } \mathcal{D}'(\mathbb{R}^N) \\ \chi_\lambda \quad Y\text{-periodic} \\ \mathcal{M}_Y(\chi_\lambda) = 0. \end{cases} \tag{6.20} $$

Also, if for any $\lambda \in \mathbb{R}^N$, we set

$$ w_\lambda = -\chi_\lambda + \lambda\cdot y, \tag{6.21} $$

then, from (6.17) and (6.18), w_λ satisfies

$$ a^*_Y(w_\lambda, v) = 0, \quad \forall v \in W_{\mathrm{per}}(Y). \tag{6.22} $$

and is the unique solution of

$$\begin{cases} -\mathrm{div}({}^tA(y)\nabla w_\lambda) = 0 & \text{in } Y \\ w_\lambda - \lambda \cdot y & Y\text{-periodic} \\ \mathcal{M}_Y(w_\lambda - \lambda \cdot y) = 0. \end{cases} \tag{6.23}$$

The corresponding variational formulation is

$$\begin{cases} \text{Find } w_\lambda \text{ such that } w_\lambda - \lambda \cdot y \in W_{\text{per}}(Y) \text{ and} \\ a^*_Y(w_\lambda, v) = 0 \\ \forall v \in W_{\text{per}}(Y). \end{cases} \tag{6.24}$$

From (6.15) and (6.16), w_λ satisfies

$$\begin{cases} -\mathrm{div}({}^tA\,\nabla w_\lambda) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^N) \\ w_\lambda - \lambda \cdot y & Y\text{-periodic} \\ \mathcal{M}_Y(w_\lambda - \lambda \cdot y) = 0. \end{cases} \tag{6.25}$$

As before, we also introduce the functions χ_i and w_i defined by

$$\begin{cases} \chi_i = \chi_{e_i} \\ w_i = w_{e_i} = y_i - \chi_i, \end{cases} \tag{6.26}$$

for any $i = 1, \dots, N$. They satisfy, respectively,

$$\begin{cases} \text{Find } \chi_i \in W_{\text{per}}(Y) \text{ such that} \\ a^*_Y(\chi_i, v) = \int_Y {}^tAe_i \nabla v \, dy \\ \forall v \in W_{\text{per}}(Y), \end{cases} \tag{6.27}$$

and

$$\begin{cases} \text{Find } w_i \text{ such that } w_i - y_i \in W_{\text{per}}(Y) \text{ and} \\ a^*_Y(w_i, v) = 0 \\ \forall v \in W_{\text{per}}(Y). \end{cases} \tag{6.28}$$

By linearity one has

$$\chi_\lambda = \sum_{i=1}^N \lambda_i \chi_i, \quad w_\lambda = \sum_{i=1}^N \lambda_i w_i, \quad \forall \lambda \in \mathbb{R}^N.$$

The functions $\widehat{\chi}_\lambda$, $\widehat{w}_\lambda$, χ_λ and w_λ play an essential role in the homogenization of problem (6.1). Indeed, the homogenized matrix A_0 from system (5.16) is expressed in terms of these functions. In the following sections we give explicit formulas for its coefficients a^0_{ij}.

6.2 The main convergence result

We have the following result which is now classical and can be found in Sanchez-Palencia (1970b, 1980), Bakhvalov (1974), Bensoussan, Lions and Papanicolaou (1978):

Theorem 6.1. *Let $f \in H^{-1}(\Omega)$ and u^ε be the solution of (6.1) with A^ε defined by (6.2)–(6.4). Then,*

$$\begin{cases} i) & u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega), \\ ii) & A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0 \quad \text{weakly in } (L^2(\Omega))^N, \end{cases}$$

where u^0 is the unique solution in $H_0^1(\Omega)$ of the homogenized problem

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^0 \frac{\partial u^0}{\partial x_j}\right) = f & \text{in } \Omega \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.29}$$

The matrix $A^0 = (a_{ij}^0)_{1 \le i,j \le N}$ is constant, elliptic and given by

$$A^0\lambda = \mathcal{M}_Y(A\nabla\widehat{w}_\lambda), \quad \forall \lambda \in \mathbb{R}^N \tag{6.30}$$

or, equivalently by

$${}^tA^0\lambda = \mathcal{M}_Y({}^tA\nabla w_\lambda), \quad \forall \lambda \in \mathbb{R}^N, \tag{6.31}$$

where $\widehat{w}_\lambda$ and w_λ are defined by (6.12) and (6.23), respectively.

Remark 6.2. Let $f \in H^{-1}(\Omega)$ and u^ε be the solution of (6.1) with A^ε defined by (6.2)–(6.4). As can be seen in the proof of Theorem 6.1 (see Section 8.1), convergence (ii) is deduced from convergence (i). This fact is a particularity of the periodic case due to the explicit computation of the homogenized coefficients. Let us mention that in the general non-periodic case, convergence (ii) is not a consequence of convergence (i) and has to be proved separately. We mentioned this convergence in the statement of the theorem, since it is one of the important homogenization results. ◇

The well-known result stated in Theorem 6.1 can be proved by different methods. We will present in this book two of them, the variational method of oscillating test functions due to Tartar (1977b, 1978) and the two-scale method of Nguetseng (1989) and Allaire (1992). We also present the formal method of asymptotic expansions (known as the multiple-scale method).

Tartar's method is a general one, and is based on the construction of a suitable set of oscillating test functions which allows us to pass to the limit in problem (5.6). This is related to the notion of compensated compactness, which is presented in Chapter 13. In particular, for the case of periodic coefficients (problem (6.1)–(6.4)), the test functions are periodic and are explicitly constructed in terms of $\widehat{w}_\lambda$. By passing to the limit one obtains the homogenized matrix A^0 given by expression (6.31). This method is described in Chapter 8, where we also give further convergence properties as, for instance the convergence of energies and a corrector result.

In Chapter 9 we prove again the convergence result by the two-scale method stated in the periodic framework. This method, taking into account the two scales of the problem, introduces a new notion of convergence, the 'two-scale convergence', tested on functions of the form $\psi(x, x/\varepsilon)$. The convergence in this sense implies the weak convergence.

Before presenting Tartar's method, we will turn our attention in Chapter 7, to the multiple-scale method. This a classical one, widely used in Mechanics and Physics for problems containing several small parameters describing different scalings. It consists in searching the solution as a formal asymptotic expansion, in terms of these parameters. It turns out that this method is particularly well-adapted to the periodic framework, as witnessed by the results obtained in this direction by Sanchez-Palencia (1970a, b), Lions (1978), Bensoussan, Lions, and Papanicolaou (1978).

As already mentioned in Section 5.3, two scales characterize problem (6.1), the macroscopic scale x and the microscopic one x/ε, describing the micro-oscillations. So, one is led to look for a development of u^ε of the form

$$u^\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots = \sum_{i=0}^{\infty} \varepsilon^i u_i\left(x, \frac{x}{\varepsilon}\right),$$

where $u_j = u_j(x, y)$ are Y-periodic in the second variable y. From (6.1), one first obtains that u_0 depends on x only, and then one shows that this u_0 is actually the solution of the homogenized problem (6.29) with A_0 defined by (6.30). The interest in this is that in general it permits us to 'guess' formally the homogenized problem.

Some natural questions arise at this point: how 'far' is u^ε from u_0, i.e. what is the error (in a suitable norm) when replacing u^ε by u_0? What is the estimate when replacing u^ε by a finite sum $\sum_{i=0}^{M} \varepsilon^i u_i(x, x/\varepsilon)$? We give here an error estimate for the case $M = 2$ under some additional regularity assumptions on the data and on the boundary of Ω. We refer the reader to Bensoussan, Lions, and Papanicolaou (1978), Oleinik, Shamaev, and Yosifian (1992) for other details.

The next result will be proved in Chapter 7:

Theorem 6.3. *Let $f \in H^{-1}(\Omega)$ and u^ε be the solution of (6.1) with A^ε defined by (6.2)–(6.4). Then, u^ε admits the following asymptotic expansion:*

$$u^\varepsilon = u_0 - \varepsilon \sum_{k=1}^{N} \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right) \frac{\partial u_0}{\partial x_k} + \varepsilon^2 \sum_{k,\ell=1}^{N} \widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 u_0}{\partial x_k \partial x_\ell} + \cdots$$

where u_0 *is solution of (6.29),* $\widehat{\chi}_k \in W_{\text{per}}(Y)$ *is defined by (6.15) and* $\widehat{\theta}^{k\ell}$ *by*

$$\begin{cases} -\text{div}\,(A(y)\nabla\widehat{\theta}^{k\ell}) = -\,a^0_{k\ell} - \displaystyle\sum_{i,j=1}^{N} \frac{\partial(a_{ij}\delta_{ki}\,\widehat{\chi}_\ell)}{\partial y_i} - \sum_{j=1}^{N} a_{kj}\,\frac{\partial(\widehat{\chi}_\ell - y_\ell)}{\partial y_j} & \text{in } Y \\ \widehat{\theta}^{k\ell} \quad Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{\theta}^{k\ell}) = 0. \end{cases}$$

Moreover, if $f \in C^\infty(\bar{\Omega})$, $\partial\Omega$ *is of class* C^∞ *and, furthermore.*

$$\widehat{\chi}_k,\ \widehat{\theta}^{k\ell} \in W^{1,\infty}(Y), \quad \forall k,\ \ell = 1,\dots,N,$$

then, there exists a constant C *independent of* ε *such that*

$$\left\| u^\varepsilon - \left(u_0 - \varepsilon \sum_{k=1}^{N} \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_k} + \varepsilon^2 \sum_{\ell,k=1}^{N} \widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0}{\partial x_k \partial x_\ell}\right)\right\|_{H^1(\Omega)} \le C\,\varepsilon^{\frac{1}{2}}.$$

Remark 6.4. One of the main interests of Theorem 6.1 and 6.3 is related to numerical computations. Indeed, these results suggest that one could approximate by u^0 the 'physical solution' u^ε satisfying a problem with oscillating coefficients. To find u^0 one has first to solve N problems (6.15) written on Y in order to determine the correctors $\widehat{\chi}_i$ and the matrix A^0, and then to solve (6.29) in order to compute u^0. The numerical interest, when replacing u^ε by u^0, comes from the fact that (6.15) is independent of ε and that (6.29) is a problem with constant coefficients.

On the other hand, let us point out that in order to prove the above error estimate, we are obliged to seek more regularity on the functions $\widehat{\chi}_k$ and $\widehat{\theta}^{k\ell}$, namely the fact that their first derivatives are bounded functions. The question is then: under what hypotheses can we have this property? As matter of fact, this property can be deduced from classical regularity results (see Agmon, Douglis, and Nirenberg, 1959 or Ladyzhenskaya and Uraltseva, 1968, Gilbarg and Trudinger, 1977 and Troianiello, 1987), under strong regularity assumptions on the matrix A, namely at least its continuity. Unfortunately, this is not true in general for composite materials (see Chapter 5 and also Example 2.5). ◊

We end this section by showing that the two expressions in (6.30) and (6.31) define the same matrix A^0.

Proposition 6.5. *Let* B^0 *be the matrix defined by*

$$B^0\lambda = \mathcal{M}_Y({}^tA\nabla w_\lambda), \quad \forall\lambda \in \mathbb{R}^N, \tag{6.32}$$

and A^0 *defined by (6.30). Then* $A^0 = {}^tB^0$, *i.e.*

$${}^tA^0\lambda = \mathcal{M}_Y({}^tA\nabla w_\lambda). \tag{6.33}$$

Proof. To prove (6.33) it is sufficient to show that

$$B^0\lambda\mu = A^0\mu\lambda, \quad \forall\lambda,\mu \in \mathbb{R}^N.$$

From the definition of B^0 and (6.21) one has

$$B^0\lambda\,\mu = \frac{1}{|Y|}\int_Y {}^tA(\lambda - \nabla\chi_\lambda)\,\mu\,dy = \frac{1}{|Y|}\int_Y A\mu\,\lambda\,dy - \frac{1}{|Y|}\int_Y A\mu\,\nabla\chi_\lambda\,dy.$$

By choosing $v = \chi_\lambda$ in (6.6) one has

$$\begin{aligned} B^0\lambda\mu &= \frac{1}{|Y|}\int_Y A\mu\lambda - \frac{1}{|Y|}\int_Y A\nabla\widehat{\chi}_\mu\nabla\chi_\lambda\,dy \\ &= \frac{1}{|Y|}\int_Y A\mu\lambda\,dy - \frac{1}{|Y|}\int_Y {}^tA\nabla\chi_\lambda\nabla\widehat{\chi}_\mu\,dy. \end{aligned}$$

From this relation, and using $v = \widehat{\chi}_\mu$ as test function in (6.18), we finally obtain from (6.9) and (6.30)

$$\begin{aligned} B^0\lambda\mu &= \frac{1}{|Y|}\int_Y A\mu\lambda\,dy - \frac{1}{|Y|}\int_{Y^*} {}^tA\lambda\nabla\widehat{\chi}_\mu\,dy \\ &= \frac{1}{|Y|}\int_Y A\mu\lambda\,dy - \frac{1}{|Y|}\int_{Y^*} A\nabla\widehat{\chi}_\mu\lambda\,dy \\ &= \frac{1}{|Y|}\int_Y A(\mu - \nabla\widehat{\chi}_\mu)\lambda\,dy = \frac{1}{|Y|}\int_Y A\nabla(\mu\cdot y - \chi_\mu)\lambda\,dy = A^0\mu\,\lambda, \end{aligned}$$

which ends the proof. □

From this proposition, it is easily seen that the following interesting result holds:

Corollary 6.6. *Let A be a matrix satisfying (6.4) and A^0 be the corresponding homogenized matrix given by Theorem 6.1. Then, the homogenized matrix $({}^tA)^0$ corresponding to ${}^tA$, is given by*

$$({}^tA)^0 = {}^t(A^0).$$

In Section 6.3 below we prove that the homogenized matrix is elliptic which implies, via the Lax–Milgram theorem, the existence and uniqueness of the solution u^0 of problem (6.29). Observe that the uniqueness of u^0 provides convergence (5.9) for the whole sequence $\{u^\varepsilon\}$ and not only for the subsequence $\{u^{\varepsilon'}\}$ introduced in (5.9).

6.3 The ellipticity of the homogenized matrix

In this Section we give some explicit formulas for the coefficients a^0_{ij} of the matrix A^0 and we prove that it is elliptic.

Observe that, from (6.30) and (6.14) one has immediately

$$A^0 e_j = \mathcal{M}_Y(A\nabla\widehat{w}_j), \quad \forall j = 1,\dots,N. \tag{6.34}$$

Since

$$(A\nabla\widehat{w}_j)_i = \sum_{k=1}^N a_{ik}\frac{\partial\widehat{w}_j}{\partial y_k},$$

from (6.14) and (6.34) one has

$$a^0_{ij} = \mathcal{M}_Y\left(\sum_{k=1}^N a_{ik}\frac{\partial\widehat{w}_j}{\partial y_k}\right) = \mathcal{M}_Y\left(a_{ij} - \sum_{k=1}^N a_{ik}\frac{\partial\widehat{\chi}_j}{\partial y_k}\right), \quad \forall i,j = 1,\dots,N.$$

Therefore,

$$\begin{cases} a^0_{ij} = \mathcal{M}_Y(a_{ij}) - \mathcal{M}_Y\left(\displaystyle\sum_{k=1}^N a_{ik}\frac{\partial\widehat{\chi}_j}{\partial y_k}\right) \\ \quad = \dfrac{1}{|Y|}\displaystyle\int_Y a_{ij}\,dy - \dfrac{1}{|Y|}\sum_{k=1}^N\int_Y a_{ik}\frac{\partial\widehat{\chi}_j}{\partial y_k}\,dy, \quad \forall i,j = 1,\dots,N. \end{cases} \tag{6.35}$$

Similarly, using (6.31) one has a second expression for a^0_{ij}, namely

$$\begin{cases} a^0_{ij} = \mathcal{M}_Y\left(\displaystyle\sum_{k=1}^N a_{kj}\frac{\partial w_i}{\partial y_k}\right) = \mathcal{M}_Y(a_{ij}) - \mathcal{M}_Y\left(\displaystyle\sum_{k=1}^N a_{kj}\frac{\partial\chi_i}{\partial y_k}\right) \\ \quad = \dfrac{1}{|Y|}\displaystyle\int_Y a_{ij}\,dy - \dfrac{1}{|Y|}\sum_{k=1}^N\int_Y a_{kj}\frac{\partial\chi_i}{\partial y_k}\,dy, \quad \forall i,j = 1,\dots,N. \end{cases} \tag{6.36}$$

Remark 6.7. As noticed in Section 5.1, in general A^0 is different from the weak limit $\mathcal{M}_Y(A)$ (usually called 'mixture low') of A^ε. The interest of formulas (6.35) and (6.36) is that they show that A^0 is actually obtained by adding to this mean value a corrector term, expressed by means of gradients of the functions $\widehat{\chi}_j$ (or χ_j). This is why the functions $\widehat{\chi}_j$, as well as χ_j, are called correctors.

Observe also that, as mentioned for the one-dimensional case in Remark 5.6 and in the case of layered materials in Remark 5.11, the homogenized coefficients a^0_{ij} do not depend on the data f and Ω of the problem. ◇

Proposition 6.8. *Let A^0 be defined by (6.30) and $\widehat{w}_i$ by (6.16), for $i = 1,\dots,N$. Then*

$$a^0_{ij} = \frac{1}{|Y|}\sum_{k,\ell=1}^N\int_Y a_{k\ell}\frac{\partial\widehat{w}_j}{\partial y_\ell}\frac{\partial\widehat{w}_i}{\partial y_k}\,dy, \quad \forall i,j = 1,\dots,N. \tag{6.37}$$

Proof. Recall that $\widehat{\chi}_j$ is solution of the following problem (see (6.15)):

$$a_{Y}(\widehat{\chi}_j, v) = \int_Y Ae_j \nabla v \, dy, \quad \forall v \in W_{\text{per}}(Y).$$

Choosing $v = \widehat{\chi}_i$ as test function one has

$$\int_Y A\nabla\widehat{\chi}_j \nabla\widehat{\chi}_i \, dy = \int_Y Ae_j \nabla\widehat{\chi}_i \, dy,$$

i.e.

$$\sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial \widehat{\chi}_j}{\partial y_\ell} \frac{\partial \widehat{\chi}_i}{\partial y_k} \, dy = \sum_{\ell-1}^{N} \int_Y a_{kj} \frac{\partial \widehat{\chi}_i}{\partial y_k} \, dy = \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial y_j}{\partial y_\ell} \frac{\partial \widehat{\chi}_i}{\partial y_k} \, dy. \quad (6.38)$$

Hence

$$\frac{1}{|Y|} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial(y_j - \widehat{\chi}_j)}{\partial y_\ell} \frac{\partial \widehat{\chi}_i}{\partial y_k} \, dy = 0. \quad (6.39)$$

On the other hand, since

$$\int_Y a_{ij} \, dy = \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial y_j}{\partial y_\ell} \frac{\partial y_i}{\partial y_k} \, dy,$$

$$\sum_{\ell=1}^{N} \int_Y a_{i\ell} \frac{\partial \widehat{\chi}_j}{\partial y_\ell} \, dy = \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial \widehat{\chi}_j}{\partial y_\ell} \frac{\partial y_i}{\partial y_k} \, dy,$$

formula (6.35) can be written as follows:

$$a^0_{ij} = \frac{1}{|Y|} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial(y_j - \widehat{\chi}_j)}{\partial y_\ell} \frac{\partial y_i}{\partial y_k} \, dy, \quad \forall i, j = 1, \ldots, N. \quad (6.40)$$

Subtracting (6.39) from (6.40) gives

$$a^0_{ij} = \frac{1}{|Y|} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial(y_j - \widehat{\chi}_j)}{\partial y_\ell} \frac{\partial(y_i - \widehat{\chi}_i)}{\partial y_k} \, dy, \quad \forall i, j = 1, \ldots, N, \quad (6.41)$$

which, together with (6.14), ends the proof of Proposition 6.8. □

By the same arguments, starting with formula (6.35) instead of (6.36), the result stated in the next proposition, is straightforward.

Proposition 6.9. *Let A^0 be defined by (6.30) and w_i by (6.28), for $i = 1, \ldots, N$. Then*

$$a^0_{ij} = \frac{1}{|Y|} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial w_i}{\partial y_k} \frac{\partial w_j}{\partial y_\ell} \, dy = \frac{1}{|Y|} a^*_Y(w_i, w_j) = \frac{1}{|Y|} a^*_Y(y_i - \chi_i, y_j - \chi_j), \quad (6.42)$$

for all $i, j = 1, \ldots, N$, where a^*_Y is defined by (6.19).

The following result is an immediate consequence of formula (6.42):

Corollary 6.10. *Suppose that the matrix A is symmetric. Then A^0 is also symmetric.*

Remark 6.11. Consider the particular case of the layered materials treated in Chapter 5. The explicit expressions of the coefficients a^0_{ij} given by Theorem 5.12 show that if A is diagonal, the matrix A^0 is diagonal too. It is easily seen that in general case the matrix A^0 is not diagonal even if A is diagonal. Indeed, when the coefficients depend on all the variables, if $a_{ij} = 0$ for $i \neq j$, from (6.36) one has

$$a^0_{ij} = -\frac{1}{|Y|}\int_Y a_{jj}\frac{\partial \chi_i}{\partial y_j}\,dy \neq 0, \quad \forall\, i,j = 1,\dots,N,\ i \neq j,$$

since, by definition, χ_i depends on all the variables y_i. ◊

We are now in position to prove the ellipticity of the homogenized matrix A^0.

Proposition 6.12. *Let A^0 be the matrix defined by (6.30). There exists a positive number α_0 such that*

$$\sum_{i,j=1}^{N} a^0_{ij}\xi_i\xi_j \geq \alpha_0|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^N. \tag{6.43}$$

Proof. Let $\xi \in \mathbb{R}^N$. Then, from (6.41) it follows that

$$\sum_{i,j=1}^{N} a^0_{ij}\xi_i\xi_j = \frac{1}{|Y|}\sum_{i,j=1}^{N}\sum_{k,\ell=1}^{N}\int_{Y} a_{k\ell}\,\xi_i\frac{\partial(y_i - \widehat{\chi}_i)}{\partial y_k}\,\xi_j\frac{\partial(y_j - \widehat{\chi}_j)}{\partial y_\ell}\,dy.$$

Setting $\zeta = \sum_{i=1}^{N}\xi_i(y_i - \widehat{\chi}_i)$ and using the ellipticity of A (assumption (6.4)), we get

$$\sum_{i,j=1}^{N} a^0_{ij}\xi_i\xi_j \geq \frac{\alpha}{|Y|}\int_{Y}|\nabla\zeta|^2\,dy \geq 0, \quad \text{for any } \xi \in \mathbb{R}^N. \tag{6.44}$$

Let us show that this inequality implies that

$$\sum_{i,j=1}^{N} a^0_{ij}\xi_i\xi_j > 0, \quad \text{for any } \xi \in \mathbb{R}^N,\ \xi \neq 0.$$

Indeed, if this were not true, from (6.44) one would have some $\xi \neq 0$ such that

$$|\nabla\zeta| = 0.$$

This means that

$$\zeta = \sum_{i=1}^{N}\xi_i(y_i - \widehat{\chi}_i) = \text{constant}.$$

i.e.

$$\sum_{i=1}^{N} \xi_i \, y_i = \sum_{i=1}^{N} \xi_i \, \widehat{\chi}_i + \text{constant},$$

and this is impossible since the right-hand side function is periodic by definition and $\xi \neq 0$.

Arguing as at the end of the proof of Theorem 5.10, one deduces inequality (6.43). □

Let us show another interesting characterization of the homogenized matrix. From (6.35) and (6.36), one can write that

$$A^0 = \mathcal{M}_Y(A) - \mathcal{M}_Y(X^0), \tag{6.45}$$

where the $N \times N$ matrix $X^0 = (X^0_{ij})_{1\leq i,j\leq N}$ is defined by

$$X^0_{ij} = \sum_{k=1}^{N} a_{ik} \frac{\partial \widehat{\chi}_j}{\partial y_k} = \sum_{k=1}^{N} a_{kj} \frac{\partial \chi_i}{\partial y_k}. \tag{6.46}$$

Formula (6.45) gives A^0 as the difference of two constant matrices. Obviously, the matrix $\mathcal{M}_Y(A)$ is elliptic. Observe that the matrix $\mathcal{M}_Y(X^0)$ is positive. Indeed, one has

Proposition 6.13. *Let X^0 be defined by (6.46). Then*

$$\sum_{i,j=1}^{N} \mathcal{M}_Y(X^0_{ij})\xi_i\xi_j \geq 0, \qquad \text{for any } \xi \in \mathbb{R}^N. \tag{6.47}$$

Proof. Notice that from (6.38), it follows that

$$\mathcal{M}_Y(X^0_{ij}) = \frac{1}{|Y|} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \frac{\partial \widehat{\chi}_i}{\partial y_k} \frac{\partial \widehat{\chi}_j}{\partial y_\ell} \, dy.$$

Hence, for any $\xi \in \mathbb{R}^N$ one has

$$\sum_{i,j=1}^{N} \mathcal{M}_Y(X^0_{ij})\xi_i\xi_j = \frac{1}{|Y|} \sum_{i,j=1}^{N} \sum_{k,\ell=1}^{N} \int_Y a_{k\ell} \, \xi_i \frac{\partial \widehat{\chi}_i}{\partial y_k} \, \xi_j \frac{\partial \widehat{\chi}_j}{\partial y_\ell} \, dy.$$

We argue as in the proof of Proposition 6.12. Setting $\zeta = \sum_{i=1}^{N} \xi_i \widehat{\chi}_i$ and using the ellipticity of A (assumption (6.4)), we get

$$\sum_{i,j=1}^{N} \mathcal{M}_Y(X^0_{ij})\xi_i\xi_j \geq \frac{\alpha}{|Y|} \int_Y |\nabla \zeta|^2 \, dy \geq 0, \quad \text{for any } \xi \in \mathbb{R}^N,$$

which proves (6.47). □

6.4 Other formulas for the homogenized matrix

Formulas (6.35) and (6.36) give the homogenized coefficients in terms of N-dimensional integrals over the domain Y. One can get rid of one integration, and hence express a^0_{ij} by integrals on a $N-1$ dimensional domain, by using the following result due to Sanchez-Palencia (1980, pp. 137–140):

Proposition 6.14. *Let $\theta = (\theta_1, \dots, \theta_N) \in L^2(Y)$ be a Y-periodic function satisfying*

$$\operatorname{div} \theta = 0 \quad \text{in } Y. \tag{6.48}$$

Set

$$Y_i =]0, \ell_1[\times \cdots \times]0, \ell_{i-1}[\times]0, \ell_{i+1}[\times \cdots \times [0, \ell_N[.$$

Then $\theta_i(y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_N)$ belongs to $H^{-1/2}(Y_i)$, for any $i = 1, \dots, N$. Moreover, one has

$$\mathcal{M}_Y(\theta_i) = \frac{1}{|Y_i|} \langle \theta_i(y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_N), 1 \rangle_{H^{-1/2}(Y_i), H^{1/2}(Y_i)}, \tag{6.49}$$

where $Y_i =]0, \ell_1[\times \cdots \times]0, \ell_{i-1}[\times]0, \ell_{i+1}[\times \cdots \times [0, \ell_N[$.

Proof. Let $\tau \in]0, \ell_i[$. Introduce the set

$$Y_i^\tau = \{ y \in Y \,|\, 0 \le y_i < \tau \}.$$

Observe that by definition, $\theta \in H(\Omega, \operatorname{div})$. From equation (6.48) and Proposition 3.47 it follows that

$$\int_{Y_i^\tau} \theta \, \nabla \varphi \, dy + \langle \theta \cdot n, \varphi \rangle_{H^{-1/2}(\partial Y_i^\tau), H^{1/2}(\partial Y_i^\tau)} = 0, \quad \forall \varphi \in H^1(Y). \tag{6.50}$$

Choosing in particular $\varphi = 1$ in this identity, one has

$$\langle \theta \cdot n, 1 \rangle_{H^{-1/2}(\partial Y_i^\tau), H^{1/2}(\partial Y_i^\tau)} = 0.$$

Observe now that $n = -e_i$ on $Y_i \cap \{y_i = 0\}$ and $n = e_i$ on $Y_i \cap \{y_i = \tau\}$ where $\{e_1, \dots, e_N\}$ is the canonical basis of $\mathbb{R}^N$. Therefore,

$$\begin{aligned} &\langle \theta_i(y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots y_N), 1 \rangle_{H^{-1/2}(Y_i \cap \{y_i = 0\}), H^{1/2}(Y_i \cap \{y_i = 0\})} \\ &= \langle \theta_i(y_1, \dots, y_{i-1}, \tau, y_{i+1}, \dots, y_N), 1 \rangle_{H^{-1/2}(Y_i \cap \{y_i = \tau\}), H^{1/2}(Y_i \cap \{y_i = \tau\})}, \end{aligned}$$

for every $\tau \in]0, \ell_i[$. Integrating (6.50) with respect to τ over $]0, \ell_i[$ and using a density argument, one deduces (6.49).

Let us also give a direct proof in the case where θ is a smooth function (for instance in $L^2(Y_i)$) so that we can integrate (6.48) over Y_i^τ. We have, due to the periodicity

$$0 = \int_{Y_i^\tau} \operatorname{div} \theta \, dy = \int_{Y_i \cap \{y_i = 0\}} \theta \cdot n \, ds + \int_{Y_i \cap \{y_i = \tau\}} \theta \cdot n \, ds.$$

Hence, by using again the fact that $n = -e_i$ on $Y_i \cap \{y_i = 0\}$ and $n = e_i$ on $Y_i \cap \{y_i = \tau\}$, one gets

$$0 = \int_{Y_i \cap \{y_i=0\}} \theta_i \, ds - \int_{Y_i \cap \{y_i=\tau\}} \theta_i \, ds$$

which leads to

$$\begin{aligned}&\int_{Y_i} \theta_i(y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_N) \, dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_N \\ &= \int_{Y_i} \theta_i(y_1, \dots, y_{i-1}, \tau, y_{i+1} \dots, y_N) \, dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_N.\end{aligned}$$

Integrating with respect to τ over $(0, \ell_i)$, one obtains

$$\ell_i \int_{Y_i} \theta_i(y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_N) \, dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_N = \int_Y \theta_i(y) \, dy.$$

Multiply this identity by $1/|Y|$. Since $\ell_i |Y_i| = |Y|$, one finally has

$$\mathcal{M}_Y(\theta_i) = \frac{1}{|Y_i|} \int_{Y_i} \theta_i(y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_N) \, dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_N.$$

This is exactly (6.49) when θ is in $L^2(Y_i)$. □

The following corollary is straightforward by using (6.14) and (6.35) (respectively, (6.26) and (6.36)). Indeed, applying Proposition 6.14 to $\theta = A\nabla\widehat{w}_j$ and to $\theta = {}^tA\nabla w_j$, one has

Corollary 6.15. *Suppose that* $\sum_{h=1}^{N} a_{ih} \frac{\partial \hat{w}_j}{\partial y_h} \in L^2(Y_i)$. *Then, for any* $i, j = 1, \dots, N$, *one has*

$$a^0_{ij} = \frac{1}{|Y_i|} \int_{Y_i} \left[a_{ij} - \sum_{h=1}^{N} a_{ih} \frac{\partial \widehat{\chi}_j}{\partial y_h} \right]_{y_i=0} dy',$$

where $dy' = dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_N$.

If, similarly, $\sum_{h=1}^{N} a_{hj} \frac{\partial w_i}{\partial y_h} \in L^2(Y_i)$, *then for all* $i, j = 1, \dots, N$, *one has*

$$a^0_{ij} = \frac{1}{|Y_i|} \int_{Y_i} \left[a_{ij} - \sum_{h=1}^{N} a_{hj} \frac{\partial \chi_i}{\partial y_h} \right]_{y_i=0} dy'.$$

6.5 The one and two-dimensional cases

In this section we want to show how the homogenized problems (5.27) and (5.39), obtained in Chapter 5 for the one-dimensional case and the case of layered materials, can be written as a particular case of Theorem 6.1.

Obviously, this is not necessary from the mathematical point of view, since (5.27) and (5.39) have been rigorously proved in Theorem 5.5 and 5.10 respectively. As a matter of fact, the homogenized coefficients in these theorems are explicit (algebraic) formulas in terms of the coefficients a_{ij} of A. It seems then natural to see what are the correctors in these cases in order to derive (5.27) and (5.39) from the general definition of A^0.

Proposition 6.16. *Under the assumptions of Theorem 5.5, one has*

$$\frac{1}{\mathcal{M}_{]0,\ell_1[}\left(\dfrac{1}{a}\right)} = \mathcal{M}_{]0,\ell_1[}\left(a - a\,\frac{d\widehat{\chi}}{dy}\right) \tag{6.51}$$

where $\widehat{\chi}$ is the solution of problem (6.15) written for $N = 1$, i.e.

$$\begin{cases} -\dfrac{d}{dy}\left(a(y)\dfrac{d\widehat{\chi}}{dy}\right) = -\dfrac{d}{dy}(a(y)) \quad \text{in }]0,\ell_1[\\ \widehat{\chi} \quad \ell_1\text{-periodic} \\ \mathcal{M}_{]0,\ell_1[}(\chi) = 0, \end{cases} \tag{6.52}$$

and is given by

$$\widehat{\chi}(y) = -\frac{1}{\mathcal{M}_{]0,\ell_1]}\left(\dfrac{1}{a}\right)} \int_0^y \frac{1}{a(t)}\,dt + y + C_0, \tag{6.53}$$

where C_0 is the constant for which $\mathcal{M}_{]0,\ell_1[}(\widehat{\chi}) = 0$.

Proof. From (6.52), one immediately has $\widehat{\chi}$ verifies

$$a(y)\frac{d\widehat{\chi}}{dy} = a(y) + C,$$

where C is a constant to be determined. Hence

$$\widehat{\chi}(y) = C\int_0^y \frac{1}{a(t)}\,dt + y + C_0.$$

where C_0 is also a constant to be determined. The periodicity condition $\widehat{\chi}(0) = \widehat{\chi}(\ell_1)$, i.e.

$$0 = C\int_0^{\ell_1} \frac{1}{a(t)}\,dt + \ell_1$$

gives the value of C, so that the solution $\widehat{\chi}$ of (6.52) is given by (6.53) with C_0 determined in such a way to have $\mathcal{M}_{]0,\ell_1[}(\widehat{\chi}) = 0$. Then, (6.51) is straightforward. □

Remark 6.17. Formula (6.51) allows to write the explicit coefficient in problem (5.27) under the general form (6.35). The interest of the approach of Chapter 5 is that problem (5.27) has been obtained directly, without any auxiliary periodic partial differential equation. ◇

In the same spirit, we can show that the explicit limit coefficients for the case of layered materials, given in Theorem 5.10, can be written under the general form (6.35) too.

Proposition 6.18. *Suppose that the hypotheses of Theorem 5.10 are fulfilled and let $\breve{A}$ be the limit matrix given herein. Then*

$$\breve{A} = A^0,$$

where A^0 is the homogenized matrix given (6.35). That is

$$\breve{a}_{ij} = \mathcal{M}_Y(a_{ij}) - \mathcal{M}_Y\left(\sum_{k=1}^{2} a_{ik}\frac{\partial \widehat{\chi}_j}{\partial y_k}\right).$$

The functions $\widehat{\chi}_1(y) = \widehat{\chi}_1(y_1)$ and $\widehat{\chi}_2(y) = \widehat{\chi}_2(y_1)$ are the solutions of problem (6.15) written for $N = 2$ and are respectively given by

$$\begin{cases} \widehat{\chi}_1(y_1) = -\dfrac{1}{\mathcal{M}_{]0,\ell_1[}\left(\dfrac{1}{a_{11}}\right)}\displaystyle\int_0^{y_1}\frac{1}{a_{11}(t)}\,dt + y_1 + C_1 \\ \widehat{\chi}_2(y_1) = \displaystyle\int_0^{y_1}\frac{a_{12}(t)}{a_{11}(t)}\,dt - \dfrac{1}{\mathcal{M}_{]0,\ell_1[}\left(\dfrac{1}{a_{11}}\right)}\mathcal{M}_{]0,\ell_1[}\left(\dfrac{a_{12}}{a_{11}}\right)\displaystyle\int_0^{y_1}\frac{1}{a_{11}(t)}\,dt + C_2, \end{cases} \tag{6.54}$$

where C_1 and C_2 are respectively, the constants for which $\mathcal{M}_Y(\widehat{\chi}_1) = 0$ and $\mathcal{M}_Y(\widehat{\chi}_2) = 0$.

Proof. Let us first compute $\widehat{\chi}_1$ which is solution of (6.15) for $i = 1$, i.e.

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2}\frac{\partial}{\partial y_i}\left(a_{ij}(y_1)\frac{\partial \widehat{\chi}_1}{\partial y_j}\right) = -\frac{\partial a_{11}}{\partial y_1} \quad \text{in } Y \\ \widehat{\chi}_1 \quad Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{\chi}_1) = 0. \end{cases}$$

Here the coefficients and the right-hand side of the equation are independent of the second variable. Then it is natural to look for $\widehat{\chi}_1$ depending on y_1 only. The same computation as that made in the proof of Proposition 6.16 leads to the first formula in (6.54). This together with (6.35) gives

$$a_{11}^0 = \mathcal{M}_Y(a_{11}) - \mathcal{M}_Y\left(a_{11}\frac{\partial \widehat{\chi}_1}{\partial y_1}\right) = \frac{1}{\mathcal{M}_{]0,\ell_1[}\left(\dfrac{1}{a_{11}}\right)} = \breve{a}_{11}.$$

Again by using (6.35) and (6.54), one has

$$a_{21}^0 = \mathcal{M}_Y(a_{21}) - \mathcal{M}_Y\left(a_{21}\frac{\partial\widehat{\chi}_1}{\partial y_1}\right) = a_{11}^0\, \mathcal{M}_{]0,\ell_1[}\left(\frac{a_{21}}{a_{11}}\right) = \breve{a}_{21}.$$

Let now compute $\widehat{\chi}_2$ which is solution of (6.15) for $i = 2$, i.e.

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial y_i}\left(a_{ij}(y_1)\frac{\partial\widehat{\chi}_2}{\partial y_j}\right) = -\frac{\partial a_{12}}{\partial y_1} \quad \text{in } Y \\ \widehat{\chi}_2 \quad Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{\chi}_2) = 0. \end{cases} \tag{6.55}$$

Again, it is natural to look for $\widehat{\chi}_2$ depending on y_1 only. Then, from (6.55), one has that

$$a_{11}(y_1)\frac{\partial\widehat{\chi}_2(y_1)}{\partial y_1} = a_{12}(y_1) + C.$$

with C a constant to be determined. Integrating once, one gets

$$\widehat{\chi}_2(y_1) = \int_0^{y_1} \frac{a_{12}(t)}{a_{11}(t)}\, dt + C\int_0^{y_1} \frac{1}{a_{11}(t)}\, dt + C_2,$$

where C_2 is also a constant to be determined. The periodicity condition which reduces to $\widehat{\chi}_2(0) = \widehat{\chi}_2(\ell_1)$ since $\widehat{\chi}_2$ depends on y_1 only, implies that

$$0 = \int_0^{\ell_1} \frac{a_{12}(t)}{a_{11}(t)}\, dt + C\int_0^{\ell_1} \frac{1}{a_{11}(t)}\, dt.$$

This gives the value of C, so that the solution $\widehat{\chi}_2$ of (6.55) is given by the second formula of (6.54) with C_2 determined in order to have $\mathcal{M}_{]0,\ell_1[}(\widehat{\chi}_2) = 0$.

One easily verifies that

$$a_{12}^0 = \mathcal{M}_Y(a_{12}) - \mathcal{M}_Y\left(a_{11}\frac{\partial\widehat{\chi}_2}{\partial y_1}\right) = a_{11}^0\, \mathcal{M}_{]0,\ell_1[}\left(\frac{a_{12}}{a_{11}}\right) = \breve{a}_{12},$$

and also

$$\begin{aligned} a_{22}^0 &= \mathcal{M}_Y(a_{22}) - \mathcal{M}_Y\left(a_{21}\frac{\partial\widehat{\chi}_2}{\partial y_1}\right) \\ &= a_{11}^0\mathcal{M}_{]0,\ell_1[}\left(\frac{a_{12}}{a_{11}}\right)\mathcal{M}_{]0,\ell_1[}\left(\frac{a_{21}}{a_{11}}\right) + \mathcal{M}_{]0,\ell_1[}\left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right) = \breve{a}_{22}. \end{aligned}$$

The proof is complete. □

Remark 6.19. Proposition 6.18 allows to rewrite (6.54) in the form

$$\begin{cases} \widehat{\chi}_1(y_1) = -a_{11}^0\displaystyle\int_0^{y_1} \frac{1}{a_{11}(t)}\, dt + y_1 + C_1 \\ \widehat{\chi}_2(y_1) = \displaystyle\int_0^{y_1} \frac{a_{12}(t)}{a_{11}(t)}\, dt - a_{12}^0\int_0^{y_1} \frac{1}{a_{11}(t)}\, dt + C_2. \end{cases} \tag{6.56}$$

◊

7

The multiple-scale method

In this chapter we apply the multiple-scale method to the study of problem (6.1). The method is presented in Section 7.1 and a formal asymptotic expansion for u^ε is obtained. Thc goal of Section 7.2 is to prove the error estimate stated in Theorem 6.3.

Recall that u^ε is the solution of

$$\begin{cases} \mathcal{A}_\varepsilon u^\varepsilon = f \text{ in } \Omega \\ u^\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{7.1}$$

where $\mathcal{A}_\varepsilon$ (see (5.4)) is defined by

$$\mathcal{A}_\varepsilon = -\text{div }(A^\varepsilon \nabla) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon \frac{\partial}{\partial x_j}\right), \tag{7.2}$$

with

$$a_{ij}^\varepsilon(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. in } \mathbb{R}^N, \quad \forall i,j = 1,\dots,N \tag{7.3}$$

and

$$\begin{cases} a_{ij} \quad Y\text{-periodic}, \quad \forall i,j = 1,\dots,N, \\ A = (a_{ij})_{1\leq i,j\leq N} \in M(\alpha,\beta,Y), \end{cases} \tag{7.4}$$

with $\alpha,\beta \in \mathbb{R}$, such that $0 < \alpha < \beta$ and $M(\alpha,\beta,Y)$ given by Definition 4.11. Here, as before, Y denotes the reference cell defined by

$$Y =]0,\ell_1[\times \cdots \times]0,\ell_N[,$$

where $\ell_1,\dots,\ell_N$ are given positive numbers.

7.1 The asymptotic expansion

As mentioned in Chapter 6, two scales describe the model: the variable x is the 'macroscopic' one, while x/ε describes the 'microscopic' one. Indeed, if $x \in \Omega$, by the definition of Y, there exists $k \in \mathbb{Z}^N$ such that $x/\varepsilon = (k_\ell + y)$ with $y \in Y$ and where $k_\ell = (k_1\ell_1,\dots,k_N\ell_N)$. Hence, x gives the position of a point in the domain Ω while y gives its position in the reference cell Y.

This suggests looking for a formal asymptotic expansion of the form

$$u^\varepsilon(x) = u_0\left(x,\frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x,\frac{x}{\varepsilon}\right) + \cdots \tag{7.5}$$

with $u_j(x,y)$ for $j = 1, 2, \ldots$, such that

$$\begin{cases} u_j(x,y) \text{ is defined for } x \in \Omega \text{ and } y \in Y \\ u_j(\cdot, y) \text{ is } Y\text{-periodic.} \end{cases}$$

Let $\Psi = \Psi(x,y)$ be a function depending on two variables of $\mathbb{R}^N$ and denote by Ψ_ε the following:

$$\Psi_\varepsilon(x) = \Psi\left(x, \frac{x}{\varepsilon}\right),$$

which depends only on one variable. Notice that

$$\frac{\partial \Psi_\varepsilon}{\partial x_i}(x) = \frac{1}{\varepsilon}\frac{\partial \Psi}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right) + \frac{\partial \Psi}{\partial x_i}\left(x, \frac{x}{\varepsilon}\right).$$

Consequently, from (7.2) one can write $\mathcal{A}_\varepsilon \Psi_\varepsilon$ as follows:

$$\mathcal{A}_\varepsilon \Psi_\varepsilon(x) = \left[(\varepsilon^{-2}\mathcal{A}_0 + \varepsilon^{-1}\mathcal{A}_1 + \mathcal{A}_2)\Psi\right]\left(x, \frac{x}{\varepsilon}\right) \tag{7.6}$$

where

$$\begin{cases} \mathcal{A}_0 = -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) \\ \mathcal{A}_1 = -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) - \sum_{i,j=1}^{N} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) \\ \mathcal{A}_2 = -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right). \end{cases} \tag{7.7}$$

Using (7.5)–(7.7) into (7.1) and equating the power-like terms of ε, we have to solve the following infinite system of equations:

$$\begin{cases} \mathcal{A}_0 u_0 = 0 \quad \text{in } Y \\ u_0 \quad Y\text{-periodic in } y, \end{cases} \tag{7.8}$$

$$\begin{cases} \mathcal{A}_0 u_1 = -\mathcal{A}_1 u_0 \quad \text{in } Y \\ u_1 \quad Y\text{-periodic in } y, \end{cases} \tag{7.9}$$

$$\begin{cases} \mathcal{A}_0 u_2 = f - \mathcal{A}_1 u_1 - \mathcal{A}_2 u_0 \quad \text{in } Y \\ u_2 \quad Y\text{-periodic in } y, \end{cases} \tag{7.10}$$

and

$$\begin{cases} \mathcal{A}_0 u_{s+2} = -\mathcal{A}_1 u_{s+1} - \mathcal{A}_2 u_s \quad \text{in } Y \\ u_{s+2} \quad Y\text{-periodic in } y, \end{cases} \tag{7.11}$$

for $s \geq 1$.

Remark 7.1. First of all, notice the special structure of this system. The unknowns u_j can be determined successively. Indeed, the first equation (7.8) contains only the unknown u_0. If u_0 is known, the second equation (7.9) allows us to determine u_1 in terms of u_0. Similarly, the third equation (7.10) determines u_2 in terms of u_0 and u_1, and so on. ◊

Remark 7.2. Note also that the operator $\mathcal{A}_0$ which appears in each equation has the same form as $\mathcal{A}_\varepsilon$ with x/ε replaced by y. It is a second order operator in y, and in each equation above, x plays the role of a parameter. ◊

Remark 7.3. All the equations above are of the form (4.61). We saw in Section 4.7 that problem (4.61) can be understood either in the sense of variational formulation (4.62) (and in this case its solution, given by Theorem 4.26, is a class of equivalence) or in the sense of variational formulation (4.65). In this last case, its solution, given by Theorem 4.27, is a function with zero mean value. In the sequel we will use both of these formulations. ◊

Let us now solve successively systems (7.9)–(7.10), by applying the results contained in Section 4.7.

We begin with system (7.8) whose variational formulation is (see (4.62))

$$\begin{cases} \text{Find } \dot{u}_0 \in \mathcal{W}_{\text{per}}(Y) \text{ such that} \\ \dot{a}_Y(\dot{u}_0, \dot{v}) = 0 \\ \forall \dot{v} \in \mathcal{W}_{\text{per}}(Y), \end{cases} \tag{7.12}$$

where

$$\dot{a}_Y(\dot{u}, \dot{v}) = \int_Y A\nabla u \, \nabla v \, dy, \quad \forall u \in \dot{u}, \forall v \in \dot{v}, \ \forall \dot{u}, \forall \dot{v} \in \mathcal{W}_{\text{per}}(Y), \tag{7.13}$$

and

$$\mathcal{W}_{\text{per}}(Y) = H^1_{\text{per}}(Y)/\mathbb{R}.$$

Recall that $\mathcal{W}_{\text{per}}(Y)$ is the space of classes of equivalence with respect to the relation

$$u \simeq v \iff u - v \text{ is a constant}, \quad \forall u, v \in H^1_{\text{per}}(Y),$$

introduced in Definition 3.51 and that $\dot{v}$ denotes the class of equivalence of v.

We can apply Theorem 4.26 to problem (7.12) to obtain

$$\dot{u}_0 = \dot{0}, \quad \text{in } \mathcal{W}_{\text{per}}(Y)$$

as the unique solution. Recalling that, by definition $u_0 = u_0(x, y)$, this implies that the solution of (7.12) is independent of y, so that

$$u_0(x, y) = u_0(x), \quad \text{for any } u_0 \in \dot{u}_0. \tag{7.14}$$

Remark 7.4. In asymptotic expansion (7.5), the first element u_0 is a priori an oscillating function, since it depends on x/ε. Relation (7.13) shows that u_0 actually depends only on x, it does not depend on ε and hence, does not oscillate 'rapidly' with x/ε. This is why we now expect u_0 to be the 'homogenized solution'. It remains to find if there is an equation in Ω satisfied by u_0, in which case we would have found the 'homogenized equation' too. ◊

We now turn to equation (7.9). Using (7.7) and (7.14), this equation can be rewritten as

$$\begin{cases} \mathcal{A}_0 u_1 = -\displaystyle\sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial y_i}\frac{\partial u_0}{\partial x_j} & \text{in } Y \\ u_1 \quad Y\text{-periodic in } y. \end{cases} \tag{7.15}$$

Its variational formulation is

$$\begin{cases} \text{Find } \dot{u}_1 \in \mathcal{W}_{\text{per}}(Y) \text{ such that} \\ \dot{a}_Y(\dot{u}_1, \dot{v}) = \langle F, \dot{v}\rangle_{(\mathcal{W}_{\text{per}}(Y))', \mathcal{W}_{\text{per}}(Y)} \\ \forall \dot{v} \in \mathcal{W}_{\text{per}}(Y), \end{cases} \tag{7.16}$$

where $\dot{a}_Y$ is given by (7.13) and F is defined by

$$\langle F, \dot{\psi}\rangle_{(\mathcal{W}_{\text{per}}(Y))', \mathcal{W}_{\text{per}}(Y)} = \sum_{i,j=1}^{N} \frac{\partial u_0}{\partial x_j}\int_Y a_{ij}(y)\frac{\partial \psi}{\partial y_i}\,dy. \quad \forall \psi \in \dot{\psi},\ \dot{\psi} \in \mathcal{W}_{\text{per}}(Y). \tag{7.17}$$

Observe that if $\psi_1, \psi_2 \in \dot{\psi}$ then

$$\frac{\partial \psi_1}{\partial y_i} = \frac{\partial \psi_2}{\partial y_i}$$

and so

$$\langle F, \psi_1\rangle_{(H^1_{\text{per}}(Y))', H^1_{\text{per}}(Y)} = \langle F, \psi_2\rangle_{(H^1_{\text{per}}(Y))', H^1_{\text{per}}(Y)}.$$

This, due to Proposition 3.52, defines F as an element of $(\mathcal{W}_{\text{per}}(Y))'$ and hence (7.17) makes sense.

Theorem 4.26 gives then a unique solution $\dot{u}_1 \in \mathcal{W}_{\text{per}}(Y)$ of (7.16). The linearity of (7.15) where $\mathcal{A}_0$ involves the variable y only, together with the fact that $\partial u_0/\partial x_j$ is independent of y, suggests to look for this $\dot{u}_1$ under the following particular form:

$$\dot{u}_1(x,y) = -\sum_{j=1}^{N} \hat{\dot{\chi}}_j(y)\frac{\partial \dot{u}_0}{\partial x_j}(x), \quad \text{in } \mathcal{W}_{\text{per}}(Y) \tag{7.18}$$

where $\hat{\dot{\chi}}$ satisfies

$$\begin{cases} \mathcal{A}_0 \hat{\dot{\chi}}_j = \displaystyle\sum_{i=1}^{N} \frac{\partial a_{ij}}{\partial y_i} & \text{in } Y \\ \hat{\dot{\chi}}_j \quad Y\text{-periodic}. \end{cases} \tag{7.19}$$

for $j = 1, \ldots, N$. It is easily seen that Theorem 4.26 together with Proposition 6.12 gives a unique solution $\dot{\hat{\chi}}_j \in \mathcal{W}_{\text{per}}(Y)$ of this problem. Moreover, as observed in Remark 7.3, we can choose a representative element of $\dot{\hat{\chi}}_j$ satisfying the variational formulation (4.65). Hence, Theorem 4.27 gives the existence and uniqueness of $\widehat{\chi}_j \in \dot{\hat{\chi}}_j$, the solution of

$$\begin{cases} \text{Find } \widehat{\chi}_j \in W_{\text{per}}(Y) \text{ such that} \\ a_Y(\widehat{\chi}_j, \psi) = \displaystyle\sum_{i=1}^{N} \int_Y a_{ij}(y) \frac{\partial \psi}{\partial y_i} dy \\ \forall \psi \in W_{\text{per}}(Y), \end{cases} \tag{7.20}$$

where (see Definition 3.48 and (4.66))

$$W_{\text{per}}(Y) = \left\{ v \in H^1_{\text{per}}(Y);\ \mathcal{M}_Y(v) = 0 \right\}.$$

Observe that this problem is exactly (6.15).

On the other hand, from (7.18) we see that any solution $u_1(x,y)$ of (7.9) has the form

$$u_1(x,y) = -\sum_{j=1}^{N} \widehat{\chi}_j(y) \frac{\partial u_0}{\partial x_j} + \tilde{u}_1(x), \quad \text{with } u_1 \in \dot{u}_1, \tag{7.21}$$

where $\tilde{u}_1$ is independent of y, i.e.

$$\tilde{u}_1(x) \in \dot{0} \quad \text{in } \mathcal{W}_{\text{per}}(Y).$$

We now pass to equation (7.10). Taking into account (7.14) and (7.21), one has

$$f - \mathcal{A}_1 u_1 - \mathcal{A}_2 u_0 = f + \sum_{i,j=1}^{N} \frac{\partial}{\partial y_i}\left(a_{ij}(y) \frac{\partial u_1}{\partial x_j} \right) + \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(y) \left(\frac{\partial u_1}{\partial y_j} + \frac{\partial u_0}{\partial x_j} \right) \right). \tag{7.22}$$

Consequently, the variational formulation of (7.10) is

$$\begin{cases} \text{Find } \dot{u}_2 \in \mathcal{W}_{\text{per}}(Y) \text{ such that} \\ \dot{a}_Y(\dot{u}_2, \dot{v}) = \langle F_1, \dot{v} \rangle_{(\mathcal{W}_{\text{per}}(Y))', \mathcal{W}_{\text{per}}(Y)} \\ \forall \dot{v} \in \mathcal{W}_{\text{per}}(Y). \end{cases} \tag{7.23}$$

where $\dot{a}_Y$ is given by (7.13) and F_1 is defined by

$$\begin{aligned} \langle F_1, \dot{\psi} \rangle_{(\mathcal{W}_{\text{per}}(Y))', \mathcal{W}_{\text{per}}(Y)} &= \int_Y f \psi \, dy - \sum_{i,j=1}^{N} \int_Y a_{ij}(y) \frac{\partial u_1}{\partial x_j} \frac{\partial \psi}{\partial y_i} \, dy \\ &+ \sum_{i,j=1}^{N} \int_Y \frac{\partial}{\partial x_i}\left(a_{ij}(y) \left(\frac{\partial u_1}{\partial y_j} + \frac{\partial u_0}{\partial x_j} \right) \right) \psi \, dy, \\ &\forall \psi \in \dot{\psi},\ \dot{\psi} \in \mathcal{W}_{\text{per}}(Y). \end{aligned} \tag{7.24}$$

This problem is well-posed if F_1 is an element of $(W_{per}(Y))'$, i.e. if

$$\langle F_1, 1\rangle_{(H^1_{per}(Y))', H^1_{per}(Y)} = 0,$$

which reads

$$-\sum_{i,j=1}^{N} \int_Y \frac{\partial}{\partial x_i}\left(a_{ij}(y)\left(\frac{\partial u_1}{\partial y_j} + \frac{\partial u_0}{\partial x_j}\right)\right) dy = \int_Y f\, dy.$$

This relation is a necessary and sufficient condition to insure the existence and uniqueness of $\hat{u}_2$, solution of (7.23) given by Theorem 4.26. Replacing herein u_1 by its formula (7.18), and since $f = f(x)$, we find that u_0 has to satisfy

$$-\sum_{i,j,k=1}^{N} \int_Y \frac{\partial}{\partial x_i}\left(a_{ij}(y)\left(-\frac{\partial \widehat{\chi}_k}{\partial y_j}\frac{\partial u_0}{\partial x_k} + \frac{\partial u_0}{\partial x_j}\right)\right) dy = |Y| f,$$

or equivalently, by taking into account (7.14),

$$-\sum_{i,k=1}^{N}\left[\sum_{j=1}^{N} \int_Y \left(a_{ik} - a_{ij}\frac{\partial \widehat{\chi}_k}{\partial y_j}\right) dy\right] \frac{\partial^2 u_0}{\partial x_i \partial x_k} = |Y| f. \tag{7.25}$$

By using the expression of A^0 in (6.35), one has

$$\sum_{j=1}^{N} \int_Y \left(a_{ik} - a_{ij}\frac{\partial \widehat{\chi}_k}{\partial y_j}\right) dy = |Y| a^0_{ik}, \quad \forall i, k = 1, \dots, N.$$

Consequently, (7.25) is nothing else than

$$-\sum_{i,k=1}^{N} \frac{\partial}{\partial x_i}\left(a^0_{ik}\frac{\partial u_0}{\partial x_k}\right) = f \quad \text{in } \Omega. \tag{7.26}$$

Notice that (7.26) is exactly the equation from the homogenized problem (6.29) in Theorem 6.1. The existence and uniqueness of $u_0 \in H^1_0(\Omega)$ follows from Proposition 6.12 via Lax–Milgram theorem so that,

$$u_0 = u^0,$$

with u^0 given by Theorem 6.1.

Remark 7.5. Let us point out that equation (7.26) has been obtained by a formal method arguing as if all the functions were smooth, namely the coefficients a_{ij} which for composite materials are not even continuous (see Section 5.2). Nevertheless, the mathematical study of auxiliary problems (7.12), (7.16), (7.20) and (7.23) above, is rigorous. ◊

Remark 7.6. Observe also that by this method one obtains the homogenized matrix A^0 under the form (6.30), or the equivalent one (6.35) which involves the matrix A and problems (6.5) and (6.11). This is intrinsic to the method. ◊

As observed in Remark 7.1, we can compute successively the functions u_j in asymptotic expansion (7.7). We describe here only u_2. Replacing (7.21) into (7.10), an easy computation leads to the equation

$$\begin{cases} \mathcal{A}_0 u_2 = f - \displaystyle\sum_{i,j,k=1}^{N} a_{ij}(y)\frac{\partial \widehat{\chi}_k}{\partial y_j}\frac{\partial^2 u^0}{\partial x_i \partial x_k} \\ \qquad - \displaystyle\sum_{i,j,k=1}^{N} \frac{\partial (a_{ij}(y)\widehat{\chi}_k)}{\partial y_i}\frac{\partial^2 u^0}{\partial x_j \partial x_k} + \sum_{i,j=1}^{N} a_{ij}(y)\frac{\partial^2 u^0}{\partial x_i \partial x_j} \quad \text{in } Y \\ u_2 \quad Y\text{-periodic in } y. \end{cases}$$

By using (7.26) and renaming the indices, this becomes

$$\begin{cases} \mathcal{A}_0 u_2 = - \displaystyle\sum_{k,\ell=1}^{N} a^0_{k\ell}\frac{\partial^2 u^0}{\partial x_k \partial x_\ell} - \sum_{j,k,\ell=1}^{N} a_{kj}(y)\frac{\partial \widehat{\chi}_\ell}{\partial y_j}\frac{\partial^2 u^0}{\partial x_\ell \partial x_k} \\ \qquad - \displaystyle\sum_{i,j,k=1}^{N} \frac{\partial (a_{ij}(y)\widehat{\chi}_k)}{\partial y_i}\frac{\partial^2 u^0}{\partial x_j \partial x_k} + \sum_{j,\ell=1}^{N} a_{j\ell}(y)\frac{\partial^2 u^0}{\partial x_j \partial x_\ell} \quad \text{in } Y \\ u_2 \quad Y\text{-periodic in } y, \end{cases}$$

which can be rewritten as

$$\begin{cases} \mathcal{A}_0 u_2 = - \displaystyle\sum_{k,\ell=1}^{N} a^0_{k\ell}\frac{\partial^2 u^0}{\partial x_k \partial x_\ell} - \sum_{i,j,k,\ell=1}^{N} \frac{\partial (a_{ij}\delta_{kj}\,\widehat{\chi}_\ell)}{\partial y_i}\frac{\partial^2 u^0}{\partial x_k \partial x_\ell} \\ \qquad - \displaystyle\sum_{j,k,\ell=1} a_{kj}\frac{\partial (\widehat{\chi}_\ell - y_\ell)}{\partial y_j}\frac{\partial^2 u^0}{\partial x_k \partial x_\ell} \quad \text{in } Y \\ u_2 \quad Y\text{-periodic in } y. \end{cases} \tag{7.27}$$

Then, F_1 in (7.24) can also be rewritten as follows:

$$\begin{aligned} \langle F_1, \dot{\psi}\rangle_{(\mathcal{W}_{\text{per}}(Y))',\mathcal{W}_{\text{per}}(Y)} &= \sum_{k,\ell=1}^{N}\Big[- a^0_{k\ell}\int_Y \psi\, dy + \sum_{i,j=1}^{N}\int_Y \frac{\partial (a_{ij}\delta_{kj}\,\widehat{\chi}_\ell)}{\partial y_i}\psi\, dy \\ &\quad - \sum_{j=1}^{N}\int_Y a_{kj}\frac{\partial (\widehat{\chi}_\ell - y_\ell)}{\partial y_j}\psi\, dy\Big]\frac{\partial^2 u^0}{\partial x_k \partial x_\ell}, \\ &\qquad \forall \psi \in \dot{\psi},\ \dot{\psi} \in \mathcal{W}_{\text{per}}(Y). \end{aligned}$$

The same arguments as those used to write down (7.18), suggests to look for $u_2 \in \dot{u}_2$, with $\dot{u}_2$ under the form

$$\dot{u}_2(x,y) = \sum_{k,\ell=1}^{N} \dot{\theta}^{k\ell}(y) \frac{\partial^2 \dot{u}^0}{\partial x_k \partial x_\ell}, \tag{7.28}$$

where the function $\dot{\theta}^{k\ell}$ is solution of

$$\begin{cases} \mathcal{A}_0 \dot{\theta}^{k\ell} = -a^0_{k\ell} - \sum\limits_{i,j=1}^{N} \dfrac{\partial(a_{ij}\delta_{kj}\,\widehat{\chi}_\ell)}{\partial y_i} - \sum\limits_{j=1} a_{kj} \dfrac{\partial(\widehat{\chi}_\ell - y_\ell)}{\partial y_j} \quad \text{in } Y \\ \dot{\theta}^{k\ell} \quad Y\text{-periodic.} \end{cases}$$

Again, Theorem 4.27 gives the existence and uniqueness of $\widehat{\theta}^{k\ell} \in \dot{\theta}^{k\ell}$, the solution of the problem

$$\begin{cases} \text{Find } \widehat{\theta}^{k\ell} \in W_{\text{per}}(Y) \text{ such that} \\ a_Y(\widehat{\theta}^{k\ell}j, \psi) = -a^0_{k\ell} \displaystyle\int_Y \psi\, dy - \sum_{i,j=1}^{N} \int_Y \frac{\partial(a_{ij}\delta_{kj}\,\widehat{\chi}_\ell)}{\partial y_i} \psi\, dy \\ \qquad - \displaystyle\sum_{j=1}^{N} \int_Y a_{kj} \frac{\partial(\widehat{\chi}_\ell - y_\ell)}{\partial y_j} \psi\, dy \\ \forall \psi \in W_{\text{per}}(Y). \end{cases} \tag{7.29}$$

From (7.28), one deduces that any solution $u_2 = u_2(x,y)$ of (7.10) has the form

$$u_2(x,y) = \sum_{k,\ell=1}^{N} \widehat{\theta}^{k\ell}(y) \frac{\partial^2 u^0}{\partial x_k \partial x_\ell} + \tilde{u}_2(x), \quad \text{with } u_2 \in \dot{u}_2, \tag{7.30}$$

where $\tilde{u}_2$ is independent of y.

Inserting the particular forms (7.21) and (7.30) of u_1 and u_2 (written with $\tilde{u}_1 = 0$ and $\tilde{u}_2 = 0$ respectively), into expansion (7.5), we get

$$u^\varepsilon(x) = u^0(x) - \varepsilon \sum_{k=1}^{N} \widehat{\chi}_k \left(\frac{x}{\varepsilon}\right) \frac{\partial u^0}{\partial x_k}(x) + \varepsilon^2 \sum_{k,\ell=1}^{N} \widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 u^0}{\partial x_k \partial x_\ell}(x) + \cdots \tag{7.31}$$

which is precisely the expansion from Theorem 6.3.

Remark 7.7. The functions $\widehat{\chi}_k$ are called first-order correctors and $\widehat{\theta}^{k\ell}$ second-order correctors. ◊

7.2 Proof of the error estimate

In this section we prove the following error estimate:

$$\left\| u^\varepsilon - \left(u_0 - \varepsilon \sum_{k=1}^{N} \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right) \frac{\partial u_0}{\partial x_k} + \varepsilon^2 \sum_{k,\ell=1}^{N} \widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 u_0}{\partial x_k \partial x_\ell} \right) \right\|_{H^1(\Omega)} \leq C\, \varepsilon^{\frac{1}{2}}$$

under the regularity hypotheses on the data given in Theorem 6.3.

Let us introduce

$$Z_\varepsilon(x) = u^\varepsilon(x) - \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2\right)\left(x, \frac{x}{\varepsilon}\right),$$

where u_1 and u_2 are defined respectively, by (7.21) and (7.30) for $\tilde{u}_1 = \tilde{u}_2 \equiv 0$, i.e.

$$\begin{cases} u_1(x,y) = -\displaystyle\sum_{\ell=1}^{N} \widehat{\chi}_\ell(y) \frac{\partial u_0}{\partial x_\ell} \\ u_2(x,y) = \displaystyle\sum_{k,\ell=1}^{N} \widehat{\theta}^{k\ell}(y)\, \frac{\partial^2 u^0}{\partial x_k \partial x_\ell}. \end{cases} \tag{7.32}$$

Let us calculate $\mathcal{A}_\varepsilon Z_\varepsilon$ where $\mathcal{A}_\varepsilon$ is defined by (7.2). From (7.6) and (7.7), we have

$$\begin{aligned} \mathcal{A}_\varepsilon Z_\varepsilon(x) &= \left[(\varepsilon^{-2}\mathcal{A}_0 + \varepsilon^{-1}\mathcal{A}_1 + \mathcal{A}_2) Z_\varepsilon\right]\left(x, \frac{x}{\varepsilon}\right) \\ &= \big[\mathcal{A}_\varepsilon u^\varepsilon - \varepsilon^{-2}\mathcal{A}_0 u_0 - \varepsilon^{-1}(\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0) \\ &\quad -(\mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0) - \varepsilon(\mathcal{A}_1 u_2 + \mathcal{A}_2 u_1) - \varepsilon^2 \mathcal{A}_2 u_2\big]\left(x, \frac{x}{\varepsilon}\right). \end{aligned}$$

Using (7.8), (7.9) and (7.10), we derive

$$\mathcal{A}_\varepsilon Z_\varepsilon(x) = \left[-\varepsilon(\mathcal{A}_2 u_1 + \mathcal{A}_1 u_2) - \varepsilon^2 \mathcal{A}_2 u_2\right]\left(x, \frac{x}{\varepsilon}\right). \tag{7.33}$$

From definition (7.7) of $\mathcal{A}_1$ and $\mathcal{A}_2$ and (7.32), we get

$$\begin{aligned} \mathcal{A}_2 u_1 &= \sum_{i,k,\ell=1}^{N} a_{ik}(y) \widehat{\chi}_\ell(y) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell} \\ \mathcal{A}_1 u_2 &= -\sum_{i,j,k,\ell=1}^{N} a_{ij}(y) \frac{\partial \widehat{\theta}^{k\ell}}{\partial y_j}(y) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell} \\ &\quad - \sum_{i,j,k,\ell=1}^{N} \frac{\partial}{\partial y_i}\left(a_{ij}(y) \widehat{\theta}^{k\ell}(y)\right) \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_\ell} \\ \mathcal{A}_2 u_2 &= -\sum_{i,j,k,\ell=1}^{N} a_{ij}(y) \widehat{\theta}^{k\ell}(y) \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_\ell}. \end{aligned}$$

Remark that, due to the strong regularity assumptions made on f and $\partial\Omega$, u_0 is also smooth, being solution of an elliptic problem with constant coefficients (see, for instance Nečas, 1967, Lions and Magenes, 1968a, and Gilbarg and Trudinger, 1977). Therefore, in particular all the derivatives of u_0 above, are in $L^\infty(\Omega)$.

Observe now that for the second term in the right-hand side of (7.33), we have

$$\begin{aligned}\mathcal{A}_1 u_2\left(x, \frac{x}{\varepsilon}\right) &= -\sum_{i,j,k,\ell=1}^{N} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \widehat{\theta}^{k\ell}}{\partial y_j}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_i \partial x_k \partial x_\ell} \\ &\quad -\sum_{i,j,k,\ell=1}^{N}\left[\frac{\partial}{\partial y_i}\left(a_{ij}\widehat{\theta}^{k\ell}\right)\right]\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_j \partial x_k \partial x_\ell}.\end{aligned}$$

A simple computation shows that

$$\begin{aligned}&\sum_{i,j,k,\ell=1}^{N}\left[\frac{\partial}{\partial y_i}\left(a_{ij}\widehat{\theta}^{k\ell}\right)\right]\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_j \partial x_k \partial x_\ell} \\ &= \varepsilon\sum_{i,j,k,\ell=1}^{N}\frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial^3 u_0(x)}{\partial x_j \partial x_k \partial x_\ell} \\ &= \varepsilon\sum_{i,j,k,\ell=1}^{N}\frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_j \partial x_k \partial x_\ell}\right] \\ &\quad -\varepsilon\sum_{i,j,k,\ell=1}^{N} a_{ij}\left(\frac{x}{\varepsilon}\right)\widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0(x)}{\partial x_i \partial x_j \partial x_k \partial x_\ell} \\ &= \varepsilon\sum_{i,j,k,\ell=1}^{N}\frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_j \partial x_k \partial x_\ell}\right] + \varepsilon\mathcal{A}_2 u_2\left(x, \frac{x}{\varepsilon}\right).\end{aligned}$$

Taking into account that u^ε and u_0 vanish on the boundary $\partial\Omega$, these computations show from (7.33), that Z_ε satisfies

$$\begin{cases} \mathcal{A}_\varepsilon Z_\varepsilon = -\displaystyle\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon \frac{\partial Z_\varepsilon}{\partial x_j}\right) = \varepsilon F^\varepsilon \quad \text{in } \Omega \\ Z_\varepsilon = \varepsilon G^\varepsilon \text{ on } \partial\Omega. \end{cases} \tag{7.34}$$

where

$$\begin{cases} F^\varepsilon(x) = \displaystyle\sum_{i,j,k,\ell=1}^{N} \left[-a_{ik}\left(\frac{x}{\varepsilon}\right)\widehat{\chi}_\ell\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \widehat{\theta}^{k\ell}}{\partial y_j}\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial^3 u_0(x)}{\partial x_i \partial x_k \partial x_\ell} \\ \qquad + \varepsilon \displaystyle\sum_{i,j,k,\ell=1}^{N} \frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_j \partial x_k \partial x_\ell}\right], \\ G^\varepsilon(x) = \displaystyle\sum_{k=1}^{N} \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0(x)}{\partial x_k} - \varepsilon \sum_{k,\ell=1}^{N} \widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0(x)}{\partial x_k \partial x_\ell}. \end{cases}$$

Observe that (7.34) is a nonhomogeneous Dirichlet problem to which we will apply Theorem 4.19 and in particular, estimate (4.29). To do so, let us check that the data F^ε and G^ε satisfy the requisite assumptions of this theorem, i.e. that $F^\varepsilon \in H^{-1}(\Omega)$ and $G^\varepsilon \in H^{\frac{1}{2}}(\partial\Omega)$.

Remark first that F^ε is of the form

$$F^\varepsilon = F_0^\varepsilon + \varepsilon \sum_{i=1}^{N} \frac{\partial}{\partial x_i} F_i^\varepsilon, \tag{7.35}$$

where

$$\begin{cases} F_0^\varepsilon(x) = \displaystyle\sum_{i,j,k,\ell=1}^{N} \left[-a_{ik}\left(\frac{x}{\varepsilon}\right)\widehat{\chi}_\ell\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \widehat{\theta}^{k\ell}}{\partial y_j}\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial^3 u_0(x)}{\partial x_i \partial x_k \partial x_\ell} \\ F_i^\varepsilon(x) = \displaystyle\sum_{j,k,\ell=1}^{N} a_{ij}\left(\frac{x}{\varepsilon}\right)\widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_j \partial x_k \partial x_\ell}, \qquad i = 1, \dots, N. \end{cases}$$

From the regularity of u_0 and the fact that the matrix $A \in L^\infty(Y)$, and since by definition (see (6.15) and (7.29)), $\widehat{\chi}_\ell$, $\widehat{\theta}^{k\ell} \in H^1(Y)$, one immediately has that $F_0^\varepsilon \in L^2(\Omega)$. Moreover, from Theorem 2.6 it follows that

$$\begin{aligned} \|F_0^\varepsilon\|_{L^2(\Omega)} &\le \|\partial^3 u_0\|_{L^\infty(\Omega)} \left\| \sum_{i,j,k,\ell=1}^{N} \left[-a_{ik}\left(\frac{\cdot}{\varepsilon}\right)\widehat{\chi}_\ell\left(\frac{\cdot}{\varepsilon}\right) + a_{ij}\left(\frac{\cdot}{\varepsilon}\right)\frac{\partial \widehat{\theta}^{k\ell}}{\partial y_j}\left(\frac{\cdot}{\varepsilon}\right)\right]\right\|_{L^2(\Omega)} \\ &\le c, \end{aligned}$$

where c is a constant independent of ε. Similarly, one also has

$$\|F_i^\varepsilon\|_{L^2(\Omega)} \le c, \quad i = 1, \dots, N.$$

Consequently, from (7.35) and Proposition 3.42, we obtain that $F^\varepsilon \in H^{-1}(\Omega)$ and moreover,

$$\|F^\varepsilon\|_{H^{-1}(\Omega)} \le c_1, \tag{7.36}$$

with c_1 independent of ε.

Let us now look at the function G^ε. We prove the following estimate:

$$\|G^\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \le c_2\varepsilon^{-\frac{1}{2}}. \tag{7.37}$$

It is at this point that we need the regularity assumptions on $\widehat{\chi}_k$ and $\widehat{\theta}^{k\ell}$ made in the statement of Theorem 6.3. We employ here an argument from Oleinik, Shamaev, and Yosifian (1992). Let us introduce the function m_ε defined as follows:

$$\begin{cases} m_\varepsilon \in \mathcal{D}(\Omega) \\ m_\varepsilon = 1 \quad \text{if } \operatorname{dist}(x, \partial\Omega) \le \varepsilon \\ m_\varepsilon = 0 \quad \text{if } \operatorname{dist}(x, \partial\Omega) \ge 2\varepsilon \\ \|\nabla m_\varepsilon\|_{L^\infty(\Omega)} \le \dfrac{1}{\varepsilon}C. \end{cases}$$

For the existence of such functions we refer the reader to Hopf (1957) and Lions (1969, Chapter 1, Lemma 7.2). Set

$$\psi^\varepsilon = m_\varepsilon G^\varepsilon.$$

Due to the definition of m_ε, the support of ψ^ε is a neighbourhood of $\partial\Omega$ of thickness 2ε that we denote by $\mathcal{U}_\varepsilon$. Let us now show that $\psi^\varepsilon \in H^1(\Omega)$ and that

$$\|\psi_\varepsilon\|_{H^1(\mathcal{U}_\varepsilon)} \le c_3\varepsilon^{-\frac{1}{2}}, \tag{7.38}$$

where c_3 is a constant independent of ε. Clearly, from the definition of m_ε and the regularity properties of u_0, one has that

$$\|\psi_\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)} \le c_4 \tag{7.39}$$

independently of ε. On the other hand, for $i = 1, \dots, N$, we have

$$\begin{aligned} \frac{\partial \psi_\varepsilon}{\partial x_i}(x) \;=\; & m_\varepsilon(x)\left[\frac{1}{\varepsilon}\sum_{k=1}^{N}\frac{\partial\widehat{\chi}_k}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0(x)}{\partial x_k} + \sum_{k=1}^{N}\widehat{\chi}_k\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0(x)}{\partial x_i\partial x_k}\right. \\ & \left. - \sum_{k,\ell=1}^{N}\frac{\partial\widehat{\theta}^{k\ell}}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0(x)}{\partial x_k\partial x_\ell} - \varepsilon\sum_{k,\ell=1}^{N}\widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_i\partial x_k\partial x_\ell}\right] \\ & + \frac{\partial m_\varepsilon}{\partial x_i}\left[\sum_{k=1}^{N}\widehat{\chi}_k\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0(x)}{\partial x_k} - \varepsilon\sum_{k,\ell=1}^{N}\widehat{\theta}^{k\ell}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0(x)}{\partial x_k\partial x_\ell}\right]. \end{aligned}$$

It easy to check that $\partial\psi_\varepsilon/\partial x_i \in L^2(\mathcal{U}_\varepsilon)$ for any $i = 1, \dots, N$. In order to obtain (7.38), we need to estimate the norm of $\partial\psi_\varepsilon/\partial x_i$ in $L^2(\mathcal{U}_\varepsilon)$. From the regularity assumptions on $\widehat{\chi}_k$ and $\widehat{\theta}^{k\ell}$, the definition of m_ε and the regularity of u_0, we derive that

$$\|\nabla\psi_\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)} \le \frac{1}{\varepsilon}c_5\|u_0\|_{H^1(\mathcal{U}_\varepsilon)} + c_6, \tag{7.40}$$

where the constants c_5 and c_6 are independent of ε. Since $\mathcal{U}_\varepsilon$ is a neighbourhood of $\partial\Omega$ of thickness 2ε, we can make use of a result from Oleinik, Shamaev, and Yosifian (1992, Chapter 1, Lemma 5.1) which states that there exists a constant c_7, independent of ε, such that

$$\|u_0\|_{H^1(\mathcal{U}_\varepsilon)} \leq \varepsilon^{\frac{1}{2}}\, c_7 \|\nabla u_0\|_{H^1(\Omega)}. \tag{7.41}$$

This, together with (7.39) and (7.40), proves (7.38).

Observe now that $\psi^\varepsilon = G^\varepsilon$ on $\partial\Omega$. Then, from Proposition 3.31 one has

$$\|G^\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|\psi_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_\gamma(\Omega)\|\psi_\varepsilon\|_{H^1(\Omega)} = C_\gamma(\Omega)\, \|\psi_\varepsilon\|_{H^1(\mathcal{U}_\varepsilon)}, \tag{7.42}$$

and this, together with (7.38), implies (7.37).

We can now write estimate (4.29) from Theorem 4.19. By using (7.36) and (7.37) we obtain

$$\begin{aligned} \|Z_\varepsilon\|_{H^1(\Omega)} &\leq \varepsilon C_1 \|F^\varepsilon\|_{H^{-1}(\Omega)} + C_2 \varepsilon \|G^\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq \varepsilon C_1\, c_1 + \varepsilon^{\frac{1}{2}} C_2\, c_2. \end{aligned}$$

Recalling the definition of Z_ε, this is the claimed error estimate. □

Remark 7.8. Let us make some comments about this proof. As we already mentioned, the particular assumptions on $\widehat{\chi}_i$ and $\widehat{\theta}^{k\ell}$ were used only to show estimate (7.37) on G^ε but not estimate (7.36) on F^ε. This is related to the fact that Z^ε is not an element of $H^1_0(\Omega)$, and so we had to estimate its trace G^ε on the boundary $\partial\Omega$. Observe that if for $\partial\psi_\varepsilon/\partial x_i$ we argue as for F^ε, by using only the regularity of u^0, we would have instead of (7.40) an estimate of the form

$$\|\nabla\psi_\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)} \leq \frac{1}{\varepsilon}\, c,$$

which would only imply that

$$\|Z^\varepsilon\|_{H^1(\Omega)} \leq C$$

with a constant independent of ε. This explains why it was necessary to introduce the neighbourhood $\mathcal{U}_\varepsilon$. Indeed, estimating the term $\|u_0\|_{H^1(\mathcal{U}_\varepsilon)}$ in (7.40) by (7.41) leads to the gain of a term of order $\varepsilon^{\frac{1}{2}}$. ◊

8

Tartar's method of oscillating test functions

We begin this chapter with the proof of Theorem 6.1. This is done by using the method introduced by Tartar (1977a, 1978). In Section 8.2 we prove the convergence of the energy associated to problem (6.1). This convergence allows us to show in Section 8.3 a corrector result. Sections 8.4 and 8.5 contain some further convergence properties of the solution u^ε of the model problem (6.1). Finally, in Section 8.6 we formulate the eigenvalues problem associated to (6.1) and give its asymptotic behaviour as $\varepsilon \to 0$.

Let us recall our model problem, namely

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.1}$$

where f is given in $H^{-1}(\Omega)$ and the matrix A^ε is the Y-periodic matrix defined by

$$a^\varepsilon_{ij}(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall i,j = 1,\ldots,N \tag{8.2}$$

and

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) = (a^\varepsilon_{ij}(x))_{1\le i,j\le N} \quad \text{a.e. on } \mathbb{R}^N, \tag{8.3}$$

where

$$\begin{cases} a_{ij} \text{ is } Y\text{-periodic}, & \forall i,j = 1,\ldots,N \\ A = (a_{ij})_{1\le i,j\le N} \in M(\alpha,\beta,Y), \end{cases} \tag{8.4}$$

with $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$ and $M(\alpha,\beta,Y)$ given by Definition 4.11.

8.1 Proof of the main convergence result

In this section we give a rigorous proof of Theorem 6.1, following a general method due to Tartar (1977a, 1978). This method relies on the construction of a class of oscillating test functions obtained by periodizing the solution of a problem set in the reference cell, actually problem (6.17). As we will see during the proof, the fact that (6.17) contains the adjoint operator $-\text{div}({}^tA(y)\nabla)$, is the key point in this method. Indeed, when trying to identify the limit ξ^0 in (5.15), this essential fact allows to eliminate all the terms containing a product of two weekly convergent sequences. By this method, we will naturally obtain the homogenized matrix A^0 under the form (6.31), or the equivalent (6.36), which involves the matrix ${}^tA$ and problems (6.27) and (6.28). This is one of the main features of Tartar's method.

Let us recall briefly the framework introduced in Section 5.1. Let u^ε be the solution of (8.1). We know that there exists a subsequence (still denoted by ε), such that

$$\begin{cases} i) & u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega) \\ ii) & u^\varepsilon \to u^0 \quad \text{strongly in } L^2(\Omega) \\ iii) & \xi^\varepsilon \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^N, \end{cases} \tag{8.5}$$

where ξ^ε is the vector-function

$$\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_N^\varepsilon) = \left(\sum_{j=1}^N a_{1j}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}, \dots, \sum_{j=1}^N a_{Nj}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \right) = A^\varepsilon \nabla u^\varepsilon, \tag{8.6}$$

and satisfies

$$\int_\Omega \xi^\varepsilon \, \nabla v \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{8.7}$$

Recall also that ξ^0 satisfies

$$-\text{div } \xi^0 = f \quad \text{in } \Omega,$$

i.e.

$$\int_\Omega \xi^0 \, \nabla v \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{8.8}$$

Therefore, Theorem 6.1 is proved if we show that

$$\xi^0 = A^0 \nabla u^0. \tag{8.9}$$

Indeed, this, together with (8.8), implies that u^0 satisfies problem (6.29). On the other hand, Proposition 6.12 and Theorem 4.16 provide the uniqueness of such a solution. Consequently, the convergences in (8.5) take place for the whole sequences. This is why we still denoted by ε the converging subsequences.

Set now

$$w_\lambda^\varepsilon(x) = \varepsilon w_\lambda\left(\frac{x}{\varepsilon}\right) = \lambda \cdot x - \varepsilon \chi_\lambda\left(\frac{x}{\varepsilon}\right), \tag{8.10}$$

where w_λ and χ_λ are defined by (6.21) and (6.18). Recalling that χ_λ is Y-periodic, in view of Theorem 2.6, it is obvious that

$$w_\lambda^\varepsilon \rightharpoonup \lambda \cdot x \quad \text{weakly in } L^2(\Omega).$$

Observe that from (6.21) one has

$$(\nabla_x w_\lambda^\varepsilon)(x) = (\nabla_y w_\lambda)\left(\frac{x}{\varepsilon}\right) = \lambda - \nabla_y \chi_\lambda\left(\frac{x}{\varepsilon}\right).$$

But $\nabla_y w_\lambda$ is Y-periodic, since χ_λ is Y-periodic and λ is constant. Then by using again Theorem 2.6,

$$\nabla_x w_\lambda^\varepsilon \rightharpoonup \mathcal{M}_Y(\lambda - \nabla_y \chi_\lambda) = \lambda - \mathcal{M}_Y(\nabla_y \chi_\lambda) \quad \text{weakly in } L^2(\Omega).$$

Observe now that from Theorem 3.33 written in Y for $u = 1$ and $v = \chi_\lambda$ one has

$$\int_Y \nabla_y \chi_\lambda(y)\, dy = \int_{\partial Y} \chi_\lambda \cdot n\, ds_y = 0,$$

where we have used Proposition 3.49. Hence

$$\mathcal{M}_Y(\nabla_y \chi_\lambda) = 0.$$

Consequently, we have the following convergences:

$$\begin{cases} i) & w_\lambda^\varepsilon \rightharpoonup \lambda \cdot x \quad \text{weakly in } H^1(\Omega) \\ ii) & w_\lambda^\varepsilon \to \lambda \cdot x \quad \text{strongly in } L^2(\Omega), \end{cases} \tag{8.11}$$

where we have used Theorem 3.23. Introduce the vector function

$$\eta_\lambda^\varepsilon = \left(\sum_{j=1}^N a_{j1}^\varepsilon \frac{\partial w_\lambda^\varepsilon}{\partial x_j}, \dots, \sum_{j=1}^N a_{jN}^\varepsilon \frac{\partial w_\lambda^\varepsilon}{\partial x_j} \right) = {}^t A^\varepsilon \nabla w_\lambda^\varepsilon. \tag{8.12}$$

From (8.3) and (8.10), we see that

$$\eta_\lambda^\varepsilon(x) = \frac{1}{\varepsilon}\left[{}^t A\left(\frac{x}{\varepsilon}\right) \left(\nabla_y(\varepsilon w_\lambda)\right)\left(\frac{x}{\varepsilon}\right)\right] = ({}^t A \nabla_y w_\lambda)\left(\frac{x}{\varepsilon}\right).$$

Since ${}^t A$ is Y-periodic, obviously ${}^t A \nabla_y w_\lambda$ is Y-periodic too. Hence, applying again Theorem 2.6 one derives the convergence

$$\eta_\lambda^\varepsilon \rightharpoonup \mathcal{M}_Y({}^t A \nabla w_\lambda) = {}^t A^0 \lambda \quad \text{weakly in } (L^2(\Omega))^N, \tag{8.13}$$

with A^0 defined by (6.31).

We now prove that η_λ^ε satisfies

$$\int_\Omega \eta_\lambda^\varepsilon \cdot \nabla v\, dx = 0, \quad \forall v \in H_0^1(\Omega). \tag{8.14}$$

To do so, let $\varphi \in \mathcal{D}(\Omega)$ and set

$$\varphi^\varepsilon(y) = \varphi(\varepsilon y), \quad \text{a.e. on } \mathbb{R}^N.$$

Obviously φ^ε belongs to $\mathcal{D}(\mathbb{R}^N)$. Hence, from (6.25) one has

$$\int_{\mathbb{R}^N} ({}^t A \nabla w_\lambda)(y) \nabla \varphi^\varepsilon(y)\, dy = 0.$$

By making the change of variable $x = \varepsilon y$ it follows that

$$\int_\Omega ({}^t A \nabla w_\lambda)\left(\frac{x}{\varepsilon}\right) \nabla\varphi(x)\, dy = 0, \quad \forall\varphi \in \mathcal{D}(\Omega),$$

since supp $\varphi \subset \Omega$. Recalling Definition 3.26 of $H_0^1(\Omega)$, one immediately has (8.14).

Let $\varphi \in \mathcal{D}(\Omega)$ and choose $\varphi w_\lambda^\varepsilon$ as test function in (8.7) and φu^ε as test function in (8.14). We have respectively,

$$\begin{aligned}\int_\Omega \xi^\varepsilon \cdot \nabla w_\lambda^\varepsilon \varphi\, dx + \int_\Omega \xi^\varepsilon \cdot \nabla\varphi\, w_\lambda^\varepsilon\, dx &= \langle f, \varphi\, w_\lambda^\varepsilon\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall\varphi \in \mathcal{D}(\Omega),\\ \int_\Omega \eta_\lambda^\varepsilon \cdot \nabla u^\varepsilon \varphi\, dx + \int_\Omega \eta_\lambda^\varepsilon \cdot \nabla\varphi\, u^\varepsilon\, dx &= 0, \quad \forall\varphi \in \mathcal{D}(\Omega).\end{aligned}$$

Observe that from definitions (8.6) and (8.12), one has

$$\xi^\varepsilon \cdot \nabla w_\lambda^\varepsilon = A^\varepsilon \nabla u^\varepsilon \cdot \nabla w_\lambda^\varepsilon = {}^t A^\varepsilon \nabla w_\lambda^\varepsilon \cdot \nabla u^\varepsilon = \eta_\lambda^\varepsilon \cdot \nabla u^\varepsilon.$$

Therefore by subtraction, the first integrals in the expressions above cancel and we obtain

$$\int_\Omega \xi^\varepsilon \cdot \nabla\varphi\, w_\lambda^\varepsilon\, dx - \int_\Omega \eta_\lambda^\varepsilon \cdot \nabla\varphi\, u^\varepsilon\, dx = \langle f, \varphi\, w_\lambda^\varepsilon\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall\varphi \in \mathcal{D}(\Omega). \tag{8.15}$$

We now pass to the limit in this identity.

Let us point out here the main idea of Tartar's method, namely the use of adjoint problem in the definition of w_λ. As a matter of fact, it is precisely this fact which allows one to cancel the two terms where one cannot identify the limit since they contain products of only weakly convergent sequences. Moreover, as we show below, the other terms all pass to the limit and the limit expression will easily imply the claimed equality (8.9).

Take $\varepsilon \to 0$ in (8.15). Convergences (8.5)iii and (8.11)ii give

$$\lim_{\varepsilon\to 0} \int_\Omega \xi^\varepsilon \cdot \nabla\varphi\, w_\lambda^\varepsilon\, dx = \int_\Omega \xi^0 \cdot \nabla\varphi\, (\lambda \cdot x)\, dx.$$

Next, from convergences (8.13) and (8.5)ii one has

$$\lim_{\varepsilon\to 0} \int_\Omega \eta_\lambda^\varepsilon \cdot \nabla\varphi\, u^\varepsilon\, dx = \int_\Omega {}^t A^0 \lambda \cdot \nabla\varphi\, u^0\, dx.$$

Then, from (8.15) and (8.11)i, we finally get

$$\int_\Omega \xi^0 \cdot \nabla\varphi\, (\lambda \cdot x)\, dx - \int_\Omega {}^t A^0 \lambda \cdot \nabla\varphi\, u^0\, dx = \langle f, (\lambda\cdot x)\varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall\varphi \in \mathcal{D}(\Omega),$$

which can be rewritten in the form

$$\begin{aligned}\int_\Omega \xi^0 \cdot \nabla[(\lambda\cdot x)\varphi]\, dx - \int_\Omega \xi^0 \cdot \lambda\, \varphi\, dx - \int_\Omega {}^t A^0 \lambda \cdot \nabla\varphi\, u^0\, dx\\ = \langle f, (\lambda\cdot x)\varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall\varphi \in \mathcal{D}(\Omega).\end{aligned}$$

This gives, by using (8.8) written for the test function $v = (\lambda \cdot x)\varphi$,

$$\int_\Omega \xi^0 \cdot \lambda \varphi \, dx = -\int_\Omega {}^t A^0 \lambda \cdot \nabla \varphi \, u^0 \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Due to Definition 3.11 and taking into account the fact that ${}^t A^0 \lambda$ is constant, we get

$$\int_\Omega \xi^0 \cdot \lambda \varphi \, dx = \int_\Omega {}^t A^0 \lambda \cdot \nabla u^0 \, \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence, Theorem 1.44 implies that

$$\xi^0 \cdot \lambda = {}^t A^0 \lambda \, \nabla u^0 = A^0 \nabla u^0 \cdot \lambda,$$

which gives (8.9), since λ is arbitrary in $\mathbb{R}^N$. This ends the proof of Theorem 6.1. □

8.2 Convergence of the energy

One interesting consequence of Theorem 6.1 is the convergence of the energy associated to problem (8.1), namely of the quantity

$$E^\varepsilon(u^\varepsilon) = \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, dx.$$

Actually, we prove the following result, which was originally proved by De Giorgi and Spagnolo (1973) in the context of the G-convergence (see Chapter 13):

Proposition 8.1. *Let u^ε be the solution of (8.1). Then,*

$$E^\varepsilon(u^\varepsilon) \longrightarrow E^0(u^0) = \int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, dx,$$

where u^0 and A^0 are given by Theorem 6.1.

Proof. From the variational formulation of (8.1) written for u^ε (see (5.7)), one has

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, dx = \langle f, u^\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Convergence (8.5)i implies that

$$\lim_{\varepsilon \to 0} \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, dx = \langle f, u^0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

On the other hand, choosing u^0 as test function in the variational formulation of (6.29), we have

$$\int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, dx = \langle f, u^0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

This gives the result. □

In fact, we also have a convergence in the sense of distributions. Indeed,

Proposition 8.2. *Let u^ε be the solution of (8.1). Then, the following convergence holds:*

$$A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \longrightarrow A^0 \nabla u^0 \, \nabla u^0 \quad \text{in } \mathcal{D}'(\Omega),$$

where u^0 and A^0 are given by Theorem 6.1.

Proof. From Definition 3.9, one has to prove that

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, \varphi \, dx \longrightarrow \int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, \varphi \, dx, \quad \text{for any } \varphi \in \mathcal{D}(\Omega). \tag{8.16}$$

Using $u^\varepsilon \varphi$ in the variational formulation of (8.1) (see (5.7)), yields

$$\begin{cases} \displaystyle\int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, \varphi \, dx = \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla (u^\varepsilon \varphi) \, dx - \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla \varphi \, u^\varepsilon \, dx \\ \displaystyle = \langle f, u^\varepsilon \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla \varphi \, u^\varepsilon \, dx \\ \displaystyle = \langle f, u^\varepsilon \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega \xi^\varepsilon \, \nabla \varphi \, u^\varepsilon \, dx. \end{cases} \tag{8.17}$$

Observe that from (8.5)i, we have that

$$u^\varepsilon \varphi \rightharpoonup u^0 \varphi \quad \text{weakly in } H_0^1(\Omega), \quad \text{for any } \varphi \in \mathcal{D}(\Omega).$$

This convergence, together with (8.5), (8.9) and Proposition 1.19, allows us to pass to the limit in (8.17) to obtain

$$\begin{cases} \displaystyle\lim_{\varepsilon \to 0} \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, \varphi \, dx = \langle f, u^0 \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega \xi^0 \, \nabla \varphi \, u^0 \, dx \\ \displaystyle = \langle f, u^0 \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega \xi^0 \, \nabla (\varphi \, u^0) \, dx + \int_\Omega \xi^0 \, \nabla u^0 \, \varphi \, dx. \end{cases} \tag{8.18}$$

Taking now $u^0 \varphi$ as test function in (8.8), one has

$$\int_\Omega \xi^0 \, \nabla (\varphi \, u^0) \, dx = \langle f, u^0 \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

which, used in (8.18), leads to

$$\lim_{\varepsilon \to 0} \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, \varphi \, dx = \int_\Omega \xi^0 \, \nabla u^0 \, \varphi \, dx.$$

This is exactly (8.16) since $\xi^0 = A^0 \nabla u^0$ (see (8.9)). □

We showed in Proposition 6.12 that there exists some constant $\alpha_0 > 0$ such that the matrix A^0 satisfies the ellipticity condition with this constant (see Remark 4.12). Since A^0 is constant, one then has

$$A^0 \in M(\alpha_0, \beta_0, \Omega),$$

where $\beta_0 = \max\limits_{i,j} a^0_{ij}$. Recall that we started with the matrix $A^\varepsilon \in M(\alpha, \beta, \Omega)$. A natural question is to precise the constants α_0 and β_0. The answer is given by the following result, a consequence of Proposition 8.2:

Proposition 8.3. *One has*

$$A^0 \in M\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right). \tag{8.19}$$

Proof. Due to Definition 4.11 of the set $M(\alpha, \beta^2/\alpha, \Omega)$, one has to prove that A^0 satisfies the following inequalities:

$$\begin{cases} i) & (A^0\lambda, \lambda) \geq \alpha|\lambda|^2 \\ ii) & |A^0\lambda| \leq \dfrac{\beta^2}{\alpha}|\lambda|, \end{cases} \tag{8.20}$$

for any $\lambda \in \mathbb{R}^N$.

Let us first prove (8.20)i.

To do so, let $z^0 \in H^1_0(\Omega)$ and z^ε be the solution of

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla z^\varepsilon) = -\text{div}\,(A^0 \nabla z^0) & \text{in } \Omega \\ z^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{8.21}$$

We can apply Theorem 6.1 to this problem to obtain

$$z^\varepsilon \rightharpoonup Z^0 \quad \text{weakly in } H^1_0(\Omega), \tag{8.22}$$

where Z^0 is the solution of

$$\begin{cases} -\text{div}\,(A^0 \nabla Z^0) = -\text{div}\,(A^0 \nabla z^0) & \text{in } \Omega \\ Z^0 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The uniqueness of the solution Z^0 of this problem implies

$$Z^0 = z^0.$$

From Proposition 8.2, we know that

$$A^\varepsilon \nabla z^\varepsilon \, \nabla z^\varepsilon \longrightarrow A^0 \nabla z^0 \, \nabla z^0 \quad \text{in } \mathcal{D}'(\Omega).$$

In particular, for any non-negative function φ in $\mathcal{D}(\Omega)$, we have

$$\int_\Omega A^\varepsilon \nabla z^\varepsilon \, \nabla z^\varepsilon \, \varphi \, dx \longrightarrow \int_\Omega A^0 \nabla z^0 \, \nabla z^0 \, \varphi \, dx. \tag{8.23}$$

Since $A^\varepsilon \in M(\alpha, \beta, \Omega)$, one has

$$\int_\Omega A^\varepsilon \nabla z^\varepsilon \, \nabla z^\varepsilon \, \varphi \, dx \geq \alpha \int_\Omega |\nabla z^\varepsilon|^2 \varphi \, dx. \tag{8.24}$$

From (8.22), we derive that

$$\sqrt{\varphi}\, \nabla z^\varepsilon \rightharpoonup \sqrt{\varphi}\, \nabla z^0 \quad \text{weakly in } (L^2(\Omega))^N.$$

The lower semi-continuity with respect to the weak convergence (see Proposition 1.14) implies

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla z^\varepsilon|^2 \varphi \, dx \geq \int_\Omega |\nabla z^0|^2 \varphi \, dx.$$

This, together with (8.23), allows us to pass to the lim inf in (8.24) to obtain

$$\int_\Omega A^0 \nabla z^0 \, \nabla z^0 \, \varphi \, dx \geq \alpha \int_\Omega |\nabla z^0|^2 \varphi \, dx.$$

Since z^0 is arbitrary and the support of φ is a compact contained in Ω, we can choose z^0 such that

$$z^0 = \lambda \cdot x, \quad \text{on supp } \varphi.$$

Then, as A^0 is constant, one has

$$(A^0 \lambda, \lambda) \int_\Omega \varphi \, dx \geq \alpha \, |\lambda|^2 \int_\Omega \varphi \, dx,$$

which implies (8.20)i, since φ is a non-negative function.

We now prove (8.20)ii. To do so, let us show first that

$$((A^\varepsilon)^{-1} \lambda, \lambda) \geq \frac{\alpha}{\beta^2} \, |\lambda|^2, \tag{8.25}$$

for any $\lambda \in \mathbb{R}^N$ and a.e. on Ω, where $(A^\varepsilon(x))^{-1}$ is the inverse matrix of $A^\varepsilon(x)$. Recall that $(A^\varepsilon)^{-1}$ is well defined since $A^\varepsilon \in M(\alpha, \beta, \Omega)$ (see Remark 4.12).

For λ fixed in $\mathbb{R}^N$, set $\mu = (A^\varepsilon)^{-1}(x)\lambda$ a.e. on Ω. Then, using again the fact that $A^\varepsilon \in M(\alpha, \beta, \Omega)$, one has

$$((A^\varepsilon)^{-1}(x)\lambda, \lambda) = (A^\varepsilon(x)\mu, \mu) \geq \alpha \, |\mu|^2 = \alpha \, \left|(A^\varepsilon)^{-1}(x)\lambda\right|^2. \tag{8.26}$$

Recall (see Remark 4.12) that

$$\|A^\varepsilon(x)\|_2 = \sup_{\mu \neq 0} \frac{|A^\varepsilon(x)\mu\,|}{|\mu|}.$$

Hence, for any μ in $\mathbb{R}^N$, one has

$$|A^\varepsilon(x)\mu\,| \leq |\mu| \, \|A^\varepsilon(x)\|_2.$$

This, written for $\mu = (A^\varepsilon)^{-1}(x)\lambda$, becomes

$$|(A^\varepsilon)^{-1}(x)\lambda| \geq \frac{|\lambda|}{\|A^\varepsilon(x)\|_2}. \tag{8.27}$$

From Remark 4.12, one deduces that

$$|(A^\varepsilon)^{-1}(x)\lambda| \geq \frac{|\lambda|}{\beta}.$$

This inequality, together with (8.26), gives (8.25).

To prove (8.20)ii let, as before, $z^0 \in H_0^1(\Omega)$, z^ε the solution of (8.21) and φ a non-negative function in $\mathcal{D}(\Omega)$.

Choosing $\lambda = A^\varepsilon \nabla z^\varepsilon$ in (8.25), one easily has

$$\int_\Omega \nabla z^\varepsilon \; A^\varepsilon \nabla z^\varepsilon \, \varphi \, dx \geq \frac{\alpha}{\beta^2} \int_\Omega |A^\varepsilon \nabla z^\varepsilon|^2 \, \varphi \, dx.$$

The same argument, used to pass to the limit in (8.24), gives

$$(\lambda, A^0\lambda) \int_\Omega \varphi \, dx \geq \frac{\alpha}{\beta^2} \, |A^0\lambda|^2 \int_\Omega \varphi \, dx.$$

Hence, since φ is a non-negative function,

$$\frac{\alpha}{\beta^2} \, |A^0\lambda|^2 \leq |\lambda| \, |A^0\lambda|.$$

This implies (8.20)(ii) and the proof of Proposition 8.3 is complete. □

8.3 Correctors

Let u^ε be the solution of problem (8.1) and u^0 the solution of the corresponding homogenized problem. From Theorem 6.1 one has, in particular, the following convergence:

$$\nabla u^\varepsilon - \nabla u^0 \rightharpoonup 0 \quad \text{weakly in } (L^2(\Omega))^N. \tag{8.28}$$

Remark 8.4. In general, convergence (8.28) cannot be improved. This can be seen from the examples treated in Chapter 5. For the one-dimensional case for instance, this is implicit in Remark 5.8. Indeed, if convergence (8.28) were strong, one would have (in the sense of the L^2-weak convergence)

$$\lim_{\varepsilon\to 0}\left(a^\varepsilon \frac{du^\varepsilon}{dx}\right) = \left(\lim_{\varepsilon\to 0} a^\varepsilon\right)\left(\lim_{\varepsilon\to 0} \frac{du^\varepsilon}{dx}\right),$$

and in general this is not true. *A fortiori*, a similar argument works for layered materials, in view of Theorem 5.10. ◊

This remark shows that convergence (8.28) is in general, not strong. However, we will prove that by adjusting the term ∇u^0, we get a strong convergence. To do so, we introduce the corrector matrix $C^\varepsilon = (C^\varepsilon_{ij})_{1\le i,j\le N}$, defined by

$$\begin{cases} C^\varepsilon_{ij}(x) = C_{ij}\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } \Omega \\ C_{ij}(y) = \delta_{ij} - \dfrac{\partial \widehat{\chi}_j}{\partial y_i}(y) = \dfrac{\partial \widehat{w}_j}{\partial y_i}(y) & \text{a.e. on } Y, \end{cases} \tag{8.29}$$

where $\widehat{\chi}_j$ and $\widehat{w}_j$ are given by (6.14), (6.15) and (6.16).

Some interesting properties of the corrector matrix C^ε are given by the following proposition:

Proposition 8.5. *Let C^ε be defined by (8.29). Then*

$$\begin{cases} i) & C^\varepsilon \rightharpoonup I \quad \text{weakly in } (L^2(\Omega))^{N\times N} \\ ii) & A^\varepsilon C^\varepsilon \rightharpoonup A^0 \quad \text{weakly in } (L^2(\Omega))^N, \end{cases} \tag{8.30}$$

where I is the unit $N \times N$ matrix.

Proof. Introduce, for $i = 1, \ldots, N$, the functions

$$\widehat{w}^\varepsilon_i(x) = \varepsilon \widehat{w}_i\left(\frac{x}{\varepsilon}\right) = x_i - \varepsilon\, \widehat{\chi}_i\left(\frac{x}{\varepsilon}\right). \tag{8.31}$$

The same argument, used to prove (8.11), gives

$$\begin{cases} i) & \widehat{w}^\varepsilon_i \rightharpoonup x_i \quad \text{weakly in } H^1(\Omega) \\ ii) & \widehat{w}^\varepsilon_i \to x_i \quad \text{strongly in } L^2(\Omega). \end{cases} \tag{8.32}$$

From (8.29), it is easily seen that

$$C^\varepsilon_{ij}(x) = \frac{\partial \widehat{w}_j}{\partial y_i}\left(\frac{x}{\varepsilon}\right) = \frac{\partial \widehat{w}^\varepsilon_j}{\partial x_i}(x).$$

Consequently, from (8.32) one immediately has (8.30)(i).

Let us introduce the vector function

$$\widehat{\eta}^\varepsilon_i = \left(\sum_{j=1}^N a^\varepsilon_{1j}\frac{\partial \widehat{w}^\varepsilon_i}{\partial x_j}, \ldots, \sum_{j=1}^N a^\varepsilon_{Nj}\frac{\partial \widehat{w}^\varepsilon_i}{\partial x_j}\right) = A^\varepsilon \nabla \widehat{w}^\varepsilon_i. \tag{8.33}$$

From (8.3) and (8.31), we have

$$\widehat{\eta}^\varepsilon_i(x) = \frac{1}{\varepsilon}\left[A\left(\frac{x}{\varepsilon}\right)(\nabla_y(\varepsilon \widehat{w}_i))\left(\frac{x}{\varepsilon}\right)\right] = (A\nabla_y \widehat{w}_i)\left(\frac{x}{\varepsilon}\right).$$

The same argument as that used to prove (8.14), shows that $\widehat{\eta}_i^\varepsilon$ satisfies

$$\int_\Omega \widehat{\eta}_i^\varepsilon \nabla v = 0, \quad \forall v \in H_0^1(\Omega). \tag{8.34}$$

From (6.14) one also has that

$$\nabla_y \widehat{w}_i = -\nabla_y \widehat{\chi}_i + e_i,$$

which is Y-periodic since $\widehat{\chi}_i$ is Y-periodic and e_i is constant. Therefore, we can apply Theorem 2.6 to obtain

$$\widehat{\eta}_i^\varepsilon \rightharpoonup \mathcal{M}_Y(A\nabla\widehat{w}_i) = A^0 e_i, \quad \text{weakly in } (L^2(\Omega))^N, \tag{8.35}$$

where we made use of (6.34). To conclude, observe that for any $i = 1, \dots, N$ one has

$$\widehat{\eta}_i^\varepsilon = A^\varepsilon C^\varepsilon e_i.$$

This equality, together with (8.35) implies convergence (8.30)(ii). □

A consequence of Proposition 8.5 and of convergence (8.28), is that

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \rightharpoonup 0 \quad \text{weakly in } (L^1(\Omega))^N. \tag{8.36}$$

Indeed, $C^\varepsilon \nabla u^0 \in L^1(\Omega)$ and for any $\varphi \in L^\infty(\Omega)$, from (8.30)(i) one has that

$$\int_\Omega C^\varepsilon \nabla u^0 \, \varphi \, dx \to \int_\Omega \nabla u^0 \, \varphi \, dx.$$

The interest of the corrector matrix C^ε is that convergence (8.36) is actually strong, as stated in Proposition 8.7 below. As a matter of fact, this result holds in the general non-periodic case and was proved by Murat and Tartar (1977a) (see also Cioranescu and Murat, 1982).

Theorem 8.6. *Let u^ε be the solution of problem (8.1) and u^0, A^0 given by Theorem 6.1. Then*

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } (L^1(\Omega))^N. \tag{8.37}$$

Moreover, if $C \in (L^r(Y))^{N\times N}$ for some r such that $2 \le r \le \infty$, and $\nabla u^0 \in (L^s(\Omega))^N$ for some s such that $2 \le s < \infty$, then

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } (L^t(\Omega))^N,$$

where

$$t = \min\left\{2, \frac{r\,s}{r+s}\right\}.$$

The proof of this result is based on the following proposition:

Proposition 8.7. *Let u^ε be the solution of problem (8.1) and u^0, A^0 given by Theorem 6.1. Then, there exists a positive constant c independent of ε, such that for any $\Phi \in (\mathcal{D}(\Omega))^N$, one has*

$$\limsup_{\varepsilon\to 0} \|\nabla u^\varepsilon - C^\varepsilon\Phi\|_{L^2(\Omega)} \leq c\,\|\nabla u^0 - \Phi\|_{L^2(\Omega)}.$$

Proof. Let $\Phi = (\Phi_1, \dots, \Phi_N) \in (\mathcal{D}(\Omega))^N$. From (8.3) and (8.4) one gets

$$\begin{aligned}
\alpha\,\|\nabla u^\varepsilon - C^\varepsilon\Phi\|^2_{L^2(\Omega)} &\leq \int_\Omega A^\varepsilon(\nabla u^\varepsilon - C^\varepsilon\Phi)(\nabla u^\varepsilon - C^\varepsilon\Phi)\,dx \\
&= \int_\Omega A^\varepsilon \nabla u^\varepsilon \nabla u^\varepsilon\,dx - \int_\Omega A^\varepsilon\nabla u^\varepsilon(C^\varepsilon\Phi)\,dx \\
&- \int_\Omega A^\varepsilon(C^\varepsilon\Phi)\nabla u^\varepsilon\,dx + \int_\Omega A^\varepsilon(C^\varepsilon\Phi)(C^\varepsilon\Phi)\,dx.
\end{aligned} \tag{8.38}$$

We will pass to the limit in all the terms in the right-hand side of this inequality.

The first term in the right-hand side is nothing else than the energy, so we can use Proposition 8.1 to obtain

$$\int_\Omega A^\varepsilon\nabla u^\varepsilon\;\nabla u^\varepsilon\,dx \longrightarrow \int_\Omega A^0\nabla u^0\;\nabla u^0\,dx. \tag{8.39}$$

To treat the second term, observe that from definition (8.29) of C^ε, one can write

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon\nabla u^\varepsilon\;(C^\varepsilon\Phi)\,dx &= \lim_{\varepsilon\to 0}\sum_{i=1}^N\int_\Omega A^\varepsilon\nabla u^\varepsilon\;(\Phi_i\nabla\widehat{w}_i^\varepsilon)\,dx \\
&= \sum_{i=1}^N\left(\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon\nabla u^\varepsilon\;\nabla(\Phi_i\,\widehat{w}_i^\varepsilon)\,dx - \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon\nabla u^\varepsilon\;\nabla\Phi_i\,\widehat{w}_i^\varepsilon\,dx\right).
\end{aligned}$$

Choosing $\Phi_i\,\widehat{w}_i^\varepsilon$ as test function in (8.1), one has

$$\int_\Omega A^\varepsilon\nabla u^\varepsilon\;\nabla(\Phi_i\,\widehat{w}_i^\varepsilon)\,dx = \langle f, \Phi_i\,\widehat{w}_i^\varepsilon\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Then, using convergences (8.5) and (8.32), one derives

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon\nabla u^\varepsilon\;(C^\varepsilon\Phi)\,dx \\
&= \sum_{i=1}^N\left(\lim_{\varepsilon\to 0}\langle f, \Phi_i\,\widehat{w}_i^\varepsilon\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon\nabla u^\varepsilon\;\nabla\Phi_i\,\widehat{w}_i^\varepsilon\,dx\right) \\
&= \sum_{i=1}^N\left(\langle f, \Phi_i\,x_i\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega A^0\nabla u^0\;\nabla\Phi_i\,x_i\,dx\right).
\end{aligned}$$

Using now $\Phi_i\, x_i$ as test function in (6.29), we finally get

$$\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla u^\varepsilon\ (C^\varepsilon \Phi)\, dx = \int_\Omega A^0 \nabla u^0\ \Phi\, dx. \tag{8.40}$$

To treat the third term of the right-hand side of (8.38), let us take $\Phi_i\, u^\varepsilon$ as test function in (8.34). We obtain, by taking into account convergences (8.5)ii and (8.35),

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon (C^\varepsilon \Phi)\ \nabla u^\varepsilon\, dx &= \sum_{i=1}^N \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla \widehat{w}_i^\varepsilon\ \nabla u^\varepsilon\ \Phi_i\, dx \\ &= \sum_{i=1}^N \lim_{\varepsilon\to 0}\left(\int_\Omega \widehat{\eta}_i^\varepsilon \cdot \nabla(\Phi_i u^\varepsilon)\, dx - \int_\Omega \widehat{\eta}_i^\varepsilon \cdot \nabla\Phi_i\ u^\varepsilon\, dx\right) \\ &= \sum_{i=1}^N \lim_{\varepsilon\to 0}\int_\Omega \widehat{\eta}_i^\varepsilon \cdot \nabla\Phi_i\ u^\varepsilon\, dx - \sum_{i=1}^N \int_\Omega A^0 e_i \cdot \nabla\Phi_i\ u^0\, dx \\ &= \int_\Omega A^0 \Phi \cdot \nabla u^0\, dx. \end{aligned} \tag{8.41}$$

For the last term in (8.38), we now choose $\Phi_i \Phi_j \widehat{w}_j^\varepsilon$ as test function in (8.34). Making use of (8.32) and (8.35), we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon (C^\varepsilon \Phi)(C^\varepsilon \Phi)\, dx &= \sum_{i,j=1}^N \lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon \nabla \widehat{w}_i^\varepsilon\ \nabla \widehat{w}_j^\varepsilon\ \Phi_i\ \Phi_j\, dx \\ &= \sum_{i,j=1}^N \lim_{\varepsilon\to 0}\left(\int_\Omega \widehat{\eta}_i^\varepsilon \cdot \nabla(\Phi_i\ \Phi_j\ \widehat{w}_j^\varepsilon)\, dx - \int_\Omega \widehat{\eta}_i^\varepsilon \cdot \nabla(\Phi_i\ \Phi_j)\ \widehat{w}_j^\varepsilon\, dx\right) \\ &= -\sum_{i,j=1}^N \int_\Omega A^0 e_i \cdot \nabla(\Phi_i \Phi_j)\ x_j\, dx = \int_\Omega A^0\, \Phi\, \Phi\, dx. \end{aligned} \tag{8.42}$$

Inserting (8.39), (8.40), (8.41) and (8.42) into (8.38), from Proposition 8.3 it follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \|\nabla u^\varepsilon - C^\varepsilon \Phi\|_{L^2(\Omega)} &\le \left[\frac{1}{\alpha}\int_\Omega A^0 (\nabla u^0 - \Phi)(\nabla u^0 - \Phi)\, dx\right]^{\frac12} \\ &\le \sqrt{\frac{\beta}{\alpha}}\ \|\nabla u^0 - \Phi\|_{L^2(\Omega)}, \end{aligned}$$

which ends the proof of Proposition 8.7. □

Proof of Theorem 8.6. Convergence (8.37) follows from Proposition 8.7 by a density argument. Let $\delta > 0$ and $\Phi_\delta \in (\mathcal{D}(\Omega))^N$ such that

$$\|\nabla u^0 - \Phi_\delta\|_{L^2(\Omega)} \le \delta.$$

The existence of such a Φ_δ is insured by Theorem 1.38. Consequently, by the triangular inequality, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \|\nabla u^\varepsilon - C^\varepsilon \nabla u^0\|_{L^1(\Omega)} \\
&\quad\le \limsup_{\varepsilon\to 0} [\|\nabla u^\varepsilon - C^\varepsilon \Phi_\delta\|_{L^1(\Omega)} + \|C^\varepsilon\Phi_\delta - C^\varepsilon\nabla u^0\|_{L^1(\Omega)}] \\
&\quad\le \limsup_{\varepsilon\to 0} c_1 \|\nabla u^\varepsilon - C^\varepsilon\Phi_\delta\|_{L^2(\Omega)} + c_2\|\nabla u^0 - \Phi_\delta\|_{L^2(\Omega)} \\
&\quad\le c c_1 \|\nabla u^0 - \Phi_\delta\|_{L^2(\Omega)} + c_2\,\delta \le c_3\,\delta.
\end{aligned}$$

where we have made use of convergence (8.30)(i), Proposition 1.14 and Proposition 8.7. This ends the proof of (8.37) since δ is an arbitrary constant.

To prove the second statement of Theorem 8.6, again let $\delta > 0$ and now choose $\Phi_\delta \in (\mathcal{D}(\Omega))^N$ such that

$$\|\nabla u^0 - \Phi_\delta\|_{L^s(\Omega)} \le \delta.$$

Then, taking into account the expression of t and using Proposition 8.7, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \|\nabla u^\varepsilon - C^\varepsilon \nabla u^0\|_{L^t(\Omega)} \\
&\qquad\le \limsup_{\varepsilon\to 0} \big[\|\nabla u^\varepsilon - C^\varepsilon\Phi_\delta\|_{L^t(\Omega)} + \|C^\varepsilon\Phi_\delta - C^\varepsilon\nabla u^0\|_{L^t(\Omega)}\big] \\
&\qquad\le \limsup_{\varepsilon\to 0} c_1 \big[\|\nabla u^\varepsilon - C^\varepsilon\Phi_\delta\|_{L^2(\Omega)} + \|C^\varepsilon\Phi_\delta - C^\varepsilon\nabla u^0\|_{L^{\frac{rs}{r+s}}(\Omega)}\big] \\
&\qquad\le c c_1 \|\nabla u^0 - \Phi_\delta\|_{L^2(\Omega)} + \limsup_{\varepsilon\to 0} c_1 \|C^\varepsilon\Phi_\delta - C^\varepsilon\nabla u^0\|_{L^{\frac{rs}{r+s}}(\Omega)} \\
&\qquad\le c_2 \|\nabla u^0 - \Phi_\delta\|_{L^s(\Omega)} + \limsup_{\varepsilon\to 0} c_1 \|C^\varepsilon\Phi_\delta - C^\varepsilon\nabla u^0\|_{L^{\frac{rs}{r+s}}(\Omega)},
\end{aligned}$$

since $t \le 2 \le s$ (see Corollary 1.35). Hence

$$\limsup_{\varepsilon\to 0} \|\nabla u^\varepsilon - C^\varepsilon \nabla u^0\|_{L^t(\Omega)} \le c_2\delta + \limsup_{\varepsilon\to 0} c_1 \|C^\varepsilon\Phi_\delta - C^\varepsilon\nabla u^0\|_{L^{\frac{rs}{r+s}}(\Omega)}.$$

From the assumption on C, definition (8.29), Theorem 2.6 and Proposition 1.14, it follows that C^ε is bounded in $(L^r(Y))^{N\times N}$. Consequently, making use of Hölder inequality (Proposition 1.34) with

$$p = \frac{r+s}{s}, \quad p' = \frac{r+s}{r},$$

one gets

$$\|C^\varepsilon\Phi_\delta - C^\varepsilon\nabla u^0\|_{L^{\frac{rs}{r+s}}(\Omega)} \le \|C^\varepsilon\|_{L^r(\Omega)} \|\nabla u^0 - \Phi_\delta\|_{L^s(\Omega)} \le c\,\delta.$$

This, used in the above estimate, ends the proof since δ is arbitrary. □

Remark 8.8. From regularity results due to Meyers (1963), there exists $r > 2$, depending on α, β, N and Y such that

$$C \in (L^r(Y))^{N\times N}.$$

Moreover, if $\partial\Omega$ is regular, classical results due to Agmon, Douglis, and Nirenberg (1959) (see also Ladyzhenskaya and Uraltseva, 1968, and Troianiello, 1987), imply in particular, that $\nabla u^0 \in (L^s(\Omega))^N$ for some $s > 2$. Hence, the last statement of Theorem 8.6 holds true for these r and s. ◊

Remark 8.9. For the one-dimensional case and the layered materials, studied in Sections 5.3 and 5.4, one can give the corrector matrix explicitly in view of the results of Section 6.5.

Indeed, for the one-dimensional case, Proposition 6.16 leads to

$$C(y) = \frac{1}{\mathcal{M}_{]0,\ell_1[}\left(\dfrac{1}{a}\right)} \frac{1}{a(y)} = \frac{a^0(y)}{a(y)}.$$

For the layered materials, from Proposition 6.18 (see also Remark 6.19), we have

$$C(y) = C(y_1) = \begin{pmatrix} \dfrac{a_{11}^0}{a_{11}(y_1)} & -\dfrac{a_{12}(y_1)}{a_{11}(y_1)} + \dfrac{a_{12}^0}{a_{11}(y_1)} \\ & \\ 0 & 1 \end{pmatrix}$$

◊

Remark 8.10. From definition (8.29) and Theorem 6.3, one can see that ∇u^0 can be written in the form

$$\begin{aligned} \nabla u^\varepsilon(x) &= \nabla u^0(x) - \sum_{k=1}^N \nabla_y \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right) \frac{\partial u^0}{\partial x_k}(x) - \varepsilon \sum_{k=1}^N \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right) \nabla\left(\frac{\partial u^0}{\partial x_k}\right)(x) + \cdots \\ &= C^\varepsilon(x)\nabla u^0(x) - \varepsilon \sum_{k=1}^N \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right) \nabla\left(\frac{\partial u^0}{\partial x_k}\right)(x) + \cdots \end{aligned}$$

so that $C^\varepsilon(x)\nabla u^0(x)$ is the first term in the asymptotic expansion of ∇u^ε in the sense of Theorem 6.3. ◊

8.4 Some comparison results

The aim of this section is to show how some comparison properties of two matrices in $M(\alpha, \beta, Y)$ are conserved by the homogenization process. The first one (Theorem 8.12), due to Tartar (1977a, 1978) (see also Bensoussan, Lions, and Papanicolaou, 1978, Chapter 1, Theorem 3.3), proves that under suitable assumptions, if two matrices are in a given order, this order is preserved by passing to the limit. The second one (Theorem 8.15) is a stability result due to Donato (1983a). For other comparison results we refer to Colombini and Spagnolo (1977) and Boccardo and Murat (1982).

Let us mention that all the results we prove here hold for the general non periodic case.

Definition 8.11. Let B and D be two $N \times N$ matrices. We say that B is *less than or equal to D in the matrix sense* and we write $B \le D$, iff

$$(B\lambda, \lambda) \le (D\lambda, \lambda),$$

for any $\lambda \in \mathbb{R}^N$.

Theorem 8.12. *Let B and D be two Y-periodic $N \times N$ matrices in $M(\alpha, \beta, Y)$, such that*

$$B \le D. \tag{8.43}$$

Suppose, furthermore, that B is symmetric. Then

$$B^0 \le D^0,$$

where B^0 and D^0 are the corresponding homogenized matrices given by Theorem 6.1 (all the inequalities are taken in the sense of Definition 8.11).

Proof. Let $w_{\lambda,B}$ and $w_{\lambda,D}$ be given by problem (6.25) written respectively for B and D. By Theorem 6.1 and using the symmetry of B one has, for any $\lambda \in \mathbb{R}^N$,

$$\begin{cases} {}^tD^0\lambda = \mathcal{M}_Y({}^tD\nabla w_{\lambda,D}) \\ B^0\lambda = \mathcal{M}_Y(B\nabla w_{\lambda,B}). \end{cases} \tag{8.44}$$

Set (see (8.10) and (8.12))

$$\begin{cases} w^\varepsilon_{\lambda,D}(x) = \varepsilon w_{\lambda,D}\left(\dfrac{x}{\varepsilon}\right), \quad w^\varepsilon_{\lambda,B}(x) = \varepsilon w_{\lambda,B}\left(\dfrac{x}{\varepsilon}\right) \\ \eta^\varepsilon_{\lambda,D} = {}^tD^\varepsilon \nabla w^\varepsilon_{\lambda,D}, \quad \eta^\varepsilon_{\lambda,B} = B^\varepsilon \nabla w^\varepsilon_{\lambda,B}, \end{cases} \tag{8.45}$$

where

$$D^\varepsilon(x) = D\left(\frac{x}{\varepsilon}\right), \quad B^\varepsilon(x) = B\left(\frac{x}{\varepsilon}\right), \quad \text{a.e. on } \mathbb{R}^N.$$

From (8.44) and from Section 8.1, we have the following convergences (see (8.11) and (8.13)):

$$\begin{cases} i) & w^\varepsilon_{\lambda,D} \rightharpoonup \lambda\cdot x & \text{weakly in } H^1(\Omega) \\ ii) & w^\varepsilon_{\lambda,D} \to \lambda\cdot x & \text{strongly in } L^2(\Omega) \\ iii) & \eta^\varepsilon_{\lambda,D} \rightharpoonup {}^tD^0\lambda & \text{weakly in } (L^2(\Omega))^N. \end{cases} \tag{8.46}$$

and

$$\begin{cases} i) & w^\epsilon_{\lambda,B} \rightharpoonup \lambda\cdot x & \text{weakly in } H^1(\Omega) \\ ii) & w^\epsilon_{\lambda,B} \to \lambda\cdot x & \text{strongly in } L^2(\Omega) \\ iii) & \eta^\epsilon_{\lambda,B} \rightharpoonup B^0\lambda & \text{weakly in } (L^2(\Omega))^N. \end{cases} \tag{8.47}$$

From assumption (8.43), and recalling that B is in $M(\alpha,\beta,Y)$ and is symmetric, it follows that

$$\begin{aligned} 0 &\le B^\epsilon\nabla(w^\epsilon_{\lambda,B}-w^\epsilon_{\lambda,D})\nabla(w^\epsilon_{\lambda,B}-w^\epsilon_{\lambda,D}) \\ &= B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,B} - 2B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,D} + B^\epsilon\nabla w^\epsilon_{\lambda,D}\,\nabla w^\epsilon_{\lambda,D}. \end{aligned}$$

Since from (8.43) one has

$$B^\epsilon \le D^\epsilon,$$

we get

$$0 \le B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,B} - 2B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,D} + D^\epsilon\nabla w^\epsilon_{\lambda,D}\,\nabla w^\epsilon_{\lambda,D}.$$

Consequently, for any $\varphi\in\mathcal{D}(\Omega)$, $\varphi\ge 0$ one has

$$\begin{aligned} 0 \le &\int_\Omega B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,B}\,\varphi\,dx - 2\int_\Omega B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,D}\,\varphi\,dx \\ &+\int_\Omega D^\epsilon\nabla w^\epsilon_{\lambda,D}\,\nabla w^\epsilon_{\lambda,D}\,\varphi\,dx. \end{aligned} \tag{8.48}$$

We can now pass to the limit for $\varepsilon\to 0$ in each term of this inequality. For the first term in the right-hand side, from (8.14) and (8.45) we have

$$\begin{aligned} \int_\Omega B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,B}\,\varphi\,dx &= \int_\Omega \eta^\epsilon_{\lambda,B}\,\nabla(w^\epsilon_{\lambda,B}\,\varphi)\,dx - \int_\Omega \eta^\epsilon_{\lambda,B}\,\nabla\varphi\; w^\epsilon_{\lambda,B}\,dx \\ &= -\int_\Omega \eta^\epsilon_{\lambda,B}\,\nabla\varphi\; w^\epsilon_{\lambda,B}\,dx. \end{aligned}$$

Hence, from convergences (8.47)ii and iii and integrating by parts, we have

$$\lim_{\epsilon\to 0}\int_\Omega B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,B}\,\varphi\,dx = -\int_\Omega B^0\lambda\nabla\varphi\,(\lambda\cdot x)\,dx = \int_\Omega (B^0\lambda,\lambda)\,\varphi\,dx, \tag{8.49}$$

since B^0 is a constant matrix.

Similarly, for the second term we get

$$\begin{aligned} \int_\Omega B^\epsilon\nabla w^\epsilon_{\lambda,B}\,\nabla w^\epsilon_{\lambda,D}\,\varphi\,dx &= \int_\Omega \eta^\epsilon_{\lambda,B}\,\nabla(w^\epsilon_{\lambda,D}\,\varphi)\,dx - \int_\Omega \eta^\epsilon_{\lambda,B}\,\nabla\varphi\; w^\epsilon_{\lambda,D}\,dx \\ &= -\int_\Omega \eta^\epsilon_{\lambda,B}\,\nabla\varphi\; w^\epsilon_{\lambda,D}\,dx, \end{aligned}$$

so that, from convergences (8.46)ii and (8.47)iii, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega B^\varepsilon \nabla w^\varepsilon_{\lambda,B}\,\nabla w^\varepsilon_{\lambda,D}\,\varphi\,dx = -\int_\Omega B^0\lambda\nabla\varphi\,(\lambda\cdot x)\,dx = \int_\Omega (B^0\lambda,\lambda)\,\varphi\,dx. \tag{8.50}$$

Finally, since

$$\int_\Omega D^\varepsilon \nabla w^\varepsilon_{\lambda,D}\,\nabla w^\varepsilon_{\lambda,D}\,\varphi\,dx = \int_\Omega {}^tD^\varepsilon \nabla w^\varepsilon_{\lambda,D}\,\nabla w^\varepsilon_{\lambda,D}\,\varphi\,dx,$$

from (8.14) and (8.45), we get by a similar computation

$$\int_\Omega D^\varepsilon \nabla w^\varepsilon_{\lambda,D}\,\nabla w^\varepsilon_{\lambda,D}\,\varphi\,dx = -\int_\Omega \eta^\varepsilon_{\lambda,D}\,\nabla\varphi\,w^\varepsilon_{\lambda,D}\,dx.$$

Consequently, from (8.46)ii and iii,

$$\lim_{\varepsilon\to 0}\int_\Omega D^\varepsilon \nabla w^\varepsilon_{\lambda,D}\,\nabla w^\varepsilon_{\lambda,D}\,\varphi\,dx = \int_\Omega (D^0\lambda,\lambda)\,\varphi\,dx. \tag{8.51}$$

Passing to the limit in (8.48) and using (8.49), (8.50) and (8.51), one obtains

$$0 \le -(B^0\lambda,\lambda)\int_\Omega \varphi\,dx + (D^0\lambda,\lambda)\int_\Omega \varphi\,dx.$$

This gives the result, since φ is a non-negative function in $\mathcal{D}(\Omega)$. □

Corollary 8.13. *Suppose that the matrix A is symmetric and let A^0 be given by Theorem 6.1. Then*

$$A^0 \in M(\alpha,\beta,\Omega).$$

Proof. The result is an immediate consequence of Theorem 8.12 applied to $B = A$ and $D = \beta I$ where I is the identity $N\times N$ identity matrix. Since $A \in M(\alpha,\beta,Y)$, (8.43) holds and since obviously, $(\beta I)^0 = \beta I$, one has

$$A^0 \le \beta I.$$

This, together with Proposition 8.3, ends the proof. □

Remark 8.14. Let us observe that the ellipticity condition (8.20)(i) proved in Proposition 8.3 can also be obtained from Theorem 8.12 applied to $B = \alpha I$ and $D = A$. ◊

Theorem 8.15. *Let B and D be two Y-periodic $N\times N$ matrices in $M(\alpha,\beta,Y)$ and B^0 and D^0 the corresponding homogenized matrices given by Theorem 6.1. Then, there exists a constant c and $q\in\mathbb{R}_+$ depending on α, β, N and Y, such that*

$$|b^0_{ij} - d^0_{ij}| \le c\left(\int_Y |a_{ij} - b_{ij}|\,dy\right)^{1/q}.$$

Proof. Set, as in the Proof of Theorem 8.12.

$$B^\varepsilon(x) = B\Big(\frac{x}{\varepsilon}\Big), \quad D^\varepsilon(x) = D\Big(\frac{x}{\varepsilon}\Big), \quad \text{a.e. on } \mathbb{R}^N,$$

and let C^ε_B and $C^\varepsilon_{{}^tD}$ be the corrector matrices corresponding respectively to B and ${}^tD$. They are defined by (8.29) and (8.45) written for $\lambda = e_i$, $i = 1, \dots, N$.

Let I be an arbitrary interval in $\mathbb{R}^N$ containing at least one translated set of Y. We prove first that

$$\begin{cases} {}^tC^\varepsilon_{{}^tD}\, B^\varepsilon\, C^\varepsilon_B \longrightarrow B^0 & \text{in } \mathcal{D}'(I) \\ {}^tC^\varepsilon_{{}^tD}\, D^\varepsilon\, C^\varepsilon_B \longrightarrow D^0 & \text{in } \mathcal{D}'(I). \end{cases} \tag{8.52}$$

(For the convergence in $\mathcal{D}'(I)$, see Definition 3.9). From the definition of corrector matrices, an easy matrix computation shows that, for any $i, j = 1, \dots, N$, the corresponding elements of matrices ${}^tC^\varepsilon_{{}^tD}\, B^\varepsilon\, C^\varepsilon_B$ and ${}^tC^\varepsilon_{{}^tD}\, D^\varepsilon\, C^\varepsilon_B$ are respectively,

$$\begin{cases} \big({}^tC^\varepsilon_{{}^tD}\, B^\varepsilon\, C^\varepsilon_B\big)_{ij} = \nabla w^\varepsilon_{i,{}^tD}\, B^\varepsilon\, \nabla w^\varepsilon_{j,B} = B^\varepsilon\, \nabla w^\varepsilon_{j,B}\, \nabla w^\varepsilon_{i,{}^tD} \\ \big({}^tC^\varepsilon_{{}^tD}\, D^\varepsilon\, C^\varepsilon_B\big)_{ij} = \nabla w^\varepsilon_{i,{}^tD}\, D^\varepsilon\, \nabla w^\varepsilon_{j,B} = {}^tD^\varepsilon\, \nabla w^\varepsilon_{i,{}^tD}\, \nabla w^\varepsilon_{j,B}. \end{cases}$$

The same computation as that used to prove (8.50) gives

$$\lim_{\varepsilon\to 0} \int_I B^\varepsilon \nabla w^\varepsilon_{j,B}\, \nabla w^\varepsilon_{i,{}^tD}\, \varphi\, dx = \int_I (B^0 e_j, e_i)\, \varphi\, dx = \int_I a^0_{ij}\, \varphi\, dx$$

$$\lim_{\varepsilon\to 0} \int_I {}^tD^\varepsilon \nabla w^\varepsilon_{i,{}^tD}\, \nabla w^\varepsilon_{j,B}\, \varphi\, dx = \int_I ({}^tD^0 e_i, e_j)\, \varphi\, dx = \int_I b^0_{ij}\, \varphi\, dx,$$

for any $i, j = 1, \dots, N$ and $\varphi \in \mathcal{D}(I)$. Hence, (8.52) is proved.

From Remark 8.8, we know that there exists $r > 2$ (depending on α, β, N and Y) such that C_B and $C_{{}^tD}$ are in $L^r(Y)^{N\times N}$. Consequently, from Theorem 2.6 and Remark 2.10, one deduces that there exists a constant c depending on α, β, N and Y, such that

$$\|C^\varepsilon_B\|^r_{L^r(I)} \le c|I|, \quad \|{}^tC^\varepsilon_{{}^tD}\|^r_{L^r(I)} \le c|I|.$$

Let η such that $1 < \eta < r$. Applying Hölder's inequality (Proposition 1.34), we have

$$\begin{aligned} \int_I |{}^tC^\varepsilon_{{}^tD}\, (B^\varepsilon - D^\varepsilon)\, C^\varepsilon_B|^\eta\, dx &\le \|({}^tC^\varepsilon_{{}^tD})^\eta\|_{L^{\frac{r}{\eta}}(I)}\, \|(B^\varepsilon - D^\varepsilon)^\eta\|_{L^s(I)}\, \|(C^\varepsilon_B)^\eta\|_{L^{\frac{r}{\eta}}(I)} \\ &= \|{}^tC^\varepsilon_{{}^tD}\|^\eta_{L^r(I)}\, \|(B^\varepsilon - D^\varepsilon)^\eta\|_{L^s(I)}\, \|C^\varepsilon_B\|^\eta_{L^r(I)} \\ &\le c|I|^{\frac{2\eta}{r}}\, \|B^\varepsilon - D^\varepsilon\|^\eta_{L^{\eta s}(I)}. \end{aligned}$$

with

$$\frac{1}{s} + \frac{2\eta}{r} = 1. \tag{8.53}$$

Again, by Remark 2.10, one has

$$\|B^\varepsilon - D^\varepsilon\|_{L^{\eta s}(I)} \le c|I|^{\frac{1}{\eta s}} \|B - D\|_{L^{\eta s}(Y)}.$$

Therefore, making also use of (8.53), it follows that

$$\int_I |{}^tC^\varepsilon_{tD}(B^\varepsilon - D^\varepsilon)C^\varepsilon_B|^\eta \, dx \le c|I| \, \|B - D\|^\eta_{L^{\eta s}(Y)}. \tag{8.54}$$

This shows that ${}^tC^\varepsilon_{tD}(B^\varepsilon - D^\varepsilon)C^\varepsilon_B$ is bounded in $(L^\eta(I))^{N\times N}$ so that, from Remark 1.45 (up to a subsequence), there exists a matrix P such that

$${}^tC^\varepsilon_{tD}(B^\varepsilon - D^\varepsilon)C^\varepsilon_B \rightharpoonup P \quad \text{weakly in } (L^\eta(I))^{N\times N}.$$

But (8.52) allows us to identify the limit P with $B^0 - D^0$. Therefore, the whole sequence converges, i.e.

$${}^tC^\varepsilon_{tD}(B^\varepsilon - D^\varepsilon)C^\varepsilon_B \rightharpoonup B^0 - D^0 \quad \text{weakly in } (L^\eta(I))^{N\times N}.$$

Recalling that B^0 and D^0 are constant, the lower semi-continuity of the norm in L^η (see Proposition 1.14), gives

$$|I|\,|B^0 - D^0|^\eta = \|B^0 - D^0\|^\eta_{L^\eta(I)} \le \liminf \|{}^tC^\varepsilon_{tD}(B^\varepsilon - D^\varepsilon)C^\varepsilon_B\|^\eta_{L^\eta(I)}.$$

This, together with (8.54), implies

$$|B^0 - D^0| \le c_1\|B - D\|_{L^{\eta s}(Y)} \le c_2\|B - D\|^{\frac{1}{\eta s}}_{L^1(Y)},$$

where c_2 depends on α, β, N and Y. This is the claimed result with $q = \eta s$. □

8.5 Case of weakly converging data

Let us recall that in problem (8.1) the right-hand side f is fixed in $H^{-1}(\Omega)$. A natural question is whether one can consider the case where f depends on ε. One can easily answer this question when the right-hand side converges either strongly in $H^{-1}(\Omega)$ or weakly in $L^2(\Omega)$. The result is contained in Theorem 8.16 below.

The situation is much more complicated if one has only weak convergence in $H^{-1}(\Omega)$. Theorem 8.19 deals with this case.

Theorem 8.16. *Let A^ε be defined by (8.2)–(8.4) and u^ε the solution of the problem*

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u^\varepsilon) = f^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{8.55}$$

Suppose that $\{f^\varepsilon\}$ *is a sequence verifying one of the two following assumptions:*

$$f^\varepsilon \to f \quad \text{strongly in } H^{-1}(\Omega), \tag{8.56}$$

or

$$f^\varepsilon \rightharpoonup f \quad \text{weakly in } L^2(\Omega). \tag{8.57}$$

Then,

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega), \\ A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0 & \text{weakly in } (L^2(\Omega))^N, \end{cases}$$

where u^0 *is the unique solution in* $H_0^1(\Omega)$ *of the homogenized problem*

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^0 \frac{\partial u^0}{\partial x_j}\right) = f & \text{in } \Omega \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

and the matrix $A^0 = (a_{ij}^0)_{1 \le i,j \le N}$ *is given by (6.35).*

Moreover, one has the convergence of energies, i.e.

$$E^\varepsilon(u^\varepsilon) = \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \, dx \longrightarrow E^0(u^0) = \int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, dx,$$

and also the convergence

$$A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \longrightarrow A^0 \nabla u^0 \, \nabla u^0 \quad \text{in } \mathcal{D}'(\Omega).$$

Finally, if C^ε *is the corrector matrix given by (8.29), then,*

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } (L^1(\Omega))^N.$$

If $C \in (L^r(Y))^{N\times N}$ *for some* r *such that* $2 \le r \le \infty$, *and* $\nabla u^0 \in (L^s(\Omega))^N$ *for some* s *such that* $2 \le s < \infty$, *then*

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } (L^t(\Omega))^N,$$

where

$$t = \min\left\{2, \frac{r\,s}{r+s}\right\}.$$

Proof. The proof follows exactly the same outline as that of Theorem 6.1, Propositions 8.1 and 8.2 as well as Theorem 8.6 given in the previous section of this chapter. The only difference is that in all the terms containing f, we have to replace it by f^ε. Assumptions (8.56) or (8.57) allow us to pass to the limit without any difficulty in all the terms we have to treat. □

As mentioned before, the result is completely different in the case where we have only a weak convergence of f^ε in $H^{-1}(\Omega)$. In this case two main features appear: first, we do not have a convergence result for the whole sequence $\{u^\varepsilon\}$. Secondly, the right-hand side is not the weak limit of f^ε but a function defined in a complicated way in terms of the corrector functions w_i^ε. This result is due to L. Tartar. In order to state it, we have to introduce some auxiliary problems.

Assume that f^ε is such that

$$f^\varepsilon \rightharpoonup f \quad \text{weakly in } H^{-1}(\Omega). \tag{8.58}$$

Let $\rho_\varepsilon \in H_0^1(\Omega)$ be the solution of the problem

$$\begin{cases} -\Delta\rho_\varepsilon = f^\varepsilon & \text{in } \Omega \\ \rho_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.59}$$

Let us introduce for $i = 1, \dots, N$ the following functions:

$$g_i^\varepsilon = \nabla w_\varepsilon^i \nabla \rho^\varepsilon. \tag{8.60}$$

where w_ε^i is defined by (8.10).

From Proposition 1.14, convergences (8.32) and assumption (8.58), we know that

$$\begin{cases} i) & \|f^\varepsilon\|_{H^{-1}(\Omega)} \le c \\ ii) & \|w_\varepsilon^i\|_{H^1(\Omega)} \le c. \end{cases} \tag{8.61}$$

where c is a constant independent of ε. Then. Theorem 4.16 shows that

$$\|\rho_\varepsilon\|_{H_0^1(\Omega)} \le c. \tag{8.62}$$

Then, by using Hölder inequality in definition (8.60). from (8.61)(ii) we have the estimate

$$\|g_i^\varepsilon\|_{L^1(\Omega_\varepsilon)} \le c. \quad i = 1, \dots, N.$$

From Proposition 1.48 there exists a subsequence ε' such that

$$g_i^{\varepsilon'} \rightharpoonup g_i^* \quad \text{weakly* in } M(\Omega). \tag{8.63}$$

The following result characterizes the divergence of g^*:

Proposition 8.17. *Let g^* be defined by (8.63). Under assumption (8.58) one has*

$$\begin{cases} \displaystyle\int_\Omega g^* \nabla\varphi \, dx = \lim_{\varepsilon' \to 0} \sum_{i=1}^N \left\langle f^{\varepsilon'}, w_i^{\varepsilon'} \frac{\partial \varphi}{\partial x_i} \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ \quad + \langle f, (\varphi - x \cdot \nabla\varphi)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad \text{for any } \varphi \in \mathcal{D}(\Omega). \end{cases} \tag{8.64}$$

Proof. Let $\Phi = (\Phi_1, \dots \Phi_N) \in (\mathcal{D}(\Omega))^N$. Then

$$\begin{aligned}
\int_\Omega g^\varepsilon \cdot \Phi \, dx &= \sum_{i=1}^N \int_\Omega \nabla w_i^\varepsilon \nabla \rho_\varepsilon \Phi_i \, dx \\
&= \sum_{i=1}^N \int_\Omega \nabla \rho_\varepsilon \nabla (w_i^\varepsilon \Phi_i) \, dx - \sum_{i=1}^N \int_\Omega \nabla \rho_\varepsilon \, w_i^\varepsilon \, \nabla \Phi_i \, dx \\
&= \sum_{i=1}^N \langle f^\varepsilon, w_i^\varepsilon \Phi_i \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \sum_{i=1}^N \int_\Omega \nabla \rho_\varepsilon \, w_i^\varepsilon \, \nabla \Phi_i \, dx.
\end{aligned}$$

We will now pass to the limit as $\varepsilon \to 0$. To begin with, from (8.63), one has obviously

$$\lim_{\varepsilon' \to 0} \int_\Omega g^{\varepsilon'} \cdot \Phi \, dx = \int_\Omega g^\star \cdot \Phi \, dx.$$

On the other hand, observe that due to (8.58) one can easily pass to the limit in (8.59) to get that

$$\rho_\varepsilon \rightharpoonup \rho \quad \text{weakly in } H_0^1(\Omega),$$

where ρ satisfies the limit problem

$$\begin{cases} -\Delta \rho = f & \text{in } \Omega \\ \rho = 0 & \text{on } \partial\Omega. \end{cases}$$

Recall that w_i^ε satisfies

$$\begin{cases} i) & w_i^\varepsilon \rightharpoonup x_i \quad \text{weakly in } H^1(\Omega) \\ ii) & w_i^\varepsilon \to x_i \quad \text{strongly in } L^2(\Omega). \end{cases} \tag{8.65}$$

Consequently,

$$\begin{aligned}
\lim_{\varepsilon' \to 0} \sum_{i=1}^N \int_\Omega \nabla \rho_{\varepsilon'} \, w_i^{\varepsilon'} \, \nabla \Phi_i \, dx &= \sum_{i=1}^N \int_\Omega \nabla \rho \, x_i \, \nabla \Phi_i \, dx \\
&= \sum_{i=1}^N \int_\Omega \nabla \rho \, \nabla (x_i \Phi_i) \, dx - \sum_{i=1}^N \int_\Omega \nabla \rho \, \nabla x_i \, \Phi_i \, dx \\
&= \sum_{i=1}^N \langle f, x_i \Phi_i \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \sum_{i=1}^N \int_\Omega \frac{\partial \rho}{\partial x_i} \Phi_i \, dx \\
&- \langle f, x \cdot \Phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega \nabla \rho \cdot \Phi \, dx.
\end{aligned}$$

Putting together this information we have

$$\begin{aligned}
\int_\Omega g^\star \cdot \Phi \, dx &= \lim_{\varepsilon' \to 0} \sum_{i=1}^N \langle f^{\varepsilon'}, w_i^{\varepsilon'} \Phi_i \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\
&- \langle f, x \cdot \Phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega \nabla \rho \cdot \Phi \, dx.
\end{aligned}$$

so that

$$\begin{cases} \displaystyle\int_\Omega (g^* - \nabla\rho)\cdot\Phi\,dx \\ \displaystyle\quad = \lim_{\varepsilon'\to 0}\sum_{i=1}^{N}\langle f^{\varepsilon'}, w_i^{\varepsilon'}\Phi_i\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \langle f, x\cdot\Phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \end{cases}$$

Choosing here $\Phi = \nabla\varphi$ with $\varphi \in \mathcal{D}(\Omega)$ and using the equation satisfied by ρ, we get the desired result. □

Remark 8.18. An interesting (and quite surprising) consequence of formula (8.64) is the fact that the function div g^* is independent of ρ^ε, since it only depends on $f^{\varepsilon'}$, $w_i^{\varepsilon'}$ and f. This means, in particular, that in definition (8.59) of ρ^ε we can choose any elliptic operator instead of $-\Delta$. For instance, let us define $\bar\rho_\varepsilon$ by

$$\begin{cases} -\text{div}\,(B\nabla\bar\rho_\varepsilon) = f^\varepsilon & \text{in } \Omega \\ \bar\rho_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

for an arbitrary $B \in M(\alpha, \beta, \Omega)$. Set

$$\bar g_i^\varepsilon = \nabla w_\varepsilon^i\, B\nabla\bar\rho^\varepsilon.$$

Then, as before, there exists a subsequence ε'' such that

$$g_i^{\varepsilon''} \rightharpoonup \bar g_i^* \quad \text{weakly* in } M(\Omega).$$

One can follow step by step the proof of Proposition 8.17 (replacing everywhere $\nabla\rho_\varepsilon$ by $B\nabla\bar\rho_\varepsilon$) to get again (8.64), written for div $\bar g^*$ and the subsequence ε'', i.e.

$$\begin{cases} \displaystyle\int_\Omega \bar g^*\nabla\varphi\,dx = \lim_{\varepsilon''\to 0}\sum_{i=1}^{N}\left\langle f^{\varepsilon''}, w_i^{\varepsilon''}\frac{\partial\varphi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ \qquad + \langle f, (\varphi - x\cdot\nabla\varphi)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \quad \text{for any } \varphi \in \mathcal{D}(\Omega). \end{cases}$$

□

We are now able to formulate the following result:

Theorem 8.19. *Let A^ε be defined by (8.2)–(8.4) and u^ε be the solution of problem (8.55), i.e.*

$$\begin{cases} -\text{div}\,(A^\varepsilon\nabla u^\varepsilon) = f^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\{f^\varepsilon\}$ a sequence satisfying (8.58).

Then, there exists a subsequence ε' such that

$$u^{\varepsilon'} \rightharpoonup u^* \quad \text{weakly in } H_0^1(\Omega).$$

where $u^\star$ is the unique solution in $H_0^1(\Omega)$ of the homogenized problem

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^0 \frac{\partial u^\star}{\partial x_j}\right) = -\text{div } g^\star \quad \text{in } \Omega \\ u^\star = 0 \quad \text{on } \partial\Omega. \end{cases}$$

where

$$g^\star = (g_1^\star, \dots, g_N^\star).$$

defined by (8.63), belongs to the space $(L^2(\Omega))^N$ and a_{ij}^0 is given by (6.35). Furthermore, if the sequence $\{f^\varepsilon\}$ satisfies (8.56) or (8.57), then

$$-\text{div } g^\star = f. \quad u^\star = u^0.$$

where u^0 is given by Theorem 8.16.

Proof. Obviously, from (8.61) we have the a priori estimate

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \leq c.$$

where c is independent of ε. Introduce the vector σ^ε by setting

$$\sigma^\varepsilon = A^\varepsilon \nabla u^\varepsilon - \nabla \rho_\varepsilon.$$

From (8.55) and (8.59), we have

$$-\text{div } \sigma^\varepsilon = 0 \quad \text{in } \Omega. \tag{8.66}$$

Moreover, from (8.62) it follows that

$$\|\sigma^\varepsilon\|_{(L^2(\Omega))^N} \leq c, \tag{8.67}$$

independently of ε. Consequently, there exists a subsequence of ε' still denoted by ε', such that

$$\begin{cases} i) & u^{\varepsilon'} \rightharpoonup u^\star \quad \text{weakly in } H_0^1(\Omega) \\ ii) & u^{\varepsilon'} \to u^\star \quad \text{strongly in } L^2(\Omega) \\ iii) & \sigma^{\varepsilon'} \rightharpoonup \sigma^\star \quad \text{weakly in } (L^2(\Omega))^N, \end{cases} \tag{8.68}$$

with

$$-\text{div } \sigma^\star = 0 \quad \text{in } \Omega. \tag{8.69}$$

We now will identify the limit $\sigma^\star$ by using the definition of $\sigma^{\varepsilon'}$. We will show that

$$\sigma^\star = A^0 \nabla u^\star - g^\star, \tag{8.70}$$

which will imply that actually $g^\star \in (L^2(\Omega))^N$, whence $-\text{div } g^\star \in H^{-1}(\Omega)$. This, together with (8.66), (8.69) and (8.70), proves that $u^\star$ satisfies the homogenized problem from the statement of the theorem. The Lax–Milgram theorem applied to this problem guarantees the uniqueness of its solution. Consequently, convergences (8.68) will hold for the whole sequence ε'.

Let now prove (8.70). To do so, set

$$\tau_i^{\varepsilon'} = \sigma^{\varepsilon'} \nabla w_i^{\varepsilon'}, \quad i = 1, \dots, N, \tag{8.71}$$

where $w_i^{\varepsilon'}$ are the functions defined by (8.10). Recall that, by construction, $w_i^{\varepsilon'}$ satisfy, in particular,

$$-\text{div}\,({}^tA^{\varepsilon'} \nabla w_i^{\varepsilon'}) = 0 \quad \text{in } \Omega \tag{8.72}$$

and convergences (8.65). Then, due also to (8.67), we see from (8.71) that $\tau_i^{\varepsilon'} \in L^1(\Omega)$. Then, for $\varphi \in \mathcal{D}(\Omega)$ we can consider the integral

$$I_{\varepsilon'} = \int_\Omega \tau_i^{\varepsilon'} \varphi \, dx.$$

One has, by definition

$$I_{\varepsilon'} = \int_\Omega A^{\varepsilon'} \nabla u^{\varepsilon'} \, \nabla w_i^{\varepsilon'} \varphi \, dx - \int_\Omega \nabla \rho_{\varepsilon'} \nabla w_i^{\varepsilon'} \varphi \, dx = \int_\Omega \sigma^{\varepsilon'} \nabla w_i^{\varepsilon'} \varphi \, dx. \tag{8.73}$$

On one hand, from (8.66), we have

$$I_{\varepsilon'} = \int_\Omega \sigma^{\varepsilon'} \nabla (w_i^{\varepsilon'} \varphi) \, dx - \int_\Omega \sigma^{\varepsilon'} \, w_i^{\varepsilon'} \, \nabla \varphi \, dx = - \int_\Omega \sigma^{\varepsilon'} \, w_i^{\varepsilon'} \, \nabla \varphi \, dx.$$

We can pass to the limit in the last integral by using convergences (8.65)ii and (8.66)iii to get

$$\lim_{\varepsilon' \to 0} I_{\varepsilon'} = - \int_\Omega \sigma^\star \, x_i \, \nabla \varphi \, dx = - \int_\Omega \sigma^\star \, \nabla (x_i \, \varphi) \, dx + \int_\Omega \sigma^\star \, e_i \, \varphi \, dx = \int_\Omega \sigma_i^\star \, \varphi \, dx,$$

where we have also made use of equation (8.69).

On the other hand, (8.73) can be rewritten in an other form as follows:

$$\begin{aligned}
I_{\varepsilon'} &= \int_\Omega {}^tA^{\varepsilon'} \nabla w_i^{\varepsilon'} \, \nabla u^{\varepsilon'} \, \varphi \, dx - \int_\Omega \nabla \rho_{\varepsilon'} \nabla w_i^{\varepsilon'} \, \varphi \, dx \\
&= \int_\Omega {}^tA^{\varepsilon'} \nabla w_i^{\varepsilon'} \, \nabla (u^{\varepsilon'} \, \varphi) \, dx - \int_\Omega {}^tA^{\varepsilon'} \nabla w_i^{\varepsilon'} \, u^{\varepsilon'} \, \nabla \varphi \, dx - \int_\Omega \nabla \rho_{\varepsilon'} \nabla w_i^{\varepsilon'} \, \varphi \, dx \\
&= - \int_\Omega {}^tA^{\varepsilon'} \nabla w_i^{\varepsilon'} \, u^{\varepsilon'} \, \nabla \varphi \, dx - \int_\Omega \nabla \rho_{\varepsilon'} \nabla w_i^{\varepsilon'} \, \varphi \, dx.
\end{aligned}$$

where we used (8.72) with $u^{\varepsilon'} \varphi$ as test function. In view of (8.13), (8.60), (8.63) and (8.68)ii, we can pass to the limit in all the terms above to obtain

$$\begin{aligned}
\lim_{\varepsilon' \to 0} I_{\varepsilon'} &= - \int_\Omega {}^tA^0 e_i \, u^\star \, \nabla \varphi \, dx - \int_\Omega g_i^\star \, \varphi \, dx \\
&= \int_\Omega (A^0 u^\star)_i \, \nabla \varphi \, dx - \int_\Omega g_i^\star \, \varphi \, dx \\
&= \int_\Omega (A^0 \nabla u^\star)_i \, \varphi \, dx - \int_\Omega g_i^\star \, \varphi \, dx.
\end{aligned}$$

Hence, using the former limit of I_ε, we have

$$\int_\Omega \sigma_i^* \varphi \, dx = \int_\Omega (A^0 \nabla u^*)_i \, \varphi \, dx - \int_\Omega g_i^* \, \varphi \, dx.$$

Since this is true for any $i = 1, \dots, N$ and φ is arbitrary in $\mathcal{D}(\Omega)$, this implies (8.70).

It remains to prove the last assertion of the theorem. To do so, observe now that, if either (8.56) or (8.57) holds, then in view again of (8.65),

$$\begin{aligned}
\lim_{\varepsilon' \to 0} \sum_{i=1}^N \left\langle f^{\varepsilon'}, w_i^{\varepsilon'} \frac{\partial \varphi}{\partial x_i} \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= \sum_{i=1}^N \left\langle f, x_i \frac{\partial \varphi}{\partial x_i} \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\
&= \langle f, x \cdot \nabla \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)},
\end{aligned}$$

which, used in (8.64), yields

$$\langle -\mathrm{div}\, g^*, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega g^* \, \nabla \varphi \, dx = \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Since φ is arbitrary in $\mathcal{D}(\Omega)$, this implies

$$-\mathrm{div}\, g^* = f.$$

Hence u^* solves the homogenized problem (6.29), so by uniqueness $u^* = u^0$ and the whole sequence $\{u^\varepsilon\}$ converges to u^0.

The proof of Theorem 8.19 is complete. □

Remark 8.20. Observe that under assumption (8.58), the convergence of u^ε can only be formulated for the subsequence ε'. This is due to the fact that convergence (8.63) holds in general only for a subsequence. ◊

Remark 8.21. Let us mention that the set of possible limits u^* does not depend on the choice of ρ^ε since the limit problem in Theorem 8.19 is written for the data $-\mathrm{div}\, g^*$. Indeed, as shown in Remark 8.18, $-\mathrm{div}\, g^*$ (and consequently u^*) depends only on $f^{\varepsilon'}$, $w_i^{\varepsilon'}$, and f. ◊

8.6 Convergence of eigenvalues

This section is devoted to the study of the eigenvalue problem and its behaviour as $\varepsilon \to 0$. The result we present in this section is contained in a general one, given by Boccardo and Marcellini (1976) concerning sequences of matrices in $M(\alpha, \beta, Y)$ (with no periodicity assumption). We give for our periodic frame a direct proof following that of Kesavan (1979).

In all this section, we suppose that the matrix $A \in M(\alpha, \beta, \Omega)$ is symmetric, i.e.

$$a_{ij} = a_{ji}, \quad \text{for } i, j = 1, \dots, N.$$

Let us recall the general definition of eigenvalues and eigenvectors.

Definition 8.22. Assume that B is a $N \times N$ symmetric matrix in $M(\alpha, \beta, \Omega)$. The constant λ is an *eigenvalue* of the operator $\mathcal{B} = -\text{div}\,(B\nabla)$ with Dirichlet boundary condition, if there exists $u \not\equiv 0$, a solution of the problem

$$\begin{cases} \mathcal{B}u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.74}$$

The function u is called an *eigenfunction* of $\mathcal{B}$, associated with the eigenvalue λ. The set of the eigenvalues is called the *spectrum* of $\mathcal{B}$.

The vector space of solutions of (8.74) for any fixed λ in the spectrum of $\mathcal{B}$, denoted by $\mathcal{E}(\lambda)$, is called the *eigenspace* associated with λ.

Moreover, an eigenvalue λ is called *simple* if the corresponding eigenspace is of dimension one.

It is easily seen that the symmetry assumption implies that the eigenvalues λ (if they exist) are all real. Consequently, the variational formulation of (8.74) is

$$\begin{cases} \text{Find } (u, \lambda) \in [H_0^1(\Omega) \setminus \{0\}] \times \mathbb{R} \text{ such that} \\ \displaystyle\int_\Omega B\nabla u\, \nabla v\, dx = \lambda \int_\Omega u\, v\, dx, \\ \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{8.75}$$

The following result is classical (see for instance Courant and Hilbert, 1962):

Proposition 8.23. *Assume that B is a symmetric matrix in $M(\alpha, \beta, \Omega)$. One has the following properties:*

i) The spectrum of $\mathcal{B}$ is a countable subset of $\mathbb{R}_+^*$ whose unique accumulation point is $+\infty$.

ii) For any eigenvalue λ, the corresponding eigenspace $\mathcal{E}(\lambda)$ is of finite dimension.

iii) The space $L^2(\Omega)$ is a Hilbert sum of all the eigenspaces of $\mathcal{B}$.

Remark 8.24. In view of Proposition 8.23, one describes the spectrum of $\mathcal{B}$ as a increasing sequence $\{\lambda_n\}$ with

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty,$$

where each eigenvalue is repeated as many times as the dimension of its corresponding eigenspace.

Consequently, from (iii) of Proposition 8.23 there exists a corresponding sequence of eigenfunctions $\{u_n\}$ which forms an orthonormal basis in $L^2(\Omega)$. This means (see Rudin, 1966) that

$$\int_\Omega u_i\, u_j\, dx = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol, and

$$v = \sum_{k=1}^{\infty} \left(\int_\Omega v\, u_k \, dx \right) u_k, \quad \forall v \in L^2(\Omega).$$

Observe also that two eigenfunctions corresponding to two different eigenvalues are orthogonal. ◇

One has the following characterization of the eigenvalues (see for instance Courant and Hilbert, 1962):

Proposition 8.25. *Assume that B is a symmetric matrix in $M(\alpha, \beta, \Omega)$ and let (λ_n) be the spectrum of $\mathcal{B}$ and (u_n) the basis of eigenfunctions introduced in Remark 8.24. For any $\ell \geq 1$, let W_ℓ be the space spanned by the first ℓ eigenfunctions $u_1, \ldots, u_\ell$. Then, one has the following characterization:*

$$\lambda_\ell = \max_{w \in W_\ell} \frac{\displaystyle\int_\Omega B\nabla w \, \nabla w \, dx}{\displaystyle\int_\Omega w^2 \, dx} = \min_{w \perp W_{\ell-1}} \frac{\displaystyle\int_\Omega B\nabla w \, \nabla w \, dx}{\displaystyle\int_\Omega w^2 \, dx}$$

$$= \min_{W \in D_\ell} \max_{w \in W} \frac{\displaystyle\int_\Omega B\nabla w \, \nabla w \, dx}{\displaystyle\int_\Omega w^2 \, dx}, \tag{8.76}$$

where

$$D_\ell = \{ W \subset H_0^1(\Omega) \,|\, \dim W = \ell \}.$$

Let now A^ε be defined by (8.3) and (8.4) and the corresponding homogenized matrix A_0 given by Theorem 6.1. Obviously, A^ε is symmetric and by Corollary 6.10, A_0 is symmetric too. Consequently, Propositions 8.23 and 8.25 hold for both $B = A^\varepsilon$ and $B = A_0$.

Denote by $\{\lambda_\ell^\varepsilon\}$ the sequence of eigenvalues of the operator $\mathcal{A}^\varepsilon = -\text{div}\,(A^\varepsilon \nabla)$ and let $\{u_\ell^\varepsilon\}$ be the corresponding sequence of eigenfunctions provided by Remark 8.24. They are solutions of the problem

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\ell^\varepsilon) = \lambda_\ell^\varepsilon \, u_\ell^\varepsilon & \text{in } \Omega \\ u_\ell^\varepsilon = 0 \quad \text{on } \partial\Omega, \\ \|u_\ell^\varepsilon\|_{L^2(\Omega)} = 1. \end{cases} \tag{8.77}$$

Similarly, introduce the spectrum $\{\lambda_\ell\}$ of the operator $\mathcal{A}^0 = -\text{div}\,(A^0 \nabla)$ given by Proposition 8.23.

The natural question is if $\{\lambda_\ell\}$ is the limit of $\{\lambda_\ell^\varepsilon\}$ as $\varepsilon \to 0$. The following result (see Boccardo and Marcellini, 1976, and Kesavan, 1979), gives a complete description of the asymptotic behaviour of the spectrum of $\mathcal{A}^\varepsilon$:

Theorem 8.26. *With the above notations, one has the following properties:*

i) For each ℓ fixed,

$$\lambda_\ell^\varepsilon \to \lambda_\ell.$$

ii) There exists a subsequence ε' such that

$$u_\ell^{\varepsilon'} \rightharpoonup u_\ell \quad \text{weakly in } H_0^1(\Omega),$$

where u_ℓ is an eigenfunction corresponding to λ_ℓ. The set (u_ℓ) is an orthonormal basis of $L^2(\Omega)$.

iii) If the eigenvalue λ_ℓ is simple, then the whole sequence $\{u_\ell^\varepsilon\}$ converges to u_ℓ.

Proof. The proof is done in several steps.
Step 1. We first show that the sequence $\{\lambda_\ell^\varepsilon\}$ is bounded independently of ε. To do so, we make use of the characterization (8.76) from Proposition 8.25.

Let (w_k) be an orthonormal basis, corresponding to $\mathcal{B} = \mathcal{A}^0$ as in Remark 1.18 and $W_\ell = [w_1, \dots, w_\ell]$ be the subspace generated by $w_1, \dots, w_\ell$. Then, using (8.76) for $\mathcal{B} = \mathcal{A}^\varepsilon$, we have

$$\begin{aligned}\lambda_\ell^\varepsilon &= \min_{W \in D_\ell} \max_{w \in W} \frac{\displaystyle\int_\Omega A^\varepsilon \nabla w \nabla w \, dx}{\displaystyle\int_\Omega w^2 \, dx} \le \max_{v \in W_\ell} \frac{\displaystyle\int_\Omega A^\varepsilon \nabla v \nabla v \, dx}{\displaystyle\int_\Omega v^2 \, dx} \\ &\le \beta \max_{v \in W_\ell} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx}{\displaystyle\int_\Omega v^2 \, dx} \le \frac{\beta}{\alpha} \max_{v \in W_\ell} \frac{\displaystyle\int_\Omega A^0 \nabla v \nabla v \, dx}{\displaystyle\int_\Omega v^2 \, dx},\end{aligned} \tag{8.78}$$

where we used assumption (8.4) and Proposition 8.3. This, together with (8.76), gives

$$\lambda_\ell^\varepsilon \le \frac{\beta}{\alpha} \lambda_\ell. \tag{8.79}$$

i.e. the sequence $\{\lambda_\ell^\varepsilon\}$ is bounded independently of ε. Hence, for a subsequence ε'', one has the convergence

$$\lambda_\ell^{\varepsilon''} \to \Lambda_\ell. \tag{8.80}$$

The fact that $\Lambda_\ell = \lambda_\ell$ (and that the whole sequence $\lambda_\ell^{\varepsilon''}$ converges to λ_ℓ) will be shown in Step 3.

Step 2. In this step we prove the convergence from statement (ii). Recalling (8.77) and (8.79), from Theorem 4.16 we have the estimate

$$\|u_\ell^\varepsilon\|_{H_0^1(\Omega)} \le \frac{C_\Omega}{\alpha} \lambda_\ell^\varepsilon \le \frac{\beta C_\Omega}{\alpha^2} \lambda_\ell.$$

Consequently, there exists a subsequence ε' such that

$$\begin{cases} u_\ell^{\varepsilon'} \rightharpoonup u_\ell & \text{weakly in } H_0^1(\Omega) \\ u_\ell^{\varepsilon'} \to u_\ell & \text{strongly in } L^2(\Omega) \\ \lambda_\ell^{\varepsilon'} \to \Lambda_\ell, \end{cases} \tag{8.81}$$

where we have also used (8.80).

Now, Theorem 8.16, written for $f^\varepsilon = \lambda_\ell^\varepsilon u_\ell^\varepsilon$, implies in particular, that u_ℓ satisfies

$$\begin{cases} -\text{div}\,(A^0 \nabla u_\ell) = \Lambda_\ell u_\ell & \text{in } \Omega \\ \quad u_\ell = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{8.82}$$

which proves that u_ℓ is an eigenvector of $\mathcal{A}^0$ corresponding to the eigenvalue Λ_ℓ.

On the other hand, by using (8.81), we can pass to the limit in the following identity:

$$\int_\Omega u_i^\varepsilon\, u_j^\varepsilon\, dx = \delta_{ij}.$$

So, the set (u_ℓ) is orthonormal in $L^2(\Omega)$, i.e.

$$\int_\Omega u_i^\varepsilon\, u_j^\varepsilon\, dx = \delta_{ij}. \tag{8.83}$$

Obviously, this implies that these functions are linearly independent.

Step 3. We now prove that $\mathcal{A}^0$ has no other eigenvalues except those defined by (8.80), (8.82) and (8.83). This will complete the proof of (i) and (ii) since the eigenvalues are ordered increasingly.

We argue by contradiction. Suppose that there exists an eigenfunction w corresponding to some eigenvalue λ, i.e. satisfying

$$\begin{cases} \mathcal{A}^0 w = \lambda w & \text{in } \Omega \\ w^0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.84}$$

and which is not given by (8.80), (8.82) and (8.83). Then, w does not belong to any subspace generated by a finite family of linearly independent eigenfunctions u_ℓ obtained above. Indeed, suppose that $w = \sum_{i=1}^m c_i u_i$, where $c_i \neq 0$ are constants. Then, from (8.82) and (8.84), one has

$$\lambda \sum_{i=1}^m c_i u_i = \lambda w = \mathcal{A}^0 w = \mathcal{A}^0 \sum_{i=1}^m c_i u_i = \sum_{i=1}^m c_i \Lambda_i u_i.$$

Hence,

$$\sum_{i=1}^m (\lambda - \Lambda_i) c_i u_i = 0,$$

which can hold only if $\lambda = \Lambda_i$ since u_i are linearly independent. But this is not possible due to the assumption on w.

Now, since w does not belong to any subspace generated by the family (u_ℓ), due to Remark 8.24, w is orthogonal to this family. From the property (i) of Proposition 8.23, there exists an ℓ_0 such that

$$\Lambda_{\ell_0+1} > \lambda. \tag{8.85}$$

Let us introduce U^ϵ, a solution of the following problem:

$$\begin{cases} \mathcal{A}^\epsilon U^\epsilon = -\text{div}\,(A^\epsilon \nabla U^\epsilon) = \lambda w & \text{in } \Omega \\ U^\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.86}$$

We can apply Theorem 6.1 to this problem. So,

$$U^\epsilon \rightharpoonup U^0 \quad \text{weakly in } H_0^1(\Omega),$$

and U^0 is the unique solution in $H_0^1(\Omega)$ of the homogenized problem

$$\begin{cases} \mathcal{A}^0 U^0 = \lambda w & \text{in } \Omega \\ U^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

From (8.84), the uniqueness implies that

$$w = U^0,$$

so that

$$U^\epsilon \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega). \tag{8.87}$$

Since (u_ℓ^ϵ) is an orthonormal basis in $L^2(\Omega)$, one has (see Rudin, 1966)

$$U^\epsilon = \sum_{k=1}^{\infty} \left(\int_\Omega U^\epsilon u_k^\epsilon \, dx \right) u_k^\epsilon.$$

Set

$$v_\varepsilon = U^\epsilon - \sum_{k=1}^{\ell_0} \left(\int_\Omega U^\epsilon u_k^\epsilon \, dx \right) u_k^\epsilon = \sum_{k=\ell_0+1}^{\infty} \left(\int_\Omega U^\epsilon u_k^\epsilon \, dx \right) u_k^\epsilon.$$

By construction we have

$$\int_\Omega v_\varepsilon \, u_k^\epsilon \, dx = 0, \quad k = 1, \dots, \ell_0,$$

hence $v_\varepsilon \perp W_{\ell_0}^\epsilon$, where $W_{\ell_0}^\epsilon$ is the space spanned by $u_1^\epsilon, \dots, u_{\ell_0}^\epsilon$. From Proposition 8.25, it follows that

$$\Lambda_{\ell_0+1} \leq \frac{\int_\Omega A^\epsilon \nabla v_\varepsilon \nabla v_\varepsilon \, dx}{\int_\Omega |v_\varepsilon|^2 \, dx}. \tag{8.88}$$

From the definition of v_ε, we have

$$\begin{aligned}\int_\Omega A^\varepsilon \nabla v_\varepsilon \nabla v_\varepsilon \, dx &= \int_\Omega A^\varepsilon \nabla U^\varepsilon \nabla U^\varepsilon \, dx \\ &\quad -2\sum_{k=1}^{\ell_0} \left(\int_\Omega U^\varepsilon u_k^\varepsilon \, dx \right) \int_\Omega A^\varepsilon \nabla U^\varepsilon \nabla u_k^\varepsilon \, dx \\ &\quad + \sum_{k,j=1}^{\ell_0} \left(\int_\Omega U_\varepsilon u_k^\varepsilon \, dx \right) \left(\int_\Omega U_\varepsilon u_j^\varepsilon \, dx \right) \int_\Omega A^\varepsilon \nabla u_k^\varepsilon \nabla u_j^\varepsilon \, dx.\end{aligned}$$

By using the variational formulation of (8.77) and (8.86), this can be rewritten as follows:

$$\int_\Omega A^\varepsilon \nabla v_\varepsilon \nabla v_\varepsilon \, dx = \lambda \int_\Omega w \, U^\varepsilon \, dx - 2\sum_{k=1}^{\ell_0} \left(\int_\Omega U^\varepsilon u_k^\varepsilon \, dx \right) \lambda \int_\Omega w \, u_k^\varepsilon \, dx$$
$$+ \sum_{k=1}^{\ell_0} \lambda_\varepsilon^k \left(\int_\Omega U_\varepsilon u_k^\varepsilon \, dx \right)^2 .$$

where we have used (8.83). We can pass to the limit in all the integrals in the right-hand side for the subsequence ε' from (8.81). Denote by W_{ℓ_0} the space spanned by $u_1, \dots, u_{\ell_0}$. By using (8.81), (8.80), and (8.87) and recalling that in particular, $w \perp W_{\ell_0}$, we obtain

$$\begin{aligned}\lim_{\varepsilon' \to 0} \int_\Omega A^{\varepsilon'} \nabla v_{\varepsilon'} \nabla v_{\varepsilon'} \, dx &= \lambda \int_\Omega w^2 \, dx - 2\sum_{k=1}^{\ell_0} \left(\int_\Omega w \, u_k \, dx \right) \lambda \int_\Omega w \, u_k \, dx \\ &\quad + \sum_{k=1}^{\ell_0} \Lambda^k \left(\int_\Omega w \, u_k \, dx \right)^2 = \lambda \int_\Omega w^2 \, dx. \quad (8.89)\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_\Omega v_\varepsilon^2 \, dx &= \int_\Omega (U^\varepsilon)^2 \, dx - 2\sum_{k=1}^{\ell_0} \left(\int_\Omega U^\varepsilon u_k^\varepsilon \, dx \right)^2 \\ &\quad + \sum_{k,j=1}^{\ell_0} \left(\int_\Omega U^\varepsilon u_k^\varepsilon \, dx \right) \left(\int_\Omega U^\varepsilon u_j^\varepsilon \, dx \right) \int_\Omega u_k^\varepsilon u_j^\varepsilon \, dx \\ &= \int_\Omega (U^\varepsilon)^2 \, dx - \sum_{k=1}^{\ell_0} \left(\int_\Omega U^\varepsilon u_k^\varepsilon \, dx \right)^2 .\end{aligned}$$

where we pass to the limit for the subsequence ε'. We obtain by the same arguments as before. that

$$\lim_{\varepsilon' \to 0} \int_\Omega v_{\varepsilon'}^2 \, dx = \int_\Omega w^2 \, dx - \sum_{k=1}^{\ell_0} \left(\int_\Omega w \, u_k \, dx \right)^2 = \int_\Omega w^2 \, dx.$$

Using into (8.88) this convergence as well as convergence (8.89), we obtain

$$\Lambda_{\ell_0+1} \leq \lambda.$$

which contradicts (8.85). This proves that $\mathcal{A}^0$ has no other eigenvalues except those defined by (8.80), (8.82), and (8.83). Hence, $\Lambda_\ell = \lambda_\ell$ for any ℓ.

To complete the proof of (i) and (ii), it remains to show that the sequence $\{u_\ell\}$ obtained in Step 2 is complete. This can be done by contradiction.

Indeed, if this is not true, due to Proposition 8.23(iii) there exists an eigenfunction w_λ corresponding to some λ which does not belong to any subspace generated by the family $\{u_\ell\}$. Then w is orthogonal to this family. From the property (i) of Proposition 8.23, there exists an ℓ_1 such that

$$\Lambda_{\ell_1+1} > \lambda.$$

Arguing now exactly as before with Λ_{ℓ_1+1} instead of Λ_{ℓ_0+1}, we have

$$\Lambda_{\ell_1+1} \leq \lambda.$$

which is the required contradiction.

Step 4. It remains to prove the last statement of the theorem. Let λ_ℓ be a simple eigenvalue and u_ℓ be a corresponding eigenfunction such that

$$\int_\Omega (u_\ell)^2 \, dx = 1. \tag{8.90}$$

Obviously, if the eigenvalue λ_ℓ is simple, as a consequence of (i) and (ii), the same is true for λ_ℓ^ε (for ε sufficiently small).

Let u_ℓ^ε be an eigenvector corresponding to λ_ℓ^ε, satisfying (8.77) and (8.83). We can suppose that for any ε

$$\int_\Omega u_\ell^\varepsilon \, u_\ell \, dx \geq 0. \tag{8.91}$$

From Step 2 (see (8.81)), for any subsequence ε' we have

$$u_\ell^{\varepsilon'} \rightharpoonup \widehat{u_\ell},$$

where $\widehat{u_\ell}$ is an eigenvector associated with λ_ℓ.

Observe now that $\widehat{u_\ell}$ and u_ℓ are two eigenvectors corresponding to the same simple eigenvalue λ_ℓ, so that there exists a constant c such that

$$\widehat{u_\ell} = c\, u_\ell.$$

Now, from (8.83) one has, after passing to the limit, that

$$\int_\Omega \left(\widehat{u_\ell}\right)^2 dx = 1$$

which, together with (8.90), implies that $|c| = 1$.

On the other hand, passing to the limit into (8.91) yields

$$\int_\Omega \widehat{u_\ell}\, u_\ell \, dx \geq 0,$$

so that $c = 1$, i.e.

$$\widehat{u_\ell} = u_\ell.$$

Thus the whole sequence $\{u_\ell^\varepsilon\}$ converges to u_ℓ. The proof of Theorem 8.26 is complete. □

9

The two-scale convergence method

In the first two sections of this chapter, we present the two-scale convergence method and we use it in Section 9.3 to prove again Theorem 6.1. As the multiple-scale method, it also takes into account the fact that we have a 'macroscopic' scale x and a 'microscopic' one x/ε.

The notion of two-scale convergence has been introduced by Nguetseng (1989) and developed by Allaire (1992, 1994). It deals with the convergence of integrals of the form

$$\int_\Omega v^\varepsilon(x)\,\psi\left(x, \frac{x}{\varepsilon}\right) dx,$$

where the sequence $\{v^\varepsilon\}$ is bounded in $L^2(\Omega)$ and $\psi = \psi(x, y)$ is a smooth function periodic with respect to y.

Notice that we have already met this kind of integral when applying Tartar's oscillating test functions method. Indeed, this method is based on the construction of functions of the form $\widehat{w}_i^\varepsilon(x/\varepsilon)$ (Section 8.1) whose products by a function $\phi \in \mathcal{D}(\Omega)$ were used as test functions in the variational formulation of problem (6.1).

Let us also mention that the two-scale convergence method justifies mathematically the (formal) asymptotic development obtained in Chapter 7 by the multiple-scale method. Moreover, in Section 9.4 we show that when the corrector functions are sufficiently smooth, the two-scale convergence method gives a very simple proof for the corrector result stated in Section 8.3.

9.1 The general setting

As in the previous chapters, Ω is a bounded open set in $\mathbb{R}^N$ and

$$Y =]0, \ell_1[\times \cdots \times]0, \ell_N[,$$

with $\ell_1, \ldots, \ell_N$ given positive numbers, is the reference cell.

In this chapter we will use the following spaces:

- $C_{\text{per}}(Y)$, the subspace of $C(\mathbb{R}^N)$ of Y-periodic functions.
- $C^\infty_{\text{per}}(Y)$, the subspace of $C^\infty(\overline{Y})$ of Y-periodic functions.
- $L^p_{\text{per}}(Y)$, the subspace of $L^p(Y)$ of Y-periodic functions in the sense of Definition 2.1.

- $H^1_{\mathrm{per}}(Y)$, the space introduced in Definition 3.48.
- $\mathcal{W}_{\mathrm{per}}(Y)$, the space introduced in Definition 3.51.
- $L^2(\Omega; C_{\mathrm{per}}(Y))$ and $L^2(\Omega; \mathcal{W}_{\mathrm{per}}(Y))$, the spaces given by Definition 3.54,
- $L^2_{\mathrm{per}}(Y; C(\overline{\Omega}))$, the space of measurable functions $u : y \in Y \to u(y) \in C(\overline{\Omega})$ such that $\|u(x)\|_{C(\overline{\Omega})} \in L^2_{\mathrm{per}}(Y)$.
- $\mathcal{D}(\Omega; C^\infty_{\mathrm{per}}(Y))$, the space of measurable functions on $\Omega \times \mathbb{R}^N$ such that $u(x, \cdot) \in C^\infty_{\mathrm{per}}(Y)$ for any $x \in \Omega$ and the map $x \in \Omega \mapsto u(x, \cdot) \in C^\infty_{\mathrm{per}}(Y)$ is indefinitely differentiable with a compact support included in Ω,
- $C(\overline{\Omega}; L^p_{\mathrm{per}}(Y))$ the space of measurable functions on $\overline{\Omega} \times \mathbb{R}^N$ such that $u(x, \cdot) \in L^p_{\mathrm{per}}(Y)$ for any $x \in \overline{\Omega}$ and the map $x \in \overline{\Omega} \mapsto u(x, \cdot) \in L^p(Y)$ is continuous.

Throughout this chapter, as mentioned above, we will have to work with functions of the form $\psi(x, x/\varepsilon)$. The properties of this kind of function have been investigated, in particular, by Bensoussan, Lions and Papanicolaou (1978), Donato (1983a,b, 1985), Allaire (1992). Some of these properties will be useful in the sequel, so, for the reader's convenience, we recall them here.

Lemma 9.1.

i) *Let $\varphi \in L^p(\Omega; C_{\mathrm{per}}(Y))$ with $1 \leq p < \infty$. Then $\varphi(\cdot, \cdot/\varepsilon) \in L^p(\Omega)$ with*

$$\left\| \varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{L^p(\Omega)} \leq \|\varphi(\cdot, \cdot)\|_{L^p(\Omega; C_{\mathrm{per}}(Y))}$$

and

$$\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right) \rightharpoonup \frac{1}{|Y|} \int_Y \varphi(\cdot, y)\, dy \quad \text{weakly in } L^p(\Omega). \tag{9.1}$$

In particular, if $\varphi \in L^2(\Omega; C_{\mathrm{per}}(Y))$, then

$$\lim_{\varepsilon \to 0} \int_\Omega \left[\varphi\left(x, \frac{x}{\varepsilon}\right) \right]^2 dx = \frac{1}{|Y|} \int_\Omega \int_Y [\varphi(x, y)]^2\, dy\, dx. \tag{9.2}$$

ii) *Suppose that $\varphi(x, y) = \varphi_1(x)\varphi_2(y)$, $\varphi_1 \in L^s(\Omega)$, $\varphi_2 \in L^r_{\mathrm{per}}(Y)$ with $1 \leq r, s < \infty$ and such that*

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{p}.$$

Then $\varphi(\cdot, \cdot/\varepsilon) \in L^p(\Omega)$ and

$$\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right) \rightharpoonup \frac{\varphi_1(\cdot)}{|Y|} \int_Y \varphi_2(y)\, dy \quad \text{weakly in } L^p(\Omega).$$

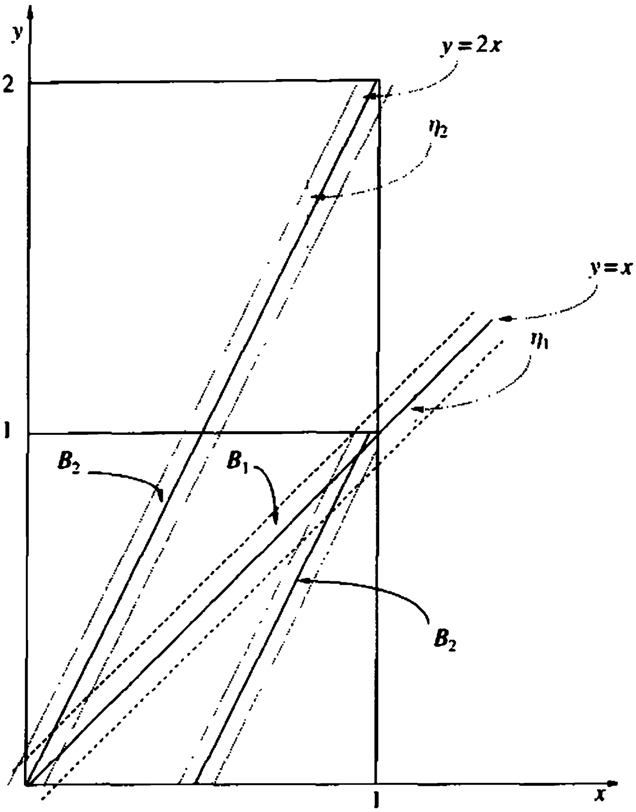


Fig. 9.1 B_k for $k = 1$ and $k = 2$.

Remark 9.2. Suppose that φ is not a product as in statement (ii). One can ask if there are other functions than that contained in $L^p(\Omega;\, C_{\text{per}}(Y))$ satisfying (9.1). As far as we know, there is no precise characterization of these functions, but some counterexamples show that one can not weaken the hypothesis from (i) too much. For instance, bounded functions do not possess property (9.1). Indeed, the following example (see Donato 1983a, 1985) exhibit a function in $L^\infty(\Omega \times Y)$ and Y-periodic in the second variable, which do not converge to its mean value.

Introduce, for $k \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, the set (see Fig. 9.1)

$$B_k = \{\ (x, y)\ \big|\, (x, y) \in [0, 1]^2,\ y \sim kx \pm c\,(\text{mod } 1),\ |c| < \eta_k\},$$

where $\eta_k = 1/(4\sqrt{2}\, k\, 2^k)$ and

$$\forall a,\ b \in \mathbb{R},\ a \sim b\,(\text{mod } 1\ ) \Longleftrightarrow \exists\ z \in \mathbb{Z} \text{ such that } a - b = z.$$

Obviously, one has that

$$|B_k| \le k\, 2\eta_k \sqrt{2} = \frac{1}{2^{k+1}}.$$

Then, if we set $B = \bigcup_{k=1}^{\infty} B_k$, we have $|B| \leq \frac{1}{2} \sum_{k=1}^{\infty} 1/2^k = \frac{1}{2}$.

Let now h be the function defined by

$$h(x,y) = \begin{cases} 1 & \text{if } (x,y) \in B \\ 0 & \text{if } (x,y) \in [0,1]^2 \setminus B. \end{cases}$$

Clearly, since B is measurable, h is a measurable function, hence $h \in L^\infty([0,1]^2)$. Let us still denote by h its extension by periodicity (of period 1) with respect to the second variable to the whole $[0,1] \times \mathbb{R}$. If property (9.1) were true, in particular one would have, for any sequence $\varepsilon_k \to 0$,

$$\lim_{\varepsilon_k \to 0} \int_0^1 h\left(x, \frac{x}{\varepsilon_k}\right) dx = \int_0^1 \int_0^1 h(x,y)\, dx\, dy = \int_B dx < \tfrac{1}{2}.$$

Consider the sequence $\varepsilon_k = 1/k$. By construction

$$h\left(x, \frac{x}{\varepsilon_k}\right) = h(x, kx) = 1, \quad \text{for } x \in [0,1],$$

so that $h(\cdot, \cdot/\varepsilon_k)$ converges to 1. Consequently,

$$\lim_{\varepsilon_k \to 0} \int_0^1 h\left(x, \frac{x}{\varepsilon_k}\right) dx = 1,$$

which is in contradiction with the former inequality.

Let us finally mention that h has even more regularity than simple boundedness, namely $h \in C([0,1], L^1_{\text{per}}(Y))$, as was proved by Allaire (1992). Therefore, the counterexample shows that a $L^\infty(]0,1[\times\Omega) \cap C([0,1], L^1_{\text{per}}(Y))$-regularity is not enough to insure (9.1). ◇

9.2 Two-scale convergence

We recall now the definition of the two-scale convergence and several important results concerning this notion (see Nguetseng, 1989, and Allaire, 1992, 1994).

Definition 9.3. Let $\{v^\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$. One says that $\{v^\varepsilon\}$ *two-scale converges* to $v_0 = v_0(x,y)$ with $v_0 \in L^2(\Omega \times Y)$ if for any function $\psi = \psi(x,y) \in \mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))$, one has

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx = \frac{1}{|Y|} \int_\Omega \int_Y v_0(x,y)\, \psi(x,y)\, dy\, dx. \tag{9.3}$$

Remark 9.4. Due to density properties, it is easily seen that if $\{v^\varepsilon\}$ two-scale converges to v_0, convergence (9.3) holds also for any $\psi \in L^2_{\text{per}}(Y; C(\overline{\Omega}))$ as well as for any ψ of the form $\psi(x,y) = \psi_1(y)\, \psi_2(x,y)$ with $\psi_1 \in L^\infty(Y)$ and $\psi_2 \in L^2_{\text{per}}(Y; C(\overline{\Omega}))$.

For the same reasons, convergence (9.3) is still true for any function ψ of the form $\psi(x,y) = \varphi_1(x)\varphi_2(y)$, where φ_1 and φ_1 are as in statement (ii) of Lemma 9.1. ◇

Remark 9.5. It is easy to see that the two-scale convergence implies the weak convergence. Indeed, if in Definition 9.3 we take ψ independent of y, then (9.3) reads exactly as the following weak convergence:

$$v^\varepsilon \rightharpoonup V^0 = \frac{1}{|Y|}\int_Y v_0(\cdot, y)\,dy \quad \text{weakly in } L^2(\Omega).$$

Clearly, if the two-scale limit v_0 is independent of y, then $V^0 = v_0$, so that the weak and the two-scale limit coincide. Observe also that if a sequence $\{v^\varepsilon\}$ two-scale converges, then it is bounded in $L^2(\Omega)$. ◇

Remark 9.6. Suppose that the sequence $\{v^\varepsilon\}$ admits an asymptotic development of the form

$$v^\varepsilon(x) = v_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

where $v_0, v_1, \ldots$ are smooth Y-periodic functions. Then, applying Lemma 9.1 to $v^\varepsilon(\cdot)\psi(\cdot, \cdot/\varepsilon)$ with ψ a smooth function, one has that $\{v^\varepsilon\}$ two-scale converges to $v_0 = v_0(x,y)$, which is the first term in the development. This can justify a posteriori the multiple-scale method from Chapter 7. ◇

One of the main results on the two-scale convergence is the following compactness theorem:

Theorem 9.7. *Let $\{v^\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then, there exists a subsequence $\{v^{\varepsilon'}\}$ and a function $v_0 \in L^2(\Omega \times Y)$ such that $\{v^{\varepsilon'}\}$ two-scale converges to v_0.*

Proof. Let $\phi \in L^2(\Omega; C_{\mathrm{per}}(Y))$. Then, from the Hölder inequality and Lemma 9.1, we have

$$\left|\int_\Omega v^\varepsilon(x)\,\phi\left(x, \frac{x}{\varepsilon}\right)\,dx\right| \le C\,\|\phi\|_{L^2(\Omega; C_{\mathrm{per}}(Y))}, \tag{9.4}$$

where C is independent of ε. This means (see Definition 1.6), that v^ε can be regarded as the element V^ε of the dual space of $L^2(\Omega; C_{\mathrm{per}}(Y))$, such that

$$\langle V^\varepsilon, \phi\rangle_{[L^2(\Omega; C_{\mathrm{per}}(Y))]', L^2(\Omega; C_{\mathrm{per}}(Y))} = \int_\Omega v^\varepsilon(x)\,\phi\left(x, \frac{x}{\varepsilon}\right)\,dx, \quad \forall \phi \in L^2_{\mathrm{per}}(Y; C(\overline{\Omega})).$$

Moreover, from (9.4), we have

$$\|V^\varepsilon\|_{[L^2(\Omega; C_{\mathrm{per}}(Y))]'} \le C.$$

Consequently, recalling (see Proposition 3.61) that $L^2(\Omega; C_{\mathrm{per}}(Y))$ is separable, we can use Theorem 1.26 to extract a subsequence ε' such that

$$V^{\varepsilon'} \rightharpoonup V_0 \quad \text{weakly* in } [L^2(\Omega; C_{\mathrm{per}}(Y))]',$$

so that

$$\begin{cases} \langle V_0, \phi\rangle_{[L^2(\Omega;C_{\mathrm{per}}(Y))]',L^2(\Omega;C_{\mathrm{per}}(Y))} \\ \qquad = \lim_{\varepsilon'\to 0} \langle V^{\varepsilon}, \phi\rangle_{[L^2(\Omega;C_{\mathrm{per}}(Y))]',L^2(\Omega;C_{\mathrm{per}}(Y))} \\ \qquad = \lim_{\varepsilon'\to 0} \int_\Omega v^{\varepsilon'}(x)\,\phi\Big(x,\frac{x}{\varepsilon'}\Big)\,dx. \end{cases} \tag{9.5}$$

On the other hand, from the boundedness of $\{v^\varepsilon\}$, the Hölder inequality and convergence (9.2), one has

$$\lim_{\varepsilon'\to 0}\left|\int_\Omega v^{\varepsilon'}(x)\,\phi\Big(x,\frac{x}{\varepsilon'}\Big)\,dx\right| \le C \lim_{\varepsilon'\to 0}\left\|\phi\Big(\cdot,\frac{\cdot}{\varepsilon'}\Big)\right\|_{L^2(\Omega)} = C\|\phi\|_{L^2(\Omega\times Y)}.$$

This, combined with (9.5), gives

$$|\langle V_0, \phi\rangle_{[L^2(\Omega;C_{\mathrm{per}}(Y))]',L^2(\Omega;C_{\mathrm{per}}(Y))}| \le C\|\phi\|_{L^2(\Omega\times Y)}, \tag{9.6}$$

for any $\phi \in L^2(\Omega; C_{\mathrm{per}}(Y))$. Since, by Proposition 3.61. the space $L^2(\Omega; C_{\mathrm{per}}(Y))$ is dense in $L^2(\Omega \times Y)$, inequality (9.6) holds for any function $\psi \in L^2(\Omega \times Y)$. Therefore, V_0 can be extended continuously to $L^2(\Omega \times Y)$ and so, from the representation theorem (Theorem 1.36) the function V_0 can be identified with an element $v \in L^2(\Omega \times Y)$ such that

$$\langle V_0, \phi\rangle_{[L^2(\Omega;C_{\mathrm{per}}(Y))]',L^2(\Omega;C_{\mathrm{per}}(Y))} = \int_{\Omega\times Y} v(x,y)\phi(x,y)\,dx\,dy.$$

This, together with (9.5) leads to

$$\lim_{\varepsilon'\to 0} \int_\Omega v^{\varepsilon'}(x)\,\phi\Big(x,\frac{x}{\varepsilon'}\Big)\,dx = \int_{\Omega\times Y} v(x,y)\phi(x,y)\,dx\,dy,$$

which (see Definition 9.3) means that $v_0 = |Y|v$ is the two-scale limit of the sequence $\{v^{\varepsilon'}\}$. □

Another important result concerns the product of two sequences which two-scale converge.

Theorem 9.8. *Let $\{v^\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$ which two-scale converges to $v_0 \in L^2(\Omega \times Y)$. Suppose furthermore, that*

$$\lim_{\varepsilon\to 0} \int_\Omega [v^\varepsilon(x)]^2\,dx = \frac{1}{|Y|}\int_\Omega\int_Y [v_0(x,y)]^2\,dx\,dy. \tag{9.7}$$

Then, for any sequence $\{w^\varepsilon\}$ that two-scale converges to a limit $w_0 \in L^2(\Omega\times Y)$, we have

$$v^\varepsilon\, w^\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y v_0(\cdot,y)\, w_0(\cdot,y)\,dy \quad \text{in } \mathcal{D}'(\Omega). \tag{9.8}$$

Proof. From the density property (ii) in Proposition 3.61, there exists a sequence $\{\varphi_n\} \subset L^2(\Omega;\, C_{\text{per}}(Y))$, such that for $n \to \infty$,

$$\varphi_n \longrightarrow v_0 \quad \text{strongly in } L^2(\Omega \times Y). \tag{9.9}$$

Consider now the integral

$$\begin{aligned} I_n^\varepsilon &= \int_\Omega \left[v^\varepsilon(x) - \varphi_n\left(x, \frac{x}{\varepsilon}\right)\right]^2 dx \\ &= \int_\Omega [v^\varepsilon(x)]^2\, dx - 2\int_\Omega v^\varepsilon(x)\, \varphi_n\left(x, \frac{x}{\varepsilon}\right) dx + \int_\Omega \left[\varphi_n\left(x, \frac{x}{\varepsilon}\right)\right]^2 dx, \end{aligned}$$

where we let $\varepsilon \to 0$. For the convergence of the first term in the right-hand side, we use hypothesis (9.7). The second one converges simply by hypothesis while for the third term we make use of convergence (9.2) from Lemma 9.1. We have at the limit

$$\begin{aligned} \lim_{\varepsilon \to 0} I_n^\varepsilon &= \frac{1}{|Y|}\int_\Omega \int_Y [v_0(x,y)]^2\, dx\, dy - 2\frac{1}{|Y|}\int_\Omega \int_Y v_0(x,y)\varphi_n(x,y)\, dx\, dy \\ &\quad + \frac{1}{|Y|}\int_\Omega \int_Y \left(\varphi_n(x,y)\right)^2 dx\, dy = \frac{1}{|Y|}\int_\Omega \int_Y [v_0 - \varphi_n(x,y)]^2\, dx\, dy. \end{aligned}$$

Due to (9.9), the last integral converges to 0 as $n \to \infty$, so that

$$\lim_{n\to\infty} \lim_{\varepsilon \to 0} I_n^\varepsilon = \lim_{n\to\infty} \lim_{\varepsilon\to 0} \frac{1}{|Y|}\int_\Omega \int_Y \left[v^\varepsilon(x) - \varphi_n\left(x, \frac{x}{\varepsilon}\right)\right]^2 dx = 0. \tag{9.10}$$

On the other hand, for any $\psi \in \mathcal{D}(\Omega)$ one has

$$\begin{aligned} \int_\Omega v^\varepsilon(x)\, w^\varepsilon(x)\, \psi(x)\, dx &= \int_\Omega \left[v^\varepsilon(x) - \varphi_n\left(x, \frac{x}{\varepsilon}\right)\right] w^\varepsilon(x)\, \psi(x)\, dx \\ &\quad + \int_\Omega \varphi_n\left(x, \frac{x}{\varepsilon}\right) w^\varepsilon(x)\, \psi(x)\, dx, \end{aligned} \tag{9.11}$$

where we make first $\varepsilon \to 0$ and then $n \to \infty$. To do so, observe that by the Hölder inequality, Remark 9.5 and Proposition 1.14, one derives

$$\begin{aligned} &\lim_{n\to\infty} \lim_{\varepsilon\to 0} \left| \int_\Omega \left[v^\varepsilon(x) - \varphi_n\left(x, \frac{x}{\varepsilon}\right)\right] w^\varepsilon(x)\, \psi(x)\, dx \right| \\ &\quad \leq C \lim_{n\to\infty} \lim_{\varepsilon\to 0} \left\{ \int_\Omega \left[v^\varepsilon(x) - \varphi_n\left(x, \frac{x}{\varepsilon}\right)\right]^2 dx \right\}^{\frac{1}{2}} = 0, \end{aligned}$$

due to (9.10). Moreover, from (9.9) and the assumption that w^ε two-scale converges to w_0, we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega \varphi_n\left(x,\frac{x}{\varepsilon}\right) w^\varepsilon(x)\,\psi(x)\,dx \\
&= \lim_{n\to\infty}\frac{1}{|Y|}\int_\Omega\int_Y w_0(x,y)\varphi_n(x,y)\psi(x)\,dx\,dy \\
&= \frac{1}{|Y|}\int_\Omega\int_Y w_0(x,y)v_0(x,y)\psi(x)\,dx\,dy.
\end{aligned}$$

Therefore, from (9.11) one has

$$\lim_{\varepsilon\to 0}\int_\Omega v^\varepsilon(x)\,w^\varepsilon(x)\,\psi(x)\,dx = \frac{1}{|Y|}\int_\Omega \psi(x)\int_Y w_0(x,y)v_0(x,y)\,dy\,dx,$$

which is precisely convergence (9.8). □

The next property gives further information on the two-scale convergence of bounded sequences in $H^1(\Omega)$.

Theorem 9.9. *Let $\{v^\varepsilon\}$ be a sequence of functions in $H^1(\Omega)$ such that*

$$v^\varepsilon \rightharpoonup v_0 \quad \text{weakly in } H^1(\Omega). \tag{9.12}$$

Then $\{v^\varepsilon\}$ two-scale converges to v_0, and there exist a subsequence ε' and $v_1 = v_1(x,y)$ in $L^2(\Omega; W_{\text{per}}(Y))$ such that

$$\nabla v^{\varepsilon'} \quad \text{two-scale converges to } \nabla_x v_0 + \nabla_y v_1.$$

Proof. Due to Theorem 9.7, one has a subsequence $\{\varepsilon'\}$ such that

$$\begin{cases} v^{\varepsilon'} & \text{two-scale converges to } v \in L^2(\Omega\times Y) \\ \nabla v^{\varepsilon'} & \text{two-scale converges to } V \in [L^2(\Omega\times Y)]^N. \end{cases}$$

Hence, for any $\psi \in \left(\mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))\right)^N$, one has

$$\lim_{\varepsilon'\to 0}\int_\Omega \nabla v^{\varepsilon'}(x)\cdot\psi\left(x,\frac{x}{\varepsilon'}\right)dx = \frac{1}{|Y|}\int_\Omega\int_Y V(x,y)\cdot\psi(x,y)\,dx\,dy. \tag{9.13}$$

By the definition of a derivative in the sense of distributions (see Definition 3.11), it follows that

$$\int_\Omega \nabla v^{\varepsilon'}(x)\cdot\psi\left(x,\frac{x}{\varepsilon'}\right)dx = -\sum_{i=1}^N\int_\Omega v^{\varepsilon'}(x)\left[\frac{\partial\psi_i}{\partial x_i}\left(x,\frac{x}{\varepsilon'}\right) + \frac{1}{\varepsilon'}\frac{\partial\psi_i}{\partial y_i}\left(x,\frac{x}{\varepsilon'}\right)\right]dx.$$

Then, multiplying by ε', one has

$$\begin{cases} \displaystyle\sum_{i=1}^N\int_\Omega v^{\varepsilon'}(x)\,\frac{\partial\psi_i}{\partial y_i}\left(x,\frac{x}{\varepsilon'}\right)dx \\ \displaystyle\qquad = \varepsilon'\left[\int_\Omega \nabla v^{\varepsilon'}(x)\cdot\psi\left(x,\frac{x}{\varepsilon'}\right)dx + \sum_{i=1}^N\int_\Omega v^{\varepsilon'}(x)\frac{\partial\psi_i}{\partial x_i}\left(x,\frac{x}{\varepsilon'}\right)dx\right]. \end{cases} \tag{9.14}$$

Passing to the limit as $\varepsilon' \to 0$, by (9.13) and the two-scale convergence of $(v^{\varepsilon'})$, we get

$$\sum_{i=1}^{N} \int_\Omega \int_Y v(x,y) \frac{\partial \psi_i}{\partial y_i}(x,y)\, dx\, dy = 0.$$

This, by Green's formula (Theorem 3.33), yields in particular for any $\psi \in \mathcal{D}(\Omega \times Y)$,

$$\int_\Omega \int_Y \nabla_y v(x,y) \cdot \psi(x,y)\, dx\, dy = 0,$$

hence, by Theorem 1.44,

$$\nabla_y v = 0 \quad \text{a.e. on } \Omega \times Y.$$

Then, from Proposition 3.38 written in terms of Y we have

$$v(x,y) = \mathcal{M}_Y(v(x,\cdot)) \quad \text{a.e. on } \Omega \times Y,$$

which means that v does not depend on y. Then, due to Remark 9.5 and convergence (9.12), $v = v_0 \in H^1(\Omega)$.

Let now $\Psi \in \left(\mathcal{D}(\Omega;\, C^\infty_{\text{per}}(Y))\right)^N$ such that $\operatorname{div}_y \Psi = \sum_{i=1}^N \partial\Psi_i/\partial y_i = 0$. From (9.14) we get

$$\begin{cases} \lim\limits_{\varepsilon'\to 0} \displaystyle\int_\Omega \nabla v^{\varepsilon'}(x) \cdot \Psi\left(x, \frac{x}{\varepsilon'}\right) dx = -\lim\limits_{\varepsilon'\to 0} \sum_{i=1}^N \int_\Omega v^{\varepsilon'}(x) \frac{\partial \Psi_i}{\partial x_i}\left(x, \frac{x}{\varepsilon'}\right) dx \\ \qquad = -\dfrac{1}{|Y|} \displaystyle\sum_{i=1}^N \int_\Omega \int_Y v_0(x) \frac{\partial \Psi_i}{\partial x_i}(x,y)\, dx\, dy \\ \qquad = \dfrac{1}{|Y|} \displaystyle\int_\Omega \int_Y \nabla v_0(x) \cdot \Psi(x,y)\, dx\, dy, \end{cases} \tag{9.15}$$

where we have used the two-scale convergence of $\{v^{\varepsilon'}\}$. This, together with (9.13) written for $\psi = \Psi$, gives

$$\int_\Omega \int_Y [V(x,y) - \nabla v_0(x)] \cdot \Psi(x,y)\, dx\, dy = 0.$$

We now make use of a classical result for which we refer the reader to Girault and Raviart (1981) and Temam (1979). It states that if $(F,\varphi)_{L^2} = 0$, for any φ such that div $\varphi = 0$, then F is a gradient. This result applied here for $F(y) = V(x,y) - \nabla v_0(x)$ a.e. on Ω, implies that there exists a unique function $v_1 \in L^2(\Omega;\, \mathcal{W}_{\text{per}}(Y))$ such that

$$V(x,y) - \nabla v_0(x) = \nabla_y v_1(x,y).$$

This ends the proof of Theorem 9.9. □

Remark 9.10. From assumption (9.12), we know that the whole sequence $\{\nabla v^\varepsilon\}$ weakly converges to ∇v_0. A natural question is whether this whole sequence is two-scale convergent. The answer is negative, since the function V and consequently, $\nabla_y v_1$, can not be uniquely identified. Actually, from assumption (9.12) and Remark 9.5, all we can say is that

$$\nabla v_0(x) = \frac{1}{|Y|}\int_Y V(x,y)\,dy,$$

which is not enough to insure the uniqueness of V. ◊

9.3 Proof of the main convergence result

We prove now Theorem 6.1 by the two-scale convergence method. Let A and f be given as in Theorem 6.1 and let u^ε be the solution of (6.1), i.e.

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

whose variational formulation is

$$\begin{cases} \text{Find } u^\varepsilon \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega A^\varepsilon \nabla u^\varepsilon \nabla v\,dx = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{9.16}$$

We proved in Section 5.1 that there exists a subsequence (still denoted by ε), such that

$$\begin{cases} i) & u^\varepsilon \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega) \\ ii) & u^\varepsilon \to u^0 & \text{strongly in } L^2(\Omega). \end{cases} \tag{9.17}$$

From Theorem 9.9, we have that u^ε two-scale converges to u^0. Moreover, there exists $u_1 = u_1(x,y)$ in $L^2(\Omega; W_{\text{per}}(Y))$ such that, up to a subsequence, ∇u^ε two-scale converges to $\nabla_x u^0 + \nabla_y u_1$. We will now prove that u^0 satisfies problem (6.29). Let $v_0 \in \mathcal{D}(\Omega)$ and $v_1 \in \mathcal{D}(\Omega; C_{\text{per}}^\infty(Y))$. Clearly, $v_0(\cdot) + \varepsilon v_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) \in H_0^1(\Omega)$, so that it can be taken as test function function in (9.16). One has

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \left[\nabla v_0(x) + \varepsilon \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx$$
$$= \left\langle f, v_0(\cdot) + \varepsilon v_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},$$

which can be rewritten as follows:

$$\begin{cases} \displaystyle\int_\Omega \nabla u^\varepsilon ({}^t A^\varepsilon)\left[\nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx + \varepsilon \int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right) dx \\ \qquad = \left\langle f, v_0(\cdot) + \varepsilon v_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \end{cases} \tag{9.18}$$

where we want to pass to the limit as $\varepsilon \to 0$.

For the first term, this is possible according to Remark 9.4. Indeed, ${}^tA^\varepsilon$ is in $L^\infty(Y)$, $\nabla v_0(x) + \nabla_y v_1(x,y)$ is in $L^2_{\text{per}}(Y; C(\overline{\Omega}))$ so that ${}^tA^\varepsilon(y)\big[\nabla v_0(x) + \nabla_y v_1(x,y)\big]$ can be used as test function in the two-scale convergence of ∇u^ε. Consequently,

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla u^\varepsilon({}^tA^\varepsilon)\left[\nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx$$
$$= \frac{1}{|Y|}\int_\Omega\int_Y (\nabla u^0(x) + \nabla_y u_1(x,y))({}^tA(y))[\nabla v_0(x) + \nabla_y v_1(x,y)]\, dx\, dy.$$

For the second term in (9.18). by using Lemma 9.1 written for $\varphi(x,y) = \nabla_x v_1(x,y)$, the Hölder inequality and the fact that $A^\varepsilon\nabla u^\varepsilon$ is bounded in $L^2(\Omega)$ (see 5.12), one derives that

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega A^\varepsilon\nabla u^\varepsilon\, \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right) dx = 0.$$

To pass to the limit in the last term, notice that by the definition of v_0 and v_1 one has that

$$v_0(\cdot) + \varepsilon v_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) \rightharpoonup v_0 \quad \text{weakly in } H^1_0(\Omega).$$

Hence, passing to the limit in (9.18) as $\varepsilon \to 0$, we finally get

$$\frac{1}{|Y|}\int_\Omega\int_Y (\nabla u^0(x) + \nabla_y u_1(x,y))\,({}^tA(y))\big[\nabla v_0(x) + \nabla_y v_1(x,y)\big]\, dx\, dy$$
$$= \langle f, v_0\rangle_{H^{-1}(\Omega), H^1_0(\Omega)},$$

which can be rewritten as

$$\left\{\begin{aligned} &\frac{1}{|Y|}\int_\Omega\int_Y A(y)\,\big(\nabla u^0(x) + \nabla_y u_1(x,y)\big)\,\big(\nabla v_0(x) + \nabla_y v_1(x,y)\big)\, dx\, dy \\ &\qquad = \langle f, v_0\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}. \end{aligned}\right. \tag{9.19}$$

Let us show that this equation is a variational equation in the space

$$\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega; W_{\text{per}}(Y))$$

and that the hypotheses of the Lax Milgram theorem are fulfilled. Indeed, endowing the space $\mathcal{H}$ with the norm

$$\|V\|^2_{\mathcal{H}} = \|v_0\|^2_{H^1_0(\Omega)} + \|v_1\|^2_{L^2(\Omega; W_{\text{per}}(Y))}, \quad \forall\, V = (v_0, v_1) \in \mathcal{H},$$

the bilinear form defined by

$$a(U,V) = \frac{1}{|Y|}\int_\Omega\int_Y A(y)\,\big(\nabla u^0(x) + \nabla_y u_1(x,y)\big)\,\big(\nabla v_0(x) + \nabla_y v_1(x,y)\big)\, dx\, dy$$

for any $U = (u^0, u_1) \in \mathcal{H}$ and $V = (v_0, v_1) \in \mathcal{H}$, is clearly continuous on $\mathcal{H}$.

Observe now that one has

$$a(V,V) \geq \frac{\alpha}{|Y|} \|\nabla v_0(x) + \nabla_y v_1(x,y)\|^2_{L^2(\Omega\times Y)}, \quad \forall\, V = (v_0, v_1) \in \mathcal{H}, \tag{9.20}$$

since $A \in M(\alpha, \beta, Y)$. On the other hand,

$$\begin{cases} \|\nabla v_0(x) + \nabla_y v_1(x,y)\|^2_{L^2(\Omega\times Y)} = \|v_0\|^2_{H_0^1(\Omega)} + \|v_1\|^2_{L^2(\Omega;\, \mathcal{W}_{per}(Y))} \\ \qquad +2\displaystyle\int_\Omega\int_Y \nabla v_0(x)\, \nabla_y v_1(x,y)\, dx\, dy = \|V\|^2_{\mathcal{H}}, \end{cases} \tag{9.21}$$

since, by the Green formula (Theorem 3.33) and the periodicity of v_1 (Proposition 3.42),

$$\begin{aligned} \int_\Omega\int_Y \nabla v_0(x)\, \nabla_y v_1(x,y)\, dx\, dy &= \int_\Omega \left[\int_Y \nabla_y\big(\nabla v_0(x)\, v_1(x,y)\big)\, dy\right] dx \\ &= \int_\Omega \left[\int_{\partial Y} \nabla v_0(x)\, v_1(x,y)\, n(y)\, ds_y\right] dx = 0. \end{aligned}$$

The coerciveness of a on $\mathcal{H}$ is then established due to (9.19) and (9.20).

Furthermore, the map

$$F : V = (v_0, v_1) \longmapsto \langle f, v_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

is obviously linear and continuous on $\mathcal{H}$.

Hence, we can apply the Lax-Milgram theorem (Theorem 4.6) to obtain the existence and uniqueness of $(u^0, u_1) \in H_0^1(\Omega) \times L^2(\Omega; \mathcal{W}_{per}(Y))$, the solution of (9.19), for any $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega; \mathcal{W}_{per}(Y))$.

Choosing now first $v_0 \equiv 0$ and after $v_1 \equiv 0$, we see that (9.19) is equivalent to the problem

$$\begin{cases} -\mathrm{div}_y\big(A(y)\, \nabla_y u_1(x,y)\big) = \mathrm{div}_y\big(A(y)\big)\, \nabla u^0(x) & \text{in } \Omega \times Y \\ -\mathrm{div}_x \left[\displaystyle\int_Y A(y)\big(\nabla u^0(x) + \nabla_y u_1(x,y)\big)\, dy\right] = |Y|\, f & \text{in } \Omega \\ u^0 = 0 \quad \text{on } \partial\Omega \\ u_1(x,\cdot) \quad Y\text{-periodic}. \end{cases} \tag{9.22}$$

To end the proof of Theorem 6.1. we will argue exactly as in Section 7.1. Observe that the first line in (9.22) is precisely problem (7.15) and we proved in Chapter 7 that its solution is of the form (7.21), i.e.

$$u_1(x,y) = -\sum_{j=1}^{N} \widehat{\chi}_j(y) \frac{\partial u_0}{\partial x_j} + \tilde{u}_1(x), \tag{9.23}$$

where $\tilde{u}_1 \in \dot{0}$ in $\mathcal{W}_{\mathrm{per}}(Y)$. The functions $\widehat{\chi}_j$ satisfy

$$\begin{cases} -\mathrm{div}_y(A(y)\nabla_y \widehat{\chi}_j) = -\sum_{i=1}^{N} \dfrac{\partial a_{ij}(y)}{\partial y_i} & \text{in } Y \\ \mathcal{M}_Y(\widehat{\chi}_j) = 0 \\ \widehat{\chi}_j \quad Y\text{-periodic,} \end{cases}$$

for $j = 1, \ldots, N$. Replacing u_1 given (9.23) in the second line in (9.22), one obtains that u^0 satisfies (7.25), namely

$$-\sum_{i,k=1}^{N} \left[\sum_{j=1}^{N} \int_Y \left(a_{ik} - a_{ij} \frac{\partial \widehat{\chi}_k}{\partial y_j} \right) dy \right] \frac{\partial^2 u^0}{\partial x_i \partial x_k} = |Y| f.$$

Reasoning as in Section 7.1, this implies that u^0 is the unique solution of (6.29). Consequently, the whole sequence in (9.17) converges to u^0. The proof of Theorem 6.1 is complete. □

Remark 9.11. Let us point out the main difference between Tartar's oscillating test functions and the two-scale convergence one. The first method is based on the use of oscillating functions, constructed specially for the matrix A under consideration. The two-scale convergence method uses general oscillating test functions which are not related to A but it needs to introduce special functional spaces as done in Sections 9.1 and 9.2. ◇

9.4 A corrector result

In this section, we place ourselves in the particular case where the homogenized solution or the correcting term u_1 are more regular. In this case the following corrector result can be proved by the two-scale convergence method:

Proposition 9.12. *Let u_1 be given by (9.23) and suppose that $\nabla_y \widehat{\chi}_i \in (L^r(Y))^N$, $i = 1, \ldots, N$ and $\nabla u^0 \in (L^s(\Omega))^N$ with $1 \leq r, s < \infty$ and such that*

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{2}.$$

Then

$$\nabla u^\varepsilon - \nabla u^0 - \nabla_y u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) \to 0 \quad \text{strongly in } (L^2(\Omega))^N.$$

Remark 9.13. Observe that $\nabla u^\varepsilon - \nabla u^0 - \nabla_y u_1(\cdot, \cdot/\varepsilon)$ is nothing else than $\nabla u^\varepsilon - C^\varepsilon \nabla u^0$, where C^ε is the corrector matrix introduced in Section 8.3. Observe also that Theorem 8.6 in the particular case $t = 2$ is exactly Proposition 9.12. ◇

Proof of Proposition 9.12. Due to the regularity assumption and the ellipticity of matrix A, we have

$$\begin{aligned}
&\alpha\left\|\nabla u^\varepsilon - \nabla u^0 - \nabla_y u_1(\cdot, \frac{\cdot}{\varepsilon})\right\|_{L^2(\Omega)} \\
&\leq \int_\Omega A\left(\frac{x}{\varepsilon}\right)\left(\nabla u^\varepsilon(x) - \nabla u^0(x) - \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right) \\
&\quad \cdot\left(\nabla u^\varepsilon(x) - \nabla u^0(x) - \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right) dx \\
&= \langle f, u^\varepsilon\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\
&\quad - \int_\Omega (A + {}^tA)\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon(x)\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right] dx \\
&\quad + \int_\Omega A\left(\frac{x}{\varepsilon}\right)\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right]\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right] dx.
\end{aligned}$$

Let us show that the right-hand side of this inequality goes to zero as $\varepsilon \to 0$. First, from Theorem 6.1 we have that

$$\langle f, u^\varepsilon\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad \to \quad \langle f, u^0\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{9.24}$$

Secondly, observe that due to Remark 9.4 the function $(A + {}^tA)(y)\,[\nabla u^0(x) + \nabla_y u_1(x, y)]$ can be chosen as test function in the two-scale convergence of ∇u^ε to $\nabla u^0 + \nabla_y u_1$. Then, using the symmetry of $A + {}^tA$, we obtain

$$\left\{\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega (A + {}^tA)\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon(x)\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right] dx \\
&\quad = \lim_{\varepsilon\to 0}\int_\Omega \nabla u^\varepsilon(x)\left[\left(A + {}^tA\right)\left(\frac{x}{\varepsilon}\right)\left(\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right)\right] dx \\
&\quad = \frac{1}{|Y|}\int_\Omega\int_Y (A + {}^tA)(y)\,[\nabla u^0(x) + \nabla_y u_1(x, y)]\,[\nabla u^0(x) + \nabla_y u_1(x, y)]\, dx\, dy \\
&\quad = \frac{1}{|Y|}\int_\Omega\int_Y 2A(y)\,[\nabla u^0(x) + \nabla_y u_1(x, y)]\,[\nabla u^0(x) + \nabla_y u_1(x, y)]\, dx\, dy.
\end{aligned}\right. \tag{9.25}$$

Lastly, set

$$\begin{aligned}
\phi(x, y) &= A(y)\,[\nabla u^0(x) + \nabla_y u_1(x, y)]\,[\nabla u^0(x) + \nabla_y u_1(x, y)] \\
&= A(y)\nabla u^0(x)\,\nabla u^0(x) - (A + {}^tA)(y)\nabla u^0(x)\left(\sum_{j=1}^N \nabla_y \chi_j(y)\frac{\partial u^0(x)}{\partial x_j}\right) \\
&\quad + A(y)\left(\sum_{j=1}^N \nabla_y \chi_j(y)\frac{\partial u^0(x)}{\partial x_j}\right)\left(\sum_{k=1}^N \nabla_y \chi_k(y)\frac{\partial u^0}{\partial x_k}(x)\right).
\end{aligned}$$

We can apply to this function the statement (ii) from Lemma 9.1 written for $p = 1$, to obtain

$$\begin{cases} \lim_{\varepsilon\to 0} \int_\Omega A\left(\frac{x}{\varepsilon}\right)\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right]\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right] dx \\ \quad = \frac{1}{|Y|}\int_\Omega\int_Y A(y)\left[\nabla u^0(x) + \nabla_y u_1(x,y)\right]\left[\nabla u^0(x) + \nabla_y u_1(x,y)\right] dx\, dy. \end{cases} \tag{9.26}$$

Taking into account convergences (9.24), (9.25) and (9.26), we finally get

$$\begin{aligned} &\lim_{\varepsilon\to 0}\Big\{ \langle f, u^\varepsilon\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega (A + {}^tA)\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(x)\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right] dx \\ &\qquad + \int_\Omega A\left(\frac{x}{\varepsilon}\right)\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right]\left[\nabla u^0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right] dx\Big\} \\ &= \langle f, u^0\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\qquad - \frac{1}{|Y|}\int_\Omega\int_Y A(y)\left[\nabla u^0(x) + \nabla_y u_1(x,y)\right]\left[\nabla u^0(x) + \nabla_y u_1(x,y)\right] dx\, dy = 0, \end{aligned}$$

where we used equation (9.19). Consequently,

$$\lim_{\varepsilon\to 0}\left\|\nabla u^\varepsilon - \nabla u^0 - \nabla_y u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} \le 0,$$

and this ends the proof of Proposition 9.12. □

Remark 9.14. Notice that the regularity assumptions on $\widehat{\chi}_i$ and u^0 are essential in the above proof. If neither $\widehat{\chi}_i$ nor u^0 satisfy them, we can still prove in this framework a convergence result in $L^1(\Omega)$. The statement of Proposition 8.7 can also be proved by two-scale convergence arguments. Then, one has to argue as in the proof of Theorem 8.6. ◊

10

Homogenization in linearized elasticity

In this chapter we are interested in the asymptotic behaviour as $\varepsilon \to 0$ of the solution of the linearized elasticity system introduced in Section 5.2. We refer to Duvaut (1978), Sanchez-Palencia (1980), Bakhvalov and Panasenko (1989), Oleinik, Shamaev, and Yosifian (1992), Sanchez-Hubert and Sanchez-Palencia (1992) for this subject and to references herein.

In this chapter, we suppose that Ω is a connected bounded open set in $\mathbb{R}^N$ such that $\partial\Omega$ is Lipschitz continuous and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are two disjoint closed sets and Γ_1 is of positive measure. Obviously, in the physical models $N \leq 3$.

Notation 10.1. *Throughout this chapter we adopt the Einstein summation convention, i.e. we sum over repeated indices.*

Furthermore, if $B = (b_{ijkh})_{1\leq i,j,k,h\leq N}$ is a fourth-order tensor, and $m = (m_{ij})_{1\leq i,j\leq N}$, $m^1 = (m^1_{ij})_{1\leq i,j\leq N}$ are square matrices, we set

$$\begin{cases} B\,m = ((B\,m)_{ij})_{1\leq i,j\leq N} = ((b_{ijkh}\,m_{kh})_{ij})_{1\leq i,j\leq N} \\ B\,m\,m^1 = b_{ijkh}\,m_{ij}\,m^1_{kh} \\ |m| = \left(\displaystyle\sum_{i,j=1}^{N} m_{ij}^2\right)^{\frac{1}{2}}. \end{cases}$$

When studying elliptic problems we defined, for any open set $\mathcal{O}$ of $\mathbb{R}^N$, the class of matrices $M(\alpha,\beta,\mathcal{O})$ (see Definition 4.11). We need to define here a class of tensors which plays an equivalent role for the elasticity system.

Definition 10.2. Let $\alpha,\beta \in \mathbb{R}$, such that $0<\alpha<\beta$ and let $\mathcal{O}$ be an open set of $\mathbb{R}^N$. We denote by $M_e(\alpha,\beta,\mathcal{O})$ the set of the tensors $B = (b_{ijkh})_{1\leq i,j,k,h\leq N}$ such that

$$\begin{cases} i) & b_{ijkh} \in L^\infty(\mathcal{O}), \quad \text{for any } i,j,k,h = 1,\dots,N \\ ii) & b_{ijkh} = b_{jikh} = b_{khij}, \quad \text{for any } i,j,k,h = 1,\dots,N \\ iii) & \alpha|m|^2 \leq B\,m\,m \quad \text{for any symmetric matrix } m \\ iv) & |B(x)m| \leq \beta|m| \quad \text{for any matrix } m, \end{cases} \tag{10.1}$$

a.e. on $\mathcal{O}$.

As classical in elasticity, let us introduce the linearized strain tensor e defined by

$$e(\varphi) = (e_{ij})_{1\le i,j\le N}, \quad e_{ij}(\varphi) = \frac{1}{2}\left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i}\right), \quad \forall\, i,j = 1,\dots,N, \tag{10.2}$$

for any $\varphi = (\varphi_1, \dots, \varphi_N)$. Clearly, $e(\varphi)$ is a symmetric matrix.

Remark 10.3. From Definition 10.2 it follows that if $B \in M_e(\alpha, \beta, \mathcal{O})$, then

$$\begin{cases} i) & \alpha |e(\varphi)|^2 \le B\, e(\varphi)\, e(\varphi) \\ ii) & |B(x) e(\varphi)| \le \beta\, |e(\varphi)|, \end{cases}$$

for any $\varphi = (\varphi_1, \dots, \varphi_N)$.

Also notice that from (10.2) and the symmetry property (10.1)(ii), the components of the matrix $B\,e(\varphi)$ read

$$\big(B\, e(\varphi)\big)_{ij} = b_{ijkh}\; e_{kh}(\varphi) = b_{ijkh}\; \frac{\partial \varphi_k}{\partial x_h}.$$

◇

Let us now describe the periodic framework in which we work in this chapter. As previously, introduce the reference cell

$$Y =]0, \ell_1[\times \cdots \times]0, \ell_N[,$$

where $\ell_1, \dots, \ell_N$ are given positive numbers.

Let $A = A(y)$ be a fourth-order tensor such that

$$\begin{cases} a_{ijkh} \text{ is } Y\text{-periodic}, \quad \forall\, i,j,k,h = 1,\dots,N \\ A = (a_{ijkh})_{1\le i,j,k,h\le N} \in M_e(\alpha, \beta, Y). \end{cases} \tag{10.3}$$

Set

$$a^{\varepsilon}_{ijkh}(x) = a_{ijkh}\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall\, i,j,k,h = 1,\dots,N \tag{10.4}$$

and

$$A^{\varepsilon}(x) = A\left(\frac{x}{\varepsilon}\right) = \big(a^{\varepsilon}_{ijkh}(x)\big)_{1\le i,j,k,h\le N} \quad \text{a.e. on } \mathbb{R}^N. \tag{10.5}$$

It is easily seen that

$$A^{\varepsilon} \in M_e(\alpha, \beta, \Omega). \tag{10.6}$$

We will study the asymptotic behaviour of the linearized elasticity system introduced in Example 5.4, namely

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a^{\varepsilon}_{ijkh}\, \dfrac{\partial u^{\varepsilon}_k}{\partial x_h}\right) = f_i & \text{in } \Omega \\ u^{\varepsilon} = 0 & \text{on } \Gamma_1 \\ a^{\varepsilon}_{ijkh}\, \dfrac{\partial u^{\varepsilon}_k}{\partial x_h}\, n_j = g_i & \text{on } \Gamma_2 \end{cases} \tag{10.7}$$

for $i = 1, \dots, N$.

In Section 10.1 below we show the existence and uniqueness of the solution of (10.7). In Section 10.2 and 10.3 we give the main homogenization results for problem (10.7).

10.1 Existence and uniqueness

Let $B \in M_e(\alpha, \beta, \Omega)$ and consider the linearized elasticity system

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(b_{ijkh}\,\dfrac{\partial u_k}{\partial x_h}\right) = f_i \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma_1 \\ b_{ijkh}\,\dfrac{\partial u_k}{\partial x_h}\, n_j = g_i \quad \text{on } \Gamma_2, \end{cases} \tag{10.8}$$

for $i = 1, \dots, N$.

Denote by $\sigma = (\sigma_{ij})_{1 \le i,j \le N} = B\,e(u)$ the stress tensor, defined by

$$\sigma_{ij} = b_{ijkh}\; e_{kh}(u). \tag{10.9}$$

Thanks to Remark 10.3, system (10.8) can be rewritten in the equivalent form

$$\begin{cases} \dfrac{\partial}{\partial x_j}\sigma_{ij} + f_i = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma_1 \\ \sigma_{ij}\; n_j = g_i \quad \text{on } \Gamma_2 \end{cases} \tag{10.10}$$

for $i = 1, \dots, N$.

This allows us to write down a variational formulation of system (10.8) to which we will be able to apply the Lax–Milgram theorem. To do so, let us introduce an appropriate functional setting.

As in Section 4.6, define the space V by

$$V = \{v \mid v \in H^1(\Omega), \gamma(v) = 0 \text{ on } \Gamma_1\},$$

and set

$$\mathcal{V} = (V)^N.$$

Due to Proposition 3.36, $\mathcal{V}$ can be equipped with the *norm*

$$\|v\|_{\mathcal{V}} = \left(\sum_{i=1}^{N} \|\nabla v_i\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \tag{10.11}$$

for $v = (v_1, \dots, v_N) \in \mathcal{V}$ and it is a Hilbert space for the scalar product

$$(u, v)_{\mathcal{V}} = \sum_{i=1}^{N} (\nabla u_i, \nabla v_i)_{L^2(\Omega)},$$

for $u, v \in \mathcal{V}$. Observe that $\mathcal{V}' = (V')^N$.

Let us make the following assumptions:

$$\begin{cases} i) & B = (b_{ijkh})_{1\leq i,j,k,h\leq N} \in M_e(\alpha, \beta, \Omega) \\ ii) & f = (f_1, \ldots, f_N) \in \mathcal{V}' \\ iii) & g = (g_1, \ldots, g_N) \in (H^{-\frac{1}{2}}(\Gamma_2))^N. \end{cases} \tag{10.12}$$

Here, the Hilbert space $(H^{-\frac{1}{2}}(\Gamma_2))^N$ is equipped with the norm

$$\|h\|_{(H^{-\frac{1}{2}}(\Gamma_2))^N} = \left(\sum_{i=1}^N \|h_i\|^2_{H^{-\frac{1}{2}}(\Gamma_2)}\right)^{\frac{1}{2}}, \quad \forall h = (h_1, \ldots, h_N) \in (H^{-\frac{1}{2}}(\Gamma_2))^N.$$

Notice that by construction

$$\langle h, v\rangle_{(H^{-\frac{1}{2}}(\Gamma_2))^N, (H^{\frac{1}{2}}(\Gamma_2))^N} = \sum_{i=1}^N \langle h_i, v_i\rangle_{H^{-\frac{1}{2}}(\Gamma_2), H^{\frac{1}{2}}(\Gamma_2)}, \quad \forall v \in (H^{\frac{1}{2}}(\Gamma_2).$$

Then, the variational formulation of problem (10.10) is the following:

$$\begin{cases} \text{Find } u \in \mathcal{V} \text{ such that} \\ \displaystyle\int_\Omega B(x)e(u)\, e(v)\, dx = \langle f, v\rangle_{\mathcal{V}',\mathcal{V}} + \langle g, v\rangle_{(H^{-\frac{1}{2}}(\Gamma_2))^N, (H^{\frac{1}{2}}(\Gamma_2))^N}, \\ \forall v \in \mathcal{V}, \end{cases} \tag{10.13}$$

which can be rewritten as

$$\begin{cases} \text{Find } u \in \mathcal{V} \text{ such that} \\ a(u, v) = \langle F, v\rangle, \quad \forall v \in \mathcal{V}, \end{cases} \tag{10.14}$$

where

$$a(u, v) = \int_\Omega B(x)\, e(u)\, e(v)\, dx, \qquad \forall u,\, v \in \mathcal{V}, \tag{10.15}$$

and

$$\langle F, v\rangle \;=\; \langle f, v\rangle_{\mathcal{V}',\mathcal{V}} + \langle g, v\rangle_{(H^{-\frac{1}{2}}(\Gamma_2))^N.(H^{\frac{1}{2}}(\Gamma_2))^N}. \tag{10.16}$$

Observe first that due to (10.12)(i) and definition (10.2), the bilinear form in (10.15) is continuous on $\mathcal{V} \times \mathcal{V}$. In order to apply Lax–Milgram theorem (Theorem 4.6) we need to show that this form satisfies a coerciveness condition (Definition 4.4). Notice that, due to Remark 10.3, we have that

$$\alpha \int_\Omega |e(v)|^2\, dx \leq a(v, v), \quad \forall v \in \mathcal{V}. \tag{10.17}$$

Then, the $\mathcal{V}$-coerciveness will be proved if we show that

$$|||\, v\, ||| = \int_\Omega |e(v)|^2\, dx \tag{10.18}$$

defines a norm on $\mathcal{V}$, equivalent to (10.11). This is easy to prove for $v \in (H_0^1(\Omega))^N$ and is known as the first Korn inequality. For functions which do not vanish on the whole boundary $\partial\Omega$, such as the elements in $\mathcal{V}$, the result is not obvious and is based on the so-called second Korn inequality. We just recall this inequality and for its proof, we refer the reader to Kondratiev and Oleinik (1989a,b, 1990) and Oleinik, Shamaev, and Yosifian (1992).

Theorem 10.4 (Second Korn inequality). *There exists a constant $c_K = c_K(\Omega)$ such that*

$$\|v\|_{(H^1(\Omega))^N} \leq c_K\left[\, \|v\|_{(L^2(\Omega))^N} + \left(\int_\Omega |e(v)|^2\, dx\right)^{\frac{1}{2}}\right], \tag{10.19}$$

for all $v \in (H^1(\Omega))^N$.

Inequality (10.19) has the following consequence:

Proposition 10.5. *The quantity $|||v|||$ in (10.18) defines on $\mathcal{V}$ a norm equivalent to the norm $\|v\|_{\mathcal{V}}$ given by (10.11).*

Proof. We follow the proof from Oleinik, Shamaev, and Yosifian (1992, Theorem 2.5) which is done in two steps.

Step 1. We prove first that $|||\,v\,|||$ is a norm on $\mathcal{V}$. To do this, it is enough to show that

$$\big(v \in \mathcal{V}, \text{ and } |e(v)| = 0\big) \implies v \equiv 0. \tag{10.20}$$

Let $|e(v)| = 0$. This means that

$$\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = 0, \qquad \forall\, i, j = 1, \ldots, N. \tag{10.21}$$

It is well-known in classical mechanics (see, for instance Love, 1944, Truesdell and Toupin, 1960), that if v is a smooth function, these relations imply that there exists a matrix m with $m_{ij} = -m_{ji}$, and a vector b such that

$$v(x) = m\,x + b \tag{10.22}$$

(actually, this follows easily by differentiating the relations in (10.21)). The boundary condition on Γ_1 gives that $m = 0$ and $b = 0$, hence (10.20) holds.

If v is in $\mathcal{V}$ (but not smoother than H^1), one still has (10.22). Its proof needs an approximation of v by smooth functions with null linearized stress tensor. For this point we refer again to Oleinik, Shamaev, and Yosifian (1992, Chapter 1, Section 2.2).

Step 2. It is obvious that

$$|||\,v\,||| \leq c\|v\|_{\mathcal{V}},$$

where the constant c depends on N. Let us show the reverse inequality,

$$\|v\|_{\mathcal{V}} \leq c_1\, |||v|||, \tag{10.23}$$

where c_1 depends only on Ω.

We follow the proof from Oleinik, Shamaev, and Yosifian (1992, Theorem 2.5) which is done by contradiction. Suppose that (10.23) is not true. Then, one can find a sequence of functions $v_n \in \mathcal{V}$ such that

$$\begin{cases} i) & \|v_n\|_{\mathcal{V}} = 1 \\ ii) & \lim_{n\to\infty} \int_\Omega |e(v_n)|^2\,dx = 0. \end{cases} \tag{10.24}$$

In view of Theorem 3.23 on Sobolev embeddings, there exists a subsequence, still denoted by n, such that

$$v_n \to v \quad \text{strongly in } L^2(\Omega). \tag{10.25}$$

Then, from (10.19) and (10.24)(ii), it follows by linearity that

$$\begin{aligned} \|v_n - v_m\|_{(H^1(\Omega))^N} &\le c_K\left[\|v_n - v_m\|_{(L^2(\Omega))^N} + \left(\int_\Omega |e(v_n - v_m)|^2\,dx\right)^{\frac{1}{2}}\right] \\ &\le c_K\left[\|v_n - v_m\|_{(L^2(\Omega))^N}\right. \\ &\quad \left. + c_2\left(\int_\Omega (|e(v_n)|^2 + |e(v_m)|^2|)\,dx\right)^{\frac{1}{2}}\right]. \end{aligned}$$

Therefore, from (10.24)(ii) and (10.25), $\{v_n\}$ is a Cauchy sequence in $H^1(\Omega)$, so that

$$v_n \to v \quad \text{strongly in } \mathcal{V},$$

with

$$\begin{cases} i) & \|v\|_{\mathcal{V}} = 1 \\ ii) & \int_\Omega |e(v)|^2\,dx = 0. \end{cases} \tag{10.26}$$

Statement (10.24)(ii), together with Step 1, implies that $v \equiv 0$, which contradicts (10.26)(i). Hence, (10.23) holds and the proof of the corollary is now complete. □

The main result of this section is the following:

Theorem 10.6. *Under assumptions (10.12), problem (10.13) has a unique solution $u \in \mathcal{V}$. Moreover,*

$$\|u\|_{\mathcal{V}} \le \frac{1}{\alpha}\left(\|f\|_{\mathcal{V}'} + C_\gamma(\Omega)\|g\|_{(H^{-\frac{1}{2}}(\Gamma_2))^N}\right), \tag{10.27}$$

where $C_\gamma(\Omega)$ is the trace constant defined by Proposition 3.31.

Proof. From (10.17) and Proposition 10.5, the bilinear form (10.15) is coercive on $\mathcal{V}$. We have already mentioned that it is also continuous on $\mathcal{V} \times \mathcal{V}$. On the other hand, arguing exactly as in the proof of Theorem 4.21, we have that (10.16) defines F as an element in $\mathcal{V}'$ with

$$\|F\|_{\mathcal{V}'} \leq \|f\|_{\mathcal{V}'} + C_\gamma(\Omega)\|g\|_{(H^{-\frac{1}{2}}(\Gamma_2))^N}.$$

All the hypotheses of Theorem 4.6 (the Lax–Milgram theorem) are fulfilled, whence the claimed result. □

Remark 10.7. It is interesting to notice that, due to symmetry properties of B, the form a given in (10.15) is symmetric. Then, from Theorem 4.8, it follows that the solution u of (10.13) is the unique solution of the minimization problem

$$\begin{cases} \text{Find } u \in \mathcal{V} \text{ such that} \\ J(u) = \inf\limits_{v \in \mathcal{V}} J(v), \end{cases}$$

where

$$J(v) = \frac{1}{2}\int_\Omega B(x)\, e(v)\, e(v)\, dx \; - \; \langle f, v\rangle_{\mathcal{V}',\mathcal{V}} - \langle g, v\rangle_{(H^{-\frac{1}{2}}(\Gamma_2))^N,(H^{\frac{1}{2}}(\Gamma_2))^N}$$

for all $v \in \mathcal{V}$. ◇

10.2 Auxiliary periodic problems

As in the scalar case (Section 6.1), we introduce a family of auxiliary periodic boundary value problems posed on the reference cell Y and related to the tensor A defined by (10.3). They are the corresponding corrector functions for the linearized elasticity system. In Section 6.1 we defined two different families of auxiliary functions since we did not supposed any symmetry. In the present case, due to symmetry properties (10.1)(ii), we need to introduce only one family of functions.

To begin with, for any $\ell, m \in \{1, \dots, N\}$, let us define the vector-valued function $P^{\ell m}(y) = (P_k^{\ell m}(y))_{1 \leq k \leq N}$ by

$$P_k^{\ell m}(y) = y_m \delta_{k\ell} \quad k = 1, \dots, N, \tag{10.28}$$

where δ_{ki} is the Kronecker symbol. Introduce, for any $\ell, m \in \{1, \dots, N\}$, the vector-valued function $\chi^{\ell m} = (\chi_k^{\ell m})_{1 \leq k \leq N}$, a solution of the system

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a_{ijkh}\,\dfrac{\partial(\chi_k^{\ell m} - P_k^{\ell m})}{\partial x_h}\right) = 0 \quad \text{in } Y, \quad i = 1, \dots, N \\ \chi_k^{\ell m} \quad Y\text{-periodic} \\ \mathcal{M}_Y(\chi_k^{\ell m}) = 0 \end{cases}$$

which can be rewritten

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a_{ijkh}\,\dfrac{\partial \chi_k^{\ell m}}{\partial x_h}\right) = -\dfrac{\partial a_{ij\ell m}}{\partial x_j} & \text{in } Y, \quad i = 1, \ldots, N, \\ \chi_k^{\ell m} \quad Y\text{-periodic} \\ \mathcal{M}_Y(\chi_k^{\ell m}) = 0. \end{cases} \tag{10.29}$$

Using definition (10.2), the variational formulation of this problem is

$$\begin{cases} \text{Find } \chi^{\ell m} \in (W_{\text{per}}(Y))^N \text{ such that} \\ a_Y(\chi^{\ell m}, v) = \displaystyle\int_Y A\, e(P^{\ell m})\, e(v)\, dy \\ \forall v \in (W_{\text{per}}(Y))^N, \end{cases} \tag{10.30}$$

where

$$a_Y(u, v) = \int_Y A(y)\, e(u)\, e(v)\, dy, \qquad \forall u,\ v \in (W_{\text{per}}(Y))^N \tag{10.31}$$

and (see 4.66)),

$$W_{\text{per}}(Y) = \left\{ v \in H^1_{\text{per}}(Y);\ \mathcal{M}_Y(v) = 0 \right\},$$

with $H^1_{\text{per}}(Y)$ given by Definition 3.48.

Recall that due to the Poincaré–Wirtinger inequality (Theorem 3.28), the space $(W_{\text{per}}(Y))^N$ can be equipped with the norm

$$\|v\|_{(W_{\text{per}}(Y))^N} = \left(\sum_{i=1}^{N} \|\nabla v_i\|^2_{L^2(Y)} \right)^{\frac{1}{2}},$$

for $v = (v_1, \ldots, v_N) \in (W_{\text{per}}(Y))^N$.

As for problem (10.13), the existence of a solution of problem (10.30) is based on the Korn inequality below corresponding to the periodic case. We refer the reader to Kondratiev and Oleinik (1989a,b) and to Oleinik, Shamaev, and Yosifian (1992) for its proof.

Theorem 10.8 (Korn inequality for the periodic case). *There exists a constant $c_K = c_K(Y)$ such that*

$$\|v\|_{(H^1(Y))^N} \leq c_K \left(\int_Y |e(v)|^2\, dy \right)^{\frac{1}{2}},$$

for all $v \in (W_{\text{per}}(Y))^N$.

Then one has immediately the following result:

Proposition 10.9. *The quantity*

$$|||v|||_Y = \left(\int_Y |e(v)|^2 \, dy \right)^{\frac{1}{2}}$$

defines on $(W_{\text{per}}(Y))^N$ *a norm equivalent to the norm* $\|v\|_{(W_{\text{per}}(Y))^N}$.

We can now prove the existence and uniqueness of the corrector functions $\chi^{\ell m}$.

Proposition 10.10. *Assume that* A *satisfies (10.3). Then, for any* $\ell, m \in \{1, \ldots, N\}$, *problem (10.30) has a unique solution* $\chi^{\ell m} \in (W_{\text{per}}(Y))^N$.

Moreover, its extension by periodicity (see (3.7)) to the whole of $\mathbb{R}^N$, *still denoted by* $\chi^{\ell m}$, *is the unique solution of the problem*

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a_{ijkh} \dfrac{\partial \chi_k^{\ell m}}{\partial x_h} \right) = -\dfrac{\partial a_{ij\ell m}}{\partial x_j} \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \quad i = 1, \ldots, N, \\ \chi_k^{\ell m} \quad Y\text{-periodic} \\ \mathcal{M}_Y(\chi_k^{\ell m}) = 0. \end{cases} \tag{10.32}$$

Proof. The existence and uniqueness of system (10.31) are straightforward by the Lax–Milgram theorem (Theorem 4.6). Due to Proposition 10.9, we can take, in Theorem 4.6, $H = (W_{\text{per}}(Y))^N$ equipped with the norm $|||v|||_Y$. The coerciveness of the form a_Y defined by (10.31) follows then from assumption (10.3) and reads

$$\alpha |||v|||_Y^2 \leq a_Y(v,\, v), \qquad \forall v \in (W_{\text{per}}(Y))^N. \tag{10.33}$$

The proof of the second statement follows the outlines of the proof of Theorem 4.28. □

Set now, for any $\ell, m \in \{1, \ldots, N\}$,

$$w^{\ell m} = -\chi^{\ell m} + P^{\ell m} \tag{10.34}$$

which, from (10.29) satisfies

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a_{ijkh} \dfrac{\partial w_k^{\ell m}}{\partial x_h} \right) = 0 \quad \text{in } Y, \quad i = 1, \ldots, N, \\ w_k^{\ell m} - P_k^{\ell m} \quad Y\text{-periodic} \\ \mathcal{M}_Y(w_k^{\ell m} - P_k^{\ell m}) = 0. \end{cases}$$

In view of (10.30), its corresponding variational formulation is

$$\begin{cases} \text{Find } w^{\ell m} \text{ with } w^{\ell m} - P^{\ell m} \in (W_{\text{per}}(Y))^N \text{ such that} \\ a_Y(w^{\ell m},\, v) = 0 \\ \forall v \in (W_{\text{per}}(Y))^N, \end{cases} \tag{10.35}$$

Let still denote by $\chi^{\ell m}$ its extension by periodicity to the whole of $\mathbb{R}^N$. Then $w^{\ell m} = -\chi^{\ell m} + P^{\ell m}$ satisfies

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a_{ijkh}\,\dfrac{\partial w_k^{\ell m}}{\partial x_h}\right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \quad i = 1,\dots,N, \\ w_k^{\ell m} - P_k^{\ell m} \quad Y\text{-periodic} \\ \mathcal{M}_Y(w_k^{\ell m} - P_k^{\ell m}) = 0. \end{cases} \tag{10.36}$$

Let us mention that further properties of functions $\chi^{\ell m}$, such as their symmetries, have been investigated by Léné (1984).

10.3 Homogenization results

Let now turn back to system (10.7) whose variational formulation is (see (10.3))

$$\begin{cases} \text{Find } u^\varepsilon \in \mathcal{V} \text{ such that} \\ \displaystyle\int_\Omega A^\varepsilon(x)\, e(u^\varepsilon)\, e(v)\, dx = \langle f, v\rangle_{\mathcal{V}',\mathcal{V}} + \langle g, v\rangle_{(H^{-\frac{1}{2}}(\Gamma_2))^N,(H^{\frac{1}{2}}(\Gamma_2))^N}, \\ \forall v \in \mathcal{V}. \end{cases} \tag{10.37}$$

The solution u^ε exists and is unique due to Theorem 10.6 applied with A^ε instead of B, which is allowed due to (10.6).

We are now interested in the behaviour of u^ε as $\varepsilon \to 0$. The homogenized problem is given by the following result:

Theorem 10.11. *Let $f \in \mathcal{V}'$, $g \in (H^{-\frac{1}{2}}(\Gamma_2))^N$ and A^ε be given by (10.3)–(10.5). Let $u^\varepsilon \in \mathcal{V}$ be the solution of (10.37). Then,*

$$\begin{cases} i) & u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } \mathcal{V}, \\ ii) & A^\varepsilon\, e(u^\varepsilon) \rightharpoonup A^0\, e(u^0) \quad \text{weakly in } (L^2(\Omega))^{N\times N}, \end{cases}$$

where $u^0 = (u_1^0, \dots, u_N^0)$ is the unique solution in $\mathcal{V}$ of the homogenized system

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\left(a_{ijkh}^0\,\dfrac{\partial u_k^0}{\partial x_h}\right) = f_i \quad \text{in } \Omega \\ u^0 = 0 \quad \text{on } \Gamma_1 \\ a_{ijkh}^0\,\dfrac{\partial u_k^0}{\partial x_h}\, n_j = g_i \quad \text{on } \Gamma_2. \end{cases} \tag{10.38}$$

for $i = 1,\dots,N$. The homogenized tensor $A^0 = (a_{ijkh}^0)_{1\le i,j,k,h\le N}$ is constant, verifies the symmetries of elasticity (10.1)(ii) and a coerciveness condition (10.1)(iii) for some α^0. Its elements are given by

$$a_{ijkh}^0 = \mathcal{M}_Y\,(a_{ij\ell m}\, e_{\ell m}(w^{kh})). \tag{10.39}$$

where w^{kh} are defined by (10.34) and (10.35).

Before proving this theorem, we give some other expressions for the tensor A^0 from which, in particular, we deduce its coerciveness.

From (10.28), one immediately has

$$e_{\ell m}(P^{kh}) = \frac{1}{2}\left(\frac{\partial P_\ell^{kh}}{\partial y_m} + \frac{\partial P_m^{kh}}{\partial y_\ell}\right) = \tfrac{1}{2}(\delta_{\ell k}\delta_{mh} + \delta_{\ell h}\delta_{mk}).$$

Then, using (10.34) in (10.39) it follows that

$$\begin{cases} a^0_{ijkh} = \mathcal{M}_Y(a_{ijkh}) - \mathcal{M}_Y(a_{ij\ell m}e_{\ell m}(\chi^{kh})) \\ \quad = \dfrac{1}{|Y|}\displaystyle\int_Y a_{ijkh}(y)\,dy - \dfrac{1}{|Y|}\int_Y a_{ij\ell m}(y)\frac{\partial \chi_\ell^{kh}}{\partial y_m}\,dy, \end{cases} \tag{10.40}$$

since

$$e_{\ell m}(P^{kh}) = \tfrac{1}{2}(\delta_{\ell k}\delta_{mh} + \delta_{\ell h}\delta_{mk}).$$

Proposition 10.12. *Let A^0 be given by (10.39). One has*

$$a^0_{ijkh} = \frac{1}{|Y|}\int_Y A(y)\,e(w^{ij})\,e(w^{kh})\,dy = \frac{1}{|Y|}\,a_Y(w^{ij}, w^{kh}), \tag{10.41}$$

where a_Y is defined by (10.31).

Proof. The proof is analogous to that of Proposition 6.8. It consists in choosing $v = \chi^{kh}$ as test function in (10.30). After some easy calculations, we derive (see (6.39))

$$\int_Y A(y)\,e(P^{\ell m} - \chi^{\ell m})\,e(\chi^{kh})\,dy = 0,$$

which, together with (10.40), implies (10.41). □

Corollary 10.13. *Let A^0 be given by (10.39). Then, there exist two positive numbers α^0 and β^0 such that*

$$A^0 \in M_e(\alpha^0, \beta^0, \Omega).$$

Proof. We have to prove (10.1). Properties (10.1)(i) and (10.1)(iv) are trivial since A^0 is constant. The symmetries (10.1)(ii) are straightforward from (10.41) and the symmetries of A.

It remains only to prove that there exists a positive number α^0, such that

$$\alpha^0|m|^2 \le A^0\,m\,m \quad \text{for any symmetric matrix } m = (m_{ij})_{1\le i,j\le N}. \tag{10.42}$$

We follow the lines of the proof of Proposition 6.12. Let m be a symmetric matrix. Then,

$$A^0\,m\,m = a^0_{ijkh}\,m_{ij}\,m_{kh} = \frac{1}{|Y|}\,a_Y(w^{ij}\,m_{ij}, w^{kh}\,m_{kh}) = \frac{1}{|Y|}\,a_Y(Z,Z), \tag{10.43}$$

where Z is the vector $w^{ij}\, m_{ij}$. The coerciveness of a_Y implies that

$$A^0\, m\, m \geq 0.$$

Let us show that this inequality holds strictly if $m \not\equiv 0$. To do so, suppose that m is a symmetric matrix such that $A^0\, m\, m = 0$. The coerciveness (10.33) of a_Y implies that

$$\alpha |||Z|||_Y = 0.$$

Hence, from Theorem 10.8, we have

$$\nabla(w_k^{ij}\, m_{ij}) = 0, \quad \forall i, j, k = 1, \ldots, N.$$

Recalling definition (10.34), this means that

$$\frac{\partial(\chi_k^{ij}\, m_{ij})}{\partial y_h} = \frac{\partial(P_k^{ij}\, m_{ij})}{\partial y_h} = m_{kh}, \quad \forall i, j, k, h = 1, \ldots, N.$$

Integrating over Y, since χ_k^{ij} is Y-periodic, one has

$$0 = |Y| m_{kh}, \quad \forall k, h = 1, \ldots, N,$$

which implies $m \equiv 0$. To prove the existence of an α^0 satisfying (10.42), we argue as at the end of the proof of Theorem 5.10. □

Proof of Theorem 10.11. We prove the result by the oscillating test functions method due to Tartar which we used for the elliptic case in Chapter 8.

Due to (10.6) and Theorem 10.6, we have the a priori estimate

$$\|u^\varepsilon\|_{(H^1(\Omega))^N} \leq \frac{1}{\alpha}\left(\|f\|_{\mathcal{V}'} + C_\gamma(\Omega)\|g\|_{(H^{-\frac{1}{2}}(\Gamma_2))^N}\right).$$

Consequently, from Proposition 10.5, it follows that

$$\int_\Omega |e(u^\varepsilon)|\, dx \leq c,$$

where c is independent of ε. Introduce the stress tensor (see (10.9)) $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon)_{1\leq i,j\leq N} = A^\varepsilon\, e(u^\varepsilon)$ defined by

$$\sigma_{ij}^\varepsilon = a_{ijkh}^\varepsilon\, e_{kh}(u^\varepsilon),$$

which satisfies

$$\int_\Omega \sigma^\varepsilon(x)\, e(v)\, dx = \langle f, v\rangle_{\mathcal{V}',\mathcal{V}} + \langle g, v\rangle_{(H^{-\frac{1}{2}}(\Gamma_2))^N,(H^{\frac{1}{2}}(\Gamma_2))^N}, \quad \forall v \in \mathcal{V}. \quad (10.44)$$

Moreover, thanks to (10.6), one also has

$$\|\sigma^\varepsilon\|_{(L^2(\Omega))^{N\times N}} \leq c.$$

From these a priori estimates, we have the following convergences (up to a subsequence):

$$\begin{cases} i) & u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } (H^1(\Omega))^N \\ ii) & u^\varepsilon \to u^0 \quad \text{strongly in } (L^2(\Omega))^N \\ iii) & \sigma^\varepsilon \rightharpoonup \sigma^0 \quad \text{weakly in } (L^2(\Omega))^{N\times N}. \end{cases} \tag{10.45}$$

We can pass to the limit in (10.44) to obtain

$$\int_\Omega \sigma^0(x)\, e(v)\, dx = \langle f, v\rangle_{\mathcal{V}',\mathcal{V}} + \langle g, v\rangle_{(H^{-\frac{1}{2}}(\Gamma_2))^N,(H^{\frac{1}{2}}(\Gamma_2))^N}, \quad \forall v \in \mathcal{V}. \tag{10.46}$$

As in Chapter 8, we have now to identify σ^0 in terms of u^0. Indeed, Theorem 10.11 is proved if we show that

$$\sigma^0 = A^0\, e(u^0), \tag{10.47}$$

since (10.46) is nothing else than the variational formulation of (10.38). On the other hand, by Corollary 10.13 and Theorem 10.6 one has the uniqueness of such a solution. This implies that (10.47) will provide the convergence for the whole sequences in (10.45).

In order to prove (10.47), let us set

$$w_\varepsilon^{kh}(x) = \varepsilon w^{kh}\left(\frac{x}{\varepsilon}\right) = P^{kh}(x) - \varepsilon\,\chi^{kh}\left(\frac{x}{\varepsilon}\right),$$

where P^{kh}, χ^{kh} and w^{kh} are defined respectively, by (10.28), (10.29), and (10.34). Recalling that χ^{kh} is Y-periodic we obtain, in view of Theorem 2.6 that

$$\begin{cases} i) & w_\varepsilon^{kh} \rightharpoonup P^{kh} \quad \text{weakly in } (H^1(\Omega))^N \\ ii) & w_\varepsilon^{kh} \to P^{kh} \quad \text{strongly in } (L^2(\Omega))^N. \end{cases} \tag{10.48}$$

Introduce the matrix

$$\eta_\varepsilon^{kh}(x) = A^\varepsilon(x)\, e(w_\varepsilon^{kh})(x) = (A\, e_y(w^{kh}))\left(\frac{x}{\varepsilon}\right),$$

where the notation e_y means that the derivatives are taken with respect to the variable y. Observe that by construction (see also the proof of (8.14)), from (10.35) one has

$$\int_\Omega \eta_\varepsilon^{kh}\, e(v)\, dx = 0, \quad \forall v \in (H_0^1(\Omega))^N. \tag{10.49}$$

By the same arguments as those used to prove (8.13), we have the convergence

$$\eta_\varepsilon^{kh} \rightharpoonup \mathcal{M}_Y(A\, e_y(w^{kh})) \quad \text{weakly in } (L^2(\Omega))^{N\times N}.$$

Recalling definition (10.39), this signifies that

$$(\eta_\varepsilon^{kh})_{ij} \rightharpoonup a_{ijkh}^0 \quad \text{weakly in } L^2(\Omega). \tag{10.50}$$

Let now $\varphi \in \mathcal{D}(\Omega)$ and choose $\varphi w_\varepsilon^{kh}$ as test function in (10.44) and φu^ε as test function in (10.49). We have

$$\int_\Omega \sigma^\varepsilon\, e(w_\varepsilon^{kh})\, \varphi\, dx + \frac{1}{2}\int_\Omega \sigma_{\ell m}^\varepsilon \left[(w_\varepsilon^{kh})_\ell \frac{\partial \varphi}{\partial x_m} + (w_\varepsilon^{kh})_m \frac{\partial \varphi}{\partial x_\ell}\right] dx = \langle f, \varphi w_\varepsilon^{kh}\rangle_{V',V},$$

$$\int_\Omega \eta_\varepsilon^{kh}\, e(u^\varepsilon)\, \varphi\, dx + \frac{1}{2}\int_\Omega (\eta_\varepsilon^{kh})_{\ell m} \left[u_\ell^\varepsilon \frac{\partial \varphi}{\partial x_m} + u_m^\varepsilon \frac{\partial \varphi}{\partial x_\ell}\right] dx = 0.$$

Observe that from the symmetry of A^ε,

$$\sigma^\varepsilon\, e(w_\varepsilon^{kh}) = \eta_\varepsilon^{kh}\, e(u^\varepsilon).$$

Consequently, by subtraction we obtain

$$\begin{cases} \displaystyle\frac{1}{2}\int_\Omega \sigma_{\ell m}^\varepsilon \left[(w_\varepsilon^{kh})_\ell \frac{\partial \varphi}{\partial x_m} + (w_\varepsilon^{kh})_m \frac{\partial \varphi}{\partial x_\ell}\right] dx \\ \displaystyle\qquad -\frac{1}{2}\int_\Omega (\eta_\varepsilon^{kh})_{\ell m} \left[u_\ell^\varepsilon \frac{\partial \varphi}{\partial x_m} + u_m^\varepsilon \frac{\partial \varphi}{\partial x_\ell}\right] dx = \langle f, \varphi w_\varepsilon^{kh}\rangle_{V',V}. \end{cases} \tag{10.51}$$

Let us now pass to the limit in this identity as $\varepsilon \to 0$ in (8.15). By using convergences (10.45) and (10.48) and definition (10.28) of P^{kh}, we have

$$\begin{cases} \displaystyle\frac{1}{2}\int_\Omega \sigma_{\ell m}^0 \left[y_h \delta_{k\ell} \frac{\partial \varphi}{\partial x_m} + y_h \delta_{km} \frac{\partial \varphi}{\partial x_\ell}\right] dx \\ \displaystyle\qquad -\frac{1}{2}\int_\Omega a_{\ell m k h}^0 \left[u_\ell^0 \frac{\partial \varphi}{\partial x_m} + u_m^0 \frac{\partial \varphi}{\partial x_\ell}\right] dx = \langle f, \varphi P^{kh}\rangle_{V',V}. \end{cases} \tag{10.52}$$

This can be rewritten in the form

$$\begin{aligned} \int_\Omega \sigma^0\, e(P^{kh}\varphi)\, dx \quad &- \quad \int_\Omega \sigma_{kh}^0\, \varphi\, dx + \int_\Omega a_{\ell m k h}^0\, e_{\ell m}(u^0)\, \varphi\, dx \\ &= \quad \langle f, \varphi P^{kh}\rangle_{V',V}. \end{aligned}$$

By using (10.46) written for the test function $v = \varphi P^{kh}$, this becomes

$$\int_\Omega \sigma_{kh}^0\, \varphi\, dx = \int_\Omega a_{\ell m k h}^0\, e_{\ell m}(u^0)\, \varphi\, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Hence Theorem 1.44 implies that

$$\sigma_{kh}^0 = a_{\ell m k h}^0\, e_{\ell m}(u^0),$$

which is exactly (10.47). This ends the proof. □

We adapted to the linearized elasticity system the proof of the homogenization result for the scalar case. In the same spirit, most of the results from Chapter 8 can easily be generalized to the present problem. We merely state here the convergence of the energy and a corrector result, whose proofs can be done by following step by step the proofs of the analogous results given in Sections 8.2 and 8.3.

Let us define the energies associated respectively to problem (10.37) and (10.38),

$$E^\varepsilon(u^\varepsilon) = \int_\Omega A^\varepsilon\, e(u^\varepsilon)\, e(u^\varepsilon)\, dx.$$

and

$$E^0(u^0) = \int_\Omega A^0\, e(u^0)\, e(u^0)\, dx.$$

Then we have

Proposition 10.14. *Let u^ε be the solution of problem (10.37) and u^0, A^0 given by Theorem 10.11. Then,*

$$E^\varepsilon(u^\varepsilon) \longrightarrow E^0(u^0).$$

Moreover,

$$A^\varepsilon\, e(u^\varepsilon)\, e(u^\varepsilon) \longrightarrow A^0\, e(u^0)\, e(u^0) \quad \text{in } \mathcal{D}'(\Omega).$$

As in the elliptic case (see Proposition 8.3), this result allows us to make precise the constants α^0 and β^0 from Corollary 10.13. As matter of fact, the following result holds:

Proposition 10.15. *The matrix A^0 given by Theorem 10.11 is such that*

$$A^0 \in M_e\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right).$$

Finally, introduce the corrector tensor $C^\varepsilon = (C^\varepsilon_{ijkh})_{1\le i,j,k,h\le N}$ defined by

$$\begin{cases} C^\varepsilon_{ijkh}(x) = C_{ijkh}\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } \Omega \\ C_{ijkh}(y) = e_{ij}(w^{kh}(y)) & \text{a.e. on } Y, \end{cases}$$

where w^{kh} is given by (10.29) and (10.34).

Theorem 10.16. *Let u^ε be the solution of problem (10.37) and u^0, A^0 given by Theorem 10.11. Then*

$$e(u^\varepsilon) - C^\varepsilon e(u^0) \;\rightarrow\; 0 \quad \text{strongly in } (L^1(\Omega))^{N\times N}.$$

Moreover, if $C \in (L^r(Y)))^{N^4}$ for some r such that $2 \le r \le \infty$, and $\nabla u^0 \in (L^s(\Omega))^{N\times N}$ for some s such that $2 \le s < \infty$, then

$$e(u^\varepsilon) - C^\varepsilon e(u^0) \rightarrow 0 \quad \text{strongly in } (L^t(\Omega)))^{N\times N},$$

where

$$t = \min\left\{2, \frac{r\,s}{r+s}\right\}.$$

11

Homogenization of the heat equation

In this chapter we are interested in the asymptotic behaviour as $\varepsilon \to 0$ of the solution $u_\varepsilon = u_\varepsilon(x,t)$ of the problem

$$\begin{cases} u'_\varepsilon - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega \times]0,T[\\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \times]0,T[\\ u_\varepsilon(x,0) = u_\varepsilon^0(x) \quad \text{in } \Omega, \end{cases} \tag{11.1}$$

where the operators div and ∇ are taken with respect to the space variable $x \in \Omega$ and $'$ denotes the derivative with respect to the time variable $t \in]0,T[$, with $T > 0$. We suppose we are given the source term f_ε and the initial state u_ε^0. Here, as in the previous chapters, the matrix A^ε is Y-periodic and defined by

$$a_{ij}^\varepsilon(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall i,j = 1,\dots,N \tag{11.2}$$

and

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) = (a_{ij}^\varepsilon(x))_{1\le i,j\le N} \quad \text{a.e. on } \mathbb{R}^N, \tag{11.3}$$

where

$$\begin{cases} a_{ij} \text{ is } Y\text{-periodic}, \quad \forall i,j = 1,\dots,N \\ A = (a_{ij})_{1\le i,j\le N} \in M(\alpha,\beta,Y), \end{cases} \tag{11.4}$$

with $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$ and $M(\alpha,\beta,Y)$ given by Definition 4.11.

As mentioned in Section 5.2, problem (11.1) is known as the heat equation, since it models the heat transfer in composite materials when the temperature u_ε is time-dependent. If u_ε and the source f_ε are independent of the time, problem (11.1) reduces to the Dirichlet elliptic problem (5.6) modelling the stationary heat diffusion (see Section 5.2). Problem (11.1) is a particular case of a large class of partial differential equations called parabolic.

As for the elliptic case, there is a very large range of results concerning parabolic problems. For general results concerning parabolic equations, we refer the reader to Lions and Magenes (1968a) (see also Pazy, 1974, Wloka, 1987, Cazenave and Haraux, 1998). For homogenization results concerning the heat equation, we refer to Bensoussan, Lions, and Papanicolaou (1978), Sanchez-Palencia (1980) for the periodic case and to Spagnolo (1967, 1968), Colombini and Spagnolo (1977) for the general non-periodic one.

In Section 11.1 below we will show the existence and uniqueness of the solution of (11.1) in a variational framework, when f_ε is in $L^2(\Omega\times]0,T[)$ and u_ε^0 in $L^2(\Omega)$. For the definition and properties of various time-dependent functional spaces used in this chapter, we refer to Section 3.5. In Section 11.2 and 11.3 we give the main homogenization results for problem (11.1).

11.1 Existence and uniqueness

Let Ω be a bounded open set in $\mathbb{R}^N$ and consider the following problem:

$$\begin{cases} u' - \operatorname{div}(B\nabla u) = f & \text{in } \Omega\times]0,T[\\ u = 0 & \text{on } \partial\Omega\times]0,T[\\ u(x,0) = u^0(x) & \text{in } \Omega, \end{cases} \tag{11.5}$$

under the following assumptions:

$$\begin{cases} i) & B \in M(\alpha,\beta,\Omega) \\ ii) & f \in L^2(\Omega\times]0,T[) \\ iii) & u^0 \in L^2(\Omega). \end{cases} \tag{11.6}$$

As in Chapter 3 (see Theorem 3.58) define

$$W = \{v \mid v \in L^2(0,T;\ H_0^1(\Omega)),\ v' \in L^2(0,T;\ H^{-1}(\Omega))\},$$

which is a Banach space with respect to the norm of the graph, i.e.

$$\|v\|_W = \|v\|_{L^2(0,T;\ H_0^1(\Omega))} + \|v'\|_{L^2(0,T;\ H^{-1}(\Omega))}.$$

Then, the variational formulation of problem (11.5) is

$$\begin{cases} \text{Find } u \in W \text{ such that} \\ \langle u'(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \displaystyle\int_\Omega B(x)\nabla u(x,t)\,\nabla v(x)\,dx \\ \qquad\qquad = \displaystyle\int_\Omega f(x,t)\,v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega), \\ u(x,0) = u^0(x) \quad \text{in } \Omega. \end{cases} \tag{11.7}$$

Remark 11.1. The initial condition has to be understood in $L^2(\Omega)$ since, due to Theorem 3.58, $u \in C([0,T];\ L^2(\Omega))$. This implies, in particular, that

$$\lim_{t\to 0} \|u(x,t)\|_{L^2(\Omega)} = \|u^0\|_{L^2(\Omega)}.$$

◇

We prove in this section the existence and uniqueness of the solution of problem (11.7). To do so, we will use the Faedo–Galerkin method (see Lions and Magenes, 1968a). This method is based on the fact that the Hilbert space $H_0^1(\Omega)$

can be approximated by a sequence of finite dimensional subspaces $\{V_m\}$ as $m \to \infty$. The proof consists in several steps. In the first one, we construct these subspaces V_m. In the second step, for any $m \in \mathbb{N}$, we formulate an approximate problem of (11.7) and show that it has a unique solution u_m. In the third step, we give a priori estimates of u_m, independent of m. In the fourth one, we pass to the limit as $m \to \infty$ and prove that u_m converges in an appropriate sense to a solution $u \in \mathcal{W}$ of (11.7). In the fifth step, we prove a priori estimate on u. In the last step we prove the uniqueness of the solution.

Let us point out that the main interest of this method is the fact that it provides a priori estimates on the solutions u. In our context, this is essential in order to study the asymptotic behaviour as $\varepsilon \to 0$ of problem (11.1).

Theorem 11.2. *Under assumptions (11.6), problem (11.7) has a unique solution $u \in \mathcal{W}$. Moreover, there exists a constant c depending on α, β, Ω, and T such that*

$$\|u\|_{\mathcal{W}} + \|u\|_{L^\infty(0,T;\,L^2(\Omega))} \leq c\big(\|f\|_{L^2(\Omega\times]0,T[)} + \|u^0\|_{L^2(\Omega)}\big). \tag{11.8}$$

Proof. As mentioned above, the proof consists in six steps.
Step 1. To construct the subspaces V_m we will make use of Proposition 8.23. Let (w_ℓ) be the orthonormal basis in $L^2(\Omega)$ given by (iii) from Proposition 8.23 and Remark 8.24 for the choice $B = I$ in problem (8.75). This means that the operator $\mathcal{B}$ is $-\Delta$. Moreover, by definition the set (w_ℓ) is orthogonal in $H_0^1(\Omega)$.

Denote by V_m be the m dimensional subspace of $H_0^1(\Omega)$, spanned by $w_1, \ldots,$ w_m.

Let us introduce also the projection operator P_m from $L^2(\Omega)$ on V_m defined by

$$P_m v = \sum_{i=1}^{m} (v, w_i)_{L^2(\Omega)}\, w_i, \quad \forall v \in L^2(\Omega). \tag{11.9}$$

From classical results concerning Hilbert spaces (see for instance Yosida, 1964 Chapter 3), one has that

$$P_m v \to v \quad \text{strongly in } L^2(\Omega), \quad \forall v \in L^2(\Omega), \tag{11.10}$$

and furthermore,

$$\|P_m\|_{\mathcal{L}(L^2(\Omega);\, L^2(\Omega))} \leq 1. \tag{11.11}$$

Moreover, the restriction of P_m to $H_0^1(\Omega)$, namely

$$P_m v = \sum_{i=1}^{m} (v, w_i)_{L^2(\Omega)}\, w_i, \quad \forall v \in H_0^1(\Omega),$$

is in $\mathcal{L}(H_0^1(\Omega);\, H_0^1(\Omega))$ and satisfies

$$\|P_m\|_{\mathcal{L}(H_0^1(\Omega);\, H_0^1(\Omega))} \leq 1. \tag{11.12}$$

Indeed,

$$\begin{aligned}\|P_m v\|^2_{H^1_0(\Omega)} &= \sum_{i=1}^{m} (v, w_i)^2_{L^2(\Omega)} \|\nabla w_i\|^2_{L^2(\Omega)} \\ &\leq \sum_{i=1}^{\infty} (v, w_i)^2_{L^2(\Omega)} \|\nabla w_i\|^2_{L^2(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)} = \|v\|^2_{H^1_0(\Omega)}.\end{aligned}$$

Moreover, as before,

$$P_m v \to v \quad \text{strongly in } H^1_0(\Omega) \quad \forall v \in H^1_0(\Omega). \tag{11.13}$$

Observe now that P_m can be extended to $H^{-1}(\Omega)$ by setting

$$P_m v = \sum_{i=1}^{m} \langle v, w_i \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} w_i, \quad \forall v \in H^{-1}(\Omega).$$

Let us show that one still has

$$\|P_m\|_{\mathcal{L}(H^{-1}(\Omega); H^{-1}(\Omega))} \leq 1. \tag{11.14}$$

Indeed, for any $z \in H^1_0(\Omega)$, due to Remark 3.44, one can write

$$\begin{aligned}|\langle P_m v, z >_{H^{-1}(\Omega), H^1_0(\Omega)}| &= \left| \sum_{i=1}^{m} \langle v, w_i \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \int_\Omega w_i \, z \, dx \right| \\ &= \left| \left\langle v, \sum_{i=1}^{m} (z, w_i)_{L^2(\Omega)} w_i \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \right| \\ &= \left| \langle v, P_m z \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \right| \leq \|v\|_{H^{-1}(\Omega)} \|z\|_{H^1_0(\Omega)}\end{aligned}$$

where we have used (11.12) and Remark 3.44. Then (11.14) is straightforward.

Step 2. Since, by assumption (11.6), $u^0 \in L^2(\Omega)$, if we set $u^0_m = P_m u^0$, from (11.10) we have

$$u^0_m \to u^0 \quad \text{strongly in } L^2(\Omega). \tag{11.15}$$

Let now introduce, for any $m \in \mathbb{N}^*$, the finite dimensional approximate problem

$$\begin{cases} \text{Find } u_m = \sum_{j=1}^{m} g^m_j(t) w_j \in V_m \text{ such that} \\ \int_\Omega u'_m(x,t)\, w_k \, dx + \int_\Omega B \nabla u_m(x,t) \, \nabla w_k \, dx \\ \qquad = \int_\Omega f(x,t)\, w_k \, dx, \quad \text{in } \mathcal{D}'(0,T) \quad \forall k = 1, \ldots, m \\ u_m(x,0) = u^0_m(x) \quad \text{in } \Omega. \end{cases} \tag{11.16}$$

From (11.9) and the initial condition in this problem, one has

$$\sum_{j=1}^{m} g_j^m(0) w_j = u_m(0) = u_m^0 = \sum_{j=1}^{m} (u^0, w_j)_{L^2(\Omega)} w_j,$$

which implies $g_j^m(0) = (u^0, w_j)_{L^2(\Omega)}$, since $w_1, \dots, w_m$ are linearly independent.

Consequently, problem (11.16) is a system of m linear ordinary differential equations of the first order with unknowns $g_1^m, \dots, g_m^m$, which reads

$$\begin{cases} \dfrac{dg_k^m}{dt} + \displaystyle\sum_{j=1}^{m} g_j^m(t) \int_\Omega B\nabla w_j \, \nabla w_k \, dx = \int_\Omega f(x,t)\, w_k \, dx \\ g_k^m(0) = (u^0, w_k), \end{cases}$$

for any $k = 1, \dots, m$. Classical results (see, for instance Coddington and Levinson, 1955, Chapter 3) give the existence and uniqueness of a continuous solution $g_1^m, \dots, g_m^m$ of this system on the interval $[0, T]$. Hence, u_m is determined and belongs to $C([0,T]; V_m)$.

Step 3. We will now prove that u_m satisfies some a priori estimates. To do so, let us multiply the kth equation in (11.16) by g_k^m and sum over k from 1 to m. We obtain

$$\int_\Omega u_m'(x,t)\, u_m(x,t)\, dx + \int_\Omega B\nabla u_m(x,t)\, \nabla u_m(x,t)\, dx = \int_\Omega f(x,t)\, u_m(x,t)\, dx.$$

Recalling the ellipticity assumption on the matrix B and applying successively the Cauchy–Schwarz inequality (Proposition 1.34) and the Poincaré inequality (Proposition 3.35) in the right-hand side term, we derive

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L^2(\Omega)} + \alpha \|u_m\|^2_{H_0^1(\Omega)} &\le \|f\|_{L^2(\Omega)} \|u_m\|_{L^2(\Omega)} \le C_\Omega \|f\|_{L^2(\Omega)} \|u_m\|_{H_0^1(\Omega)} \\ &= \Big(\frac{C_\Omega}{\sqrt{\alpha}} \|f\|_{L^2(\Omega)}\Big)\Big(\sqrt{\alpha}\|u_m\|_{H_0^1(\Omega)}\Big) \le \frac{C_\Omega^2}{2\alpha}\|f\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u_m\|^2_{H_0^1(\Omega)}, \end{aligned}$$

where C_Ω is the Poincaré constant which is obviously independent of m. Integrating over $]0, t[$ with $t \in [0, T]$, it follows that

$$\|u_m(t)\|^2_{L^2(\Omega)} + \alpha \int_0^t \|u_m(\tau)\|^2_{H_0^1(\Omega)}\, d\tau \le \|u_m^0\|^2_{L^2(\Omega)} + \frac{C_\Omega^2}{\alpha}\int_0^T \|f(\tau)\|^2_{L^2(\Omega)}\, d\tau.$$

This, together with (11.15), implies that $u_m \in L^\infty(0, T: L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ with

$$\|u_m\|_{L^\infty(0,T; L^2(\Omega))} + \|u_m\|_{L^2(0,T; H_0^1(\Omega))} \le c_0\big(\|u_m^0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega\times(0,T))}\big) \le c_1, \tag{11.17}$$

where c_0 and c_1 are constants independent of m.

We will now give an a priori estimate for u'_m. In order to do this, remark first that the equation in (11.16) implies that

$$(u'_m(t), v)_{L^2(\Omega)} = (-\text{ div } (B\nabla)u_m(t) + f, v)_{L^2(\Omega)}, \quad \forall v \in V_m.$$

This means that

$$u'_m(t) = -[P_m(\mathcal{F}(u_m) + f)](t), \tag{11.18}$$

where the operator P_m is defined by (11.9) and $\mathcal{F} = -\text{ div }(B\nabla)$.

Since B verifies (11.6)(i), it is easily seen that $\mathcal{F} \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))$. Therefore, for any $u_m \in V_m$ one has

$$\|\mathcal{F}(u_m)(t)\|_{H^{-1}(\Omega)} \leq \beta \|u_m(t)\|_{H_0^1(\Omega)}.$$

Hence, $\mathcal{F}(u_m) \in L^2(0,T;\, H^{-1}(\Omega))$ and in view of (11.17),

$$\begin{aligned} \|\mathcal{F}(u_m)\|_{L^2(0,T;\,H^{-1}(\Omega))} &\leq \beta\|u_m(t)\|_{L^2(0,T;\,H_0^1(\Omega))} \\ &\leq c_2\big(\|u_m^0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega\times(0,T))}\big), \end{aligned}$$

where c_2 is a constant independent of m. Then, using (11.14), one deduces from (11.18) the following a priori estimate:

$$\|u'_m\|_{L^2(0,T;\,H^{-1}(\Omega))} \leq c_3\big(\|u_m^0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega\times(0,T))}\big) \leq c_4, \tag{11.19}$$

where c_3 and c_4 are constants independent of m.

Step 4. By using the a priori estimates obtained in Step 3, we now pass to the limit in (11.16) as $m \to \infty$.

Thanks to estimates (11.17) and (11.19), we can extract a subsequence (still denoted by m), such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly* in } L^\infty(0,T;\, L^2(\Omega)) \\ u_m \rightharpoonup u & \text{weakly in } L^2(0,T;\, H_0^1(\Omega)) \\ u'_m \rightharpoonup u' & \text{weakly in } L^2(0,T;\, H^{-1}(\Omega)). \end{cases} \tag{11.20}$$

Indeed, the first convergence follows from Theorem 1.26 since from Proposition 3.59, one has $[L^1(0,T;\, L^2(\Omega))]' = L^\infty(0,T;\, L^2(\Omega))$ and from Proposition 3.55 one knows that the space $L^1(0,T;\, L^2(\Omega))$ is separable. The other convergences follow from Theorem 1.18 and Proposition 3.55, recalling that $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ are reflexive.

Now let ψ be given in $\mathcal{D}(0,T)$ and $v \in H_0^1(\Omega)$. Multiply the equation in (11.16) by $(v, w_k)_{L^2(\Omega)}\psi$ and sum over k from 1 to m. We get, after integration in t over $(0,T)$

$$\begin{cases} \displaystyle\int_0^T \int_\Omega u'_m(x,t)\,\psi(t)\,(P_m v)(x)\,dx\,dt \\ \displaystyle\qquad + \int_0^T \int_\Omega B(x)\nabla u_m(x,t)\,\psi(t)\,\nabla(P_m v)(x)\,dx\,dt \\ \displaystyle\qquad = \int_0^T \int_\Omega f(x,t)\,\psi(t)\,(P_m v)(x)\,dx\,dt, \end{cases} \tag{11.21}$$

where we have used definition (11.9). We now let $m \to \infty$ here. All the terms pass to the limit, thanks to convergences (11.20) and strong convergence (11.13). We finally get

$$\begin{cases} \displaystyle\int_0^T \langle u'(t), \psi(t)\, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + \int_0^T \int_\Omega B(x)\nabla u(x,t)\, \psi(t)\, \nabla v(x)\, dx\, dt \\ \displaystyle\qquad\qquad = \int_0^T \int_\Omega f(x,t)\, \psi(t)\, v(x)\, dx\, dt, \end{cases} \tag{11.22}$$

which is exactly the variational equation in (11.7) since ψ and v are arbitrary respectively, in $\mathcal{D}(0,T)$ and $H_0^1(\Omega)$.

It remains to show that u satisfies the initial condition $u(x,0) = u^0(x)$. To do so, observe that since $u_m \in W$, equation (11.21) is still valid if $\psi \in C^\infty([0,T])$. Choose a ψ such that $\psi(0) = 1$ and $\psi(T) = 0$. Then, integrating by parts with respect to t in (11.21), one has

$$\begin{cases} \displaystyle -\int_0^T \int_\Omega u_m(x,t)\, \psi'(t)\, (P_m v)(x)\, dx\, dt \\ \displaystyle\qquad + \int_0^T \int_\Omega B(x)\nabla u_m(x,t)\, \psi(t)\, \nabla(P_m v)(x)\, dx\, dt \\ \displaystyle\qquad = \int_0^T \int_\Omega f(x,t)\, \psi(t)\, (P_m v)(x)\, dx\, dt + \int_\Omega u_m^0\, (P_m v)(x)\, dx. \end{cases}$$

We can pass here to the limit by the same argument as above using the strong convergence (11.15). We obtain

$$-\int_0^T \int_\Omega u(x,t)\, \psi'(t)\, v(x)\, dx\, dt + \int_0^T \int_\Omega B(x)\nabla u(x,t)\, \psi(t)\, \nabla v(x)\, dx\, dt$$
$$= \int_0^T \int_\Omega f(x,t)\, \psi(t)\, v(x)\, dx\, dt + \int_\Omega u^0\, v\, dx.$$

Note that for the first term in this identity, due to Theorem 3.58 (iii) we have

$$\int_\Omega u(x,t)\, \psi'(t)\, v(x)\, dx = \langle u(t),\ \psi'(t) v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$
$$= -\langle u'(t), \psi(t)\, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \frac{d}{dt}\int_\Omega u(x,t)\psi(t)\, v(x)\, dx,$$

which can be integrated with respect to t. Since $u \in C([0,T];\, L^2(\Omega))$ (see Remark 11.1), we have

$$\int_0^T \langle u'(t), \psi(t)\, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt + \int_\Omega u(x,0)\, v(x)\, dx$$
$$+ \int_0^T \int_\Omega B(x)\nabla u(x,t)\, \psi(t)\, \nabla v(x)\, dx\, dt$$
$$= \int_0^T \int_\Omega f(x,t)\, \psi(t)\, v(x)\, dx\, dt + \int_\Omega u^0(x)\, v(x)\, dx.$$

Observing that (11.22) is still valid for $\psi \in C^\infty([0,T])$, we deduce that

$$\int_\Omega u(x,0)\, v\, dx = \int_\Omega u^0(x)\, v(x)\, dx, \quad \forall v \in H_0^1(\Omega),$$

which by Theorem 1.44 implies the required equality.

Step 5. We now prove estimate (11.8). We show it for the solution u obtained in the previous steps. This is not restrictive, since in Step 6 we will prove the uniqueness of the solution of problem (11.7).

Estimate (11.8) for the solution given by (11.20) is a simple consequence of estimates (11.17) and (11.19). Using again convergences (11.15) and (11.20) and the lower-semicontinuity of the norm from Propositions 1.14(ii) and 1.24(ii), we get from (11.17)

$$\begin{aligned}
&\|u\|_{L^\infty(0,T;\, L^2(\Omega))} + \|u\|_{L^2(0,T;\, H_0^1(\Omega))} \\
&\quad \le \liminf_{m\to\infty} \|u_m\|_{L^\infty(0,T;\, L^2(\Omega))} + \liminf_{m\to\infty} \|u_m\|_{L^2(0,T;\, H_0^1(\Omega))} \\
&\quad \le \liminf_{m\to\infty} \big(\|u_m\|_{L^\infty(0,T;\, L^2(\Omega))} + \|u_m\|_{L^2(0,T;\, H_0^1(\Omega))}\big) \\
&\quad \le c_0 \lim_{m\to\infty} \big(\|u_m^0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega\times(0,T))}\big) \\
&\quad = c_0\big(\|u^0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega\times(0,T))}\big).
\end{aligned}$$

Similarly, from (11.19) we obtain

$$\|u'\|_{L^2(0,T;\, H^{-1}(\Omega))} \le c_3\big(\|u^0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega\times(0,T))}\big).$$

These estimates imply the required one (11.8).

Step 6. Let u_1 and u_2 be two solutions corresponding to the same data. Their difference satisfies (11.7) with $f \equiv 0$ and $u^0 \equiv 0$, namely

$$\begin{cases}
\langle (u_1-u_2)'(t), v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \displaystyle\int_\Omega B(x)\nabla(u_1-u_2)(x,t))\,\nabla v(x)\, dx \\
\qquad\qquad = 0 \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega), \\
(u_1-u_2)(x,0) = 0 \quad \text{in } \Omega.
\end{cases}$$

Take $v = u_1 - u_2$ and use Theorem 3.58. From the ellipticity of the matrix B and Cauchy–Schwarz inequality (Proposition 1.34), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|^2_{L^2(\Omega)} + \alpha\|u_1 - u_2\|^2_{H_0^1(\Omega)} \le 0.$$

Integrating over $]0,t[$ with $t \in [0,T]$, it follows that

$$\|u_1 - u_2(t)\|^2_{L^2(\Omega)} + \alpha\int_0^t \|u_1 - u_2(\tau)\|^2_{H_0^1(\Omega)}\, d\tau \le 0.$$

This implies that $u_1 - u_2 \equiv 0$. □

Remark 11.3. Let us mention that one can also take $f \in L^2(0,T;\, H^{-1}(\Omega))$ in problem (11.5). Theorem 11.2 can easily be adapted to this case. For the sake of simplicity, we have restricted ourselves to the case $f \in L^2(\Omega\times]0,T[)$. ◊

11.2 The homogenization result

Let us now turn back to problem (11.1) with $f_\varepsilon \in L^2(\Omega\times]0,T[)$ and $u_\varepsilon^0 \in L^2(\Omega)$. The variational formulation is

$$\begin{cases} \text{Find } u_\varepsilon \in W \text{ such that} \\ \langle u_\varepsilon'(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \displaystyle\int_\Omega A^\varepsilon(x)\nabla u_\varepsilon(x,t)\,\nabla v(x)\,dx \\ \qquad\qquad = \displaystyle\int_\Omega f_\varepsilon(x,t)\,v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) \quad \text{in } \Omega. \end{cases} \tag{11.23}$$

The existence and uniqueness of u_ε is given by Theorem 11.2. We will now study what happens when $\varepsilon \to 0$. Notice that the oscillations in (11.23) are only due to the variable x. As will see below, in the homogenization process, the variable t plays the role of a parameter and consequently, the homogenized matrix is that of the elliptic case treated in the previous chapters. As a matter of fact, we have the following result:

Theorem 11.4. *Let $f_\varepsilon \in L^2(\Omega\times]0,T[)$, $u_\varepsilon^0 \in L^2(\Omega)$ and let u_ε be the solution of (11.1) with A^ε defined by (11.2)–(11.4). Suppose that*

$$\begin{cases} i) & u_\varepsilon^0 \rightharpoonup u^0 \quad \text{weakly in } L^2(\Omega) \\ ii) & f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^2(\Omega\times]0,T[), \end{cases} \tag{11.24}$$

Then u_ε satisfies

$$\begin{cases} i) & u_\varepsilon \rightharpoonup u \quad \text{weakly in } W, \\ ii) & A^\varepsilon\nabla u_\varepsilon \rightharpoonup A^0\nabla u^0 \quad \text{weakly in } (L^2(\Omega\times]0,T[))^N, \end{cases} \tag{11.25}$$

where u is the solution of the following limit problem:

$$\begin{cases} u' - \operatorname{div}\left(A^0\nabla u\right) = f & \text{in } \Omega\times]0,T[\\ u = 0 & \text{on } \partial\Omega\times]0,T[\\ u(x,0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{11.26}$$

Here, A^0 is the homogenized matrix given in Theorem 6.1 by (6.30).

Proof. For the proof, we make use of Tartar's method of oscillating test functions. We follow along the lines of the proof given in Section 8.1 for the elliptic case.

Observe first that since $A^\varepsilon \in M(\alpha,\beta,\Omega)$, using assumption (11.24) and Proposition 1.14, estimate (11.8) now reads

$$\|u_\varepsilon\|_W + \|u_\varepsilon\|_{L^\infty(0,T;\,L^2(\Omega))} \le c\big(\|f_\varepsilon\|_{L^2(\Omega\times]0,T[)} + \|u_\varepsilon^0\|_{L^2(\Omega)}\big) \le c_1, \tag{11.27}$$

where c and c_1 are independent of ε. Moreover, if we introduce the vector ξ^ε defined by

$$\xi^\varepsilon(x,t) = (\xi_1^\varepsilon(x,t),\ldots,\xi_N^\varepsilon(x,t)) = A^\varepsilon(t)\nabla u_\varepsilon(x,t), \tag{11.28}$$

from (11.27) and assumptions on A^ε, one has

$$\|\xi^\varepsilon\|_{(L^2(\Omega\times]0,T[))^N} \leq \beta c_1.$$

Consequently, there exists a subsequence, still denoted by ε such that

$$\begin{cases} i) & u_\varepsilon \rightharpoonup u & \text{weakly* in } L^\infty(0,T;\ L^2(\Omega)) \\ ii) & u_\varepsilon \rightharpoonup u & \text{weakly in } L^2(0,T;\ H_0^1(\Omega)) \\ iii) & u_\varepsilon \to u & \text{strongly in } L^2(\Omega\times]0,T[) \\ iv) & u'_\varepsilon \rightharpoonup u' & \text{weakly in } L^2(0,T;\ H^{-1}(\Omega)) \\ v) & \xi^\varepsilon \rightharpoonup \xi^0 & \text{weakly in } (L^2(\Omega\times]0,T[))^N, \end{cases} \tag{11.29}$$

where we have used the compact injection $\mathcal{W} \subset L^2(0,T;\ L^2(\Omega)) = L^2(\Omega\times]0,T[)$ (see Theorem 3.58). Then, convergence (11.25)(i) holds for this subsequence.

From its definition (11.28) and problem (11.23), it is easily seen that ξ^ε satisfies

$$\int_0^T\int_\Omega \xi^\varepsilon(x,t)\cdot\nabla v(x)\,\varphi(t)\,dx\,dt = \int_0^T\int_\Omega f_\varepsilon(x,t)\,v(x)\varphi(t)\,dx\,dt - \int_0^T \langle u'_\varepsilon(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi(t)\,dt,$$

for any $v \in H_0^1(\Omega)$ and $\varphi \in \mathcal{D}(0,T)$. According to Proposition 3.59, this is equivalent to

$$\int_0^T\int_\Omega \xi^\varepsilon(x,t)\cdot\nabla v(x)\,\varphi(t)\,dx\,dt = \int_0^T\int_\Omega f_\varepsilon(x,t)\,v(x)\varphi(t)\,dx\,dt - \langle u'_\varepsilon, v\varphi\rangle_{L^2(a,b;\ H^{-1}(\Omega)),L^2(a,b;\ H_0^1(\Omega))}, \tag{11.30}$$

where we can pass to the limit due to convergences (11.29). We obtain that ξ^0 satisfies

$$\langle u'(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega \xi^0(x,t)\cdot\nabla v(x)\,dx = \int_\Omega f(x,t)\,v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega). \tag{11.31}$$

At this point, as in the elliptic case, we only have to prove that

$$\xi^0 = A^0\nabla u. \tag{11.32}$$

We will make use, as before, of the oscillating test functions w_λ^ε defined by (8.10), i.e.

$$w_\lambda^\varepsilon(x) = \varepsilon w_\lambda\Big(\frac{x}{\varepsilon}\Big) = \lambda\cdot x - \varepsilon\,\chi_\lambda\Big(\frac{x}{\varepsilon}\Big),$$

where w_λ is defined by (6.27).

Let us recall the following convergences (see (8.11)):

$$\begin{cases} i) & w_\lambda^\varepsilon \rightharpoonup \lambda\cdot x \quad \text{weakly in } H^1(\Omega) \\ ii) & w_\lambda^\varepsilon \to \lambda\cdot x \quad \text{strongly in } L^2(\Omega). \end{cases} \tag{11.33}$$

Introduce also the vector function

$$\eta_\lambda^\varepsilon = {}^tA^\varepsilon \nabla w_\lambda^\varepsilon,$$

which satisfies the convergence (see (8.13))

$$\eta_\lambda^\varepsilon \rightharpoonup \mathcal{M}_Y({}^tA\nabla w_\lambda) = {}^tA^0\lambda \quad \text{weakly in } (L^2(\Omega))^N, \tag{11.34}$$

and the equation (see (8.14))

$$\int_\Omega \eta_\lambda^\varepsilon\cdot\nabla v\,dx = 0, \qquad \forall v\in H_0^1(\Omega).$$

Let $\psi\in\mathcal{D}(\Omega)$ and $\varphi\in\mathcal{D}(0,T)$. Choose here $v=\psi\,u_\varepsilon\varphi$ and integrate over $]0,T[$. Then

$$\int_0^T\!\!\int_\Omega \eta_\lambda^\varepsilon\cdot\nabla u_\varepsilon(x,t)\,\psi(x)\varphi(t)\,dx\,dt + \int_0^T\!\!\int_\Omega \eta_\lambda^\varepsilon\cdot\nabla\psi(x)\,u_\varepsilon(x,t)\varphi(t)\,dx\,dt = 0. \tag{11.35}$$

Choosing now $v=\psi\,w_\lambda^\varepsilon$ in (11.30) and subtracting from (11.35) we obtain

$$\begin{aligned}\int_0^T\!\!\int_\Omega \xi^\varepsilon(x,t)\cdot\nabla\psi(x)\,w_\lambda^\varepsilon\,\varphi(t)\,dx\,dt &- \int_0^T\!\!\int_\Omega \eta_\lambda^\varepsilon\cdot\nabla\psi(x)\,u_\varepsilon(x,t)\varphi(t)\,dx\,dt\\ &= \int_0^T\!\!\int_\Omega f_\varepsilon(x,t)\,\psi(x)\,w_\lambda^\varepsilon(x)\varphi(t)\,dx\,dt\\ &\quad - \langle u_\varepsilon',\psi(x)\,w_\lambda^\varepsilon(x)\varphi\rangle_{L^2(a,b;\,H^{-1}(\Omega)),L^2(a,b;\,H_0^1(\Omega))},\end{aligned}$$

where we pass to the limit by using convergences (11.24), (11.29), (11.32) and (11.33) and obtain

$$\begin{aligned}&\int_0^T\!\!\int_\Omega \xi^0(x,t)\cdot\nabla\psi(x)\,(\lambda\cdot x)\,\varphi(t)\,dx\,dt\\ &\qquad - \int_0^T\!\!\int_\Omega {}^tA^0\lambda\cdot\nabla\psi(x)\,u(x,t)\varphi(t)\,dx\,dt\\ &\qquad = \int_0^T\!\!\int_\Omega f(x,t)\,\psi(x)\,(\lambda\cdot x)\varphi(t)\,dx\,dt\\ &\qquad\quad - \langle u',\psi(x)\,(\lambda\cdot x)\varphi\rangle_{L^2(a,b;\,H^{-1}(\Omega)),L^2(a,b;\,H_0^1(\Omega))}.\end{aligned}$$

From equation (11.31), by the same computation as in Section 8.1 we deduce (11.32).

To end the proof, we have to show that u satisfies the initial condition in (11.26). We make use of the same argument as in Step 4 of the proof of Theorem 11.2. To do so, choose $\varphi \in C^\infty([0,T])$ such that $\varphi(0) = 1$ and $\varphi(T) = 0$ and $v \in \mathcal{D}(\Omega)$ in (11.30). Then, from Theorem 3.58(iii), one has

$$\int_0^T \int_\Omega \xi^\varepsilon(x,t) \cdot \nabla v(x)\, \varphi(t)\, dx\, dt = \int_0^T \int_\Omega f_\varepsilon(x,t)\, v(x)\, \varphi(t)\, dx\, dt + \int_0^T \langle u_\varepsilon(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \varphi'(t)\, dt + \int_\Omega u_\varepsilon^0\, v\, dx.$$

Thanks to convergences (11.24), (11.29) we can pass to the limit in this identity to get

$$\int_0^T \int_\Omega \xi^0(x,t) \cdot \nabla v(x)\, \varphi(t)\, dx\, dt = \int_0^T \int_\Omega f(x,t)\, v(x) \varphi(t)\, dx\, dt + \int_0^T \langle u(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, \varphi'(t)\, dt + \int_\Omega u^0\, v\, dx.$$

From this equality, multiplying (11.31) by φ and integrating with respect to t, we obtain

$$\int_0^T \langle u'(t), \varphi(t)\, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt = -\int_0^T \langle u(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, \varphi'(t)\, dt + \int_\Omega u^0(x) v\, dx.$$

This, together with Theorem 3.58(iii), implies that

$$u(x,0) = u^0(x).$$

To conclude the proof, observe that since A^0 is elliptic (see Proposition 6.12), Theorem 11.2 provides the uniqueness of the solution of problem (11.26). Consequently, the whole sequences in (11.29) converge. □

11.3 Convergence of the energy

Let u_ε be the solution of (11.1) whose variational formulation is (11.23), namely

$$\begin{cases} \text{Find } u_\varepsilon \in W \text{ such that} \\ \langle u_\varepsilon'(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \displaystyle\int_\Omega A^\varepsilon(x) \nabla u_\varepsilon(x,t)\, \nabla v(x)\, dx \\ \qquad = \displaystyle\int_\Omega f_\varepsilon(x,t)\, v(x)\, dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega), \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) \quad \text{in } \Omega. \end{cases}$$

Let u be the solution of the homogenized problem (11.26) whose variational formulation is

$$\begin{cases} \text{Find } u \in W \text{ such that} \\ \langle u'(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \displaystyle\int_\Omega A^0(x)\nabla u(x,t)\,\nabla v(x)\,dx \\ \qquad = \displaystyle\int_\Omega f(x,t)\,v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega), \\ u(x,0) = u^0(x) \quad \text{in } \Omega. \end{cases} \tag{11.36}$$

We introduce now the energies associated with these problems:

$$\begin{cases} i) \quad E^\varepsilon(u_\varepsilon)(t) = \frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega)} + \displaystyle\int_0^t\int_\Omega A^\varepsilon(x)\nabla u_\varepsilon(x,\tau)\,\nabla u_\varepsilon(x,\tau)\,dx\,d\tau \\ ii) \quad E(u)(t) = \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \displaystyle\int_0^t\int_\Omega A^0(x)\nabla u(x,\tau)\,\nabla u(x,\tau)\,dx\,d\tau. \end{cases}$$

Choosing $v = u_\varepsilon$ in (11.23) and $v = u$ in (11.36), it easily seen that

$$\begin{cases} i) \quad E^\varepsilon(u_\varepsilon)(t) = \frac{1}{2}\|u_\varepsilon^0\|^2_{L^2(\Omega)} + \displaystyle\int_0^t\int_\Omega f_\varepsilon(x,\tau)u_\varepsilon(x,\tau)\,dx\,d\tau \\ ii) \quad E(u)(t) = \frac{1}{2}\|u^0\|^2_{L^2(\Omega)} + \displaystyle\int_0^t\int_\Omega f(x,\tau)\,u(x,\tau)\,dx\,d\tau, \end{cases} \tag{11.37}$$

since from Theorem 3.58(iii)

$$\frac{d}{dt}\int_\Omega u_\varepsilon^2(x,t)\,dx = 2\langle u_\varepsilon'(t), u_\varepsilon(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

$$\frac{d}{dt}\int_\Omega u^2(x,t)\,dx = 2\langle u'(t), u(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

The following convergence of energies holds true:

Proposition 11.5. *Let $f_\varepsilon \in L^2(\Omega\times]0,T[)$, $u_\varepsilon^0 \in L^2(\Omega)$ and let u_ε be the solution of (11.1) with A^ε defined by (11.2)-(11.4). Suppose that*

$$\begin{cases} i) \quad u_\varepsilon^0 \to u^0 \quad \text{strongly in } L^2(\Omega) \\ ii) \quad f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^2(\Omega\times]0,T[). \end{cases} \tag{11.38}$$

Then

$$E^\varepsilon(u_\varepsilon) \to E(u) \quad \text{in } C([0,T]).$$

Proof. Let us prove first that $E^\varepsilon(u_\varepsilon)$ belongs to a compact set in $C([0,T])$. Due to the Ascoli Arzelà theorem (see for instance Yosida, 1964), it is sufficient to prove the following two properties:

$$\begin{cases} i) \quad |E^\varepsilon(u_\varepsilon)(t)| \le c, \quad \forall t \in [0,T] \\ ii) \quad |E^\varepsilon(u_\varepsilon)(t+h) - E^\varepsilon(u_\varepsilon)(t)| \le \theta(h), \\ \quad\ \text{uniformly with respect to } \varepsilon, \forall t \in [0,T-h], \forall h > 0. \end{cases} \tag{11.39}$$

where θ tends to zero as h goes to zero.

Applying the Hölder inequality in (11.37)(i) and using estimate (11.27), assumptions (11.4) and (11.38), one has immediately statement (i).

For the second statement, observe that (11.37)(i) yields

$$|E^\varepsilon(u_\varepsilon)(t+h) - E^\varepsilon(u_\varepsilon)(t)| = \left| \int_t^{t+h} \int_\Omega f_\varepsilon(x,\tau) u_\varepsilon(x,\tau)\, dx\, d\tau \right|$$
$$\leq h^{\frac{1}{2}} \|u_\varepsilon\|_{L^\infty(0,T;\, L^2(\Omega))} \|f_\varepsilon\|_{L^2(\Omega\times]0,T[)} \leq c_1 h^{\frac{1}{2}},$$

where we have again used the Hölder inequality, estimate (11.27), and assumption (11.38)(ii).

Hence, there exists a subsequence (still denoted by ε) and some $\zeta \in C([0,T])$ such that

$$E^\varepsilon(u_\varepsilon) \to \zeta \quad \text{in } C([0,T]). \tag{11.40}$$

We now show that $\zeta = E(u)$. Due to assumptions (11.38) and convergences (11.29)(iii), one can pass to the limit in (11.37)(i) to get

$$\lim_{\varepsilon \to 0} E^\varepsilon(u_\varepsilon)(t) = E(u)(t), \quad \forall t \in [0,T].$$

This identifies ζ in (11.40) and ends the proof. □

Remark 11.6. From the above proof, it is clear that the strong convergence of the initial data u_ε^0 is necessary in order to insure that $\zeta = E(u)$. The weak convergence would only give a compactness of $E^\varepsilon(u_\varepsilon)$ in $C([0,T])$. ◊

11.4 A corrector result

We prove here a corrector result, in the spirit of Section 8.3. The proof makes use of arguments from Brahim-Otsmane, Francfort, and Murat (1992). As for the elliptic case, the convergence of the energy plays an essential role. The corrector matrix is the same as that of the elliptic case, namely $C^\varepsilon = (C_{ij}^\varepsilon)_{1\leq i,j\leq N}$ is defined by

$$\begin{cases} C_{ij}^\varepsilon(x) = C_{ij}\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } \Omega \\[2mm] C_{ij}(y) = \delta_{ij} - \dfrac{\partial \widehat{\chi}_j}{\partial y_i}(y) = \dfrac{\partial \widehat{w}_j}{\partial y_i}(y) & \text{a.e. on } Y, \end{cases} \tag{11.41}$$

where $\widehat{\chi}_j$ and $\widehat{w}_j$ are defined by (6.15) and (6.16). One has the following result:

Theorem 11.7. *Let u_ε be the solution of problem (11.1). Let u and A^0 be given by Theorem 11.4. Under hypotheses (11.38) one has*

$$\begin{cases} i) & u_\varepsilon \to u \quad \text{strongly in } C([0,T];\, L^2(\Omega)) \\ ii) & \nabla u_\varepsilon - C^\varepsilon \nabla u \to 0 \quad \text{strongly in } (L^2(0,T;\, L^1(\Omega)))^N. \end{cases} \tag{11.42}$$

Moreover, if $C \in (L^r(Y))^{N\times N}$ *for some* r *such that* $2 \le r \le \infty$, *and* $\nabla u \in (L^s(\Omega))^N$ *for some* s *such that* $2 \le s < \infty$, *then*

$$\nabla u_\varepsilon - C^\varepsilon \nabla u \to 0 \quad \text{strongly in } (L^2(0,T;\ L^t(\Omega)))^N,$$

where

$$t = \min\left\{2, \frac{r\,s}{r+s}\right\}.$$

The proof of this result is based on the following proposition, which is analogous to the time-dependent case of Proposition 8.7.

Proposition 11.8. *Suppose that the assumptions of Theorem 11.6 are fulfilled. Set for any* $\Phi \in C^\infty([0,T];\ \mathcal{D}(\Omega))$

$$\begin{aligned}\rho_\varepsilon(t) &= \tfrac{1}{2}\|u_\varepsilon(t) - \Phi(t)\|^2_{L^2(\Omega)} \\ &\quad + \int_0^t \int_\Omega A^\varepsilon(x)(\nabla u_\varepsilon - C^\varepsilon\nabla\Phi)(x,\tau)\ (\nabla u_\varepsilon - C^\varepsilon\nabla\Phi)(x,\tau)\, dx\, d\tau.\end{aligned}$$

Then

$$\rho_\varepsilon \to \rho \quad \text{strongly in } C([0,T]),$$

where

$$\rho(t) = \tfrac{1}{2}\|u(t) - \Phi(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^0(x)(\nabla u - \nabla\Phi)(x,\tau)\ (\nabla u - \nabla\Phi)(x,\tau)\, dx\, d\tau.$$

Proof. Remark that ρ_ε can be written as follows:

$$\rho_\varepsilon = \rho_\varepsilon^1 + \rho_\varepsilon^2 + \rho_\varepsilon^3,$$

where

$$\left\{\begin{aligned}\rho_\varepsilon^1(t) &= \tfrac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon\, dx\, d\tau \\ \rho_\varepsilon^2(t) &= \tfrac{1}{2}\|\Phi(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^\varepsilon\,(C^\varepsilon\nabla\Phi)\ (C^\varepsilon\nabla\Phi)\ dx\, d\tau \\ \rho_\varepsilon^3(t) &= \int_\Omega u_\varepsilon\,\Phi\, dx + \int_0^t \int_\Omega A^\varepsilon\,(C^\varepsilon\nabla\Phi)\,\nabla u_\varepsilon\, dx\, d\tau \\ &\quad + \int_0^t \int_\Omega A^\varepsilon \nabla u_\varepsilon\,(C^\varepsilon\nabla\Phi)\ dx\, d\tau.\end{aligned}\right. \tag{11.43}$$

We now prove the convergence in $C([0,T])$ of each term of this decomposition.
First term. Notice that ρ_ε^1 is nothing else that the energy $E^\varepsilon(u_\varepsilon)$. Hence, by Proposition 11.5,

$$\rho_\varepsilon^1 \to E(u) = \tfrac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^0(x)\nabla u(x,\tau)\ \nabla u(x,\tau)\, dx\, d\tau \quad \text{in } C([0,T]). \tag{11.44}$$

Second term. We obtain first a pointwise convergence of the second term ρ_ε^2. To do so, thanks to convergences (8.32) and (8.35), we make exactly the same computation as in (8.42) (with Φ replaced by $\nabla\Phi$), the variable t playing the role of a parameter. So,

$$\rho_\varepsilon^2 \to \tfrac{1}{2}\|\Phi(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega A^0(x)\nabla\Phi(x,\tau)\,\nabla\Phi(x,\tau)\,dx\,d\tau \quad \text{for any } t\in[0,T]. \tag{11.45}$$

It remains to show that ρ_ε^2 is a compact of $C([0,T])$. Due to the compact injection $W^{1,\infty}(0,T)\subset C([0,T])$ (see Theorem 3.27(iii)), it is sufficient to prove that ρ_ε^2 is bounded in $W^{1,\infty}(0,T)$, i.e. that there exists a constant c independent of ε such that

$$\|\rho_\varepsilon^2\|_{L^\infty(0,T)} + \|(\rho_\varepsilon^2)'\|_{L^\infty(0,T)} \le c.$$

Clearly, we only have to check this estimate for the second term in definition (11.43) of ρ_ε^2, since Φ is regular and independent of ε. This estimate is a consequence of (8.30)(i) and the assumptions on A, so that

$$\|\rho_\varepsilon^2\|_{L^\infty(0,T)} + \|(\rho_\varepsilon^2)'\|_{L^\infty(0,T)} \le c_1(T+1)\,\beta\|C^\varepsilon\|^2_{L^2(\Omega)}\|\nabla\Phi\|^2_{L^\infty(\Omega\times]0,T[)} \le c.$$

This, together with (11.45), gives

$$\rho_\varepsilon^2 \to \tfrac{1}{2}\|\Phi(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega A^0(x)\nabla\Phi(x,\tau)\,\nabla\Phi(x,\tau)\,dx\,d\tau, \quad \text{in } C([0,T]). \tag{11.46}$$

Third term. We proceed as for the previous term. Remark first that the pointwise convergence

$$\rho_\varepsilon^3 \to \int_\Omega u\,\Phi\,dx + \int_0^t\int_\Omega A^0\nabla\Phi\nabla u\,dx\,d\tau + \int_0^t\int_\Omega A^0\nabla u\nabla\Phi\,dx\,d\tau$$
$$\text{for any } t\in[0,T],$$

is straightforward by using the same computations as in (8.40) and (8.41) and convergence (11.25). We now prove that ρ_ε^3 is bounded in $H^1(0,T)$, showing that

$$\|\rho_\varepsilon^3\|_{L^\infty(0,T)} + \|(\rho_\varepsilon^3)'\|_{L^2(0,T)} \le c_2.$$

A priori estimates (11.27) for u_ε, convergence (8.30)(i) and the assumptions on A give immediately the boundedness of $\|\rho_\varepsilon^3\|_{L^\infty(0,T)}$. One the other hand, from Theorem 3.58 (iii), one has

$$\begin{aligned}(\rho_\varepsilon^3)'(t) &= \langle u_\varepsilon'(\cdot,t),\,\Phi(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega u_\varepsilon(x,t)\,\Phi'(x,t)\,dx\\ &+ \int_\Omega A^\varepsilon\,(C^\varepsilon\nabla\Phi)\,\nabla u_\varepsilon\,dx + \int_\Omega A^\varepsilon\nabla u_\varepsilon\,(C^\varepsilon\nabla\Phi)\,dx.\end{aligned}$$

From this expression, the boundedness of $\|(\rho_\varepsilon^3)'\|_{L^2(0,T)}$ is obvious by the same arguments as above. Then, from the compactness of the injection $H^1(0,T)\subset$

$C([0,T])$ (see Theorem 3.27(iii)), one has the convergence

$$\rho_\varepsilon^3 \to \int_\Omega u\,\Phi\,dx + \int_0^t \int_\Omega A^0 \nabla\Phi\nabla u\,dx\,d\tau + \int_0^t \int_\Omega A^0 \nabla u \nabla\Phi\,dx\,d\tau \quad \text{in } C([0,T]). \tag{11.47}$$

Recalling that $\rho_\varepsilon = \rho_\varepsilon^1 + \rho_\varepsilon^2 + \rho_\varepsilon^3$, from definition (11.43) and convergences (11.44), (11.46), and (11.47), an easy computation gives the claimed result. □

Proof of Theorem 11.7. We will prove here only convergences (11.42), since the last statement of Theorem 11.8 follows by the same arguments as the last statement in Theorem 8.6.

Let $\delta > 0$ be given. From Proposition 3.60, there exists $\Phi_\delta \in C^\infty([0,T];\mathcal{D}(\Omega))$, such that

$$\begin{cases} i) & \|u - \Phi_\delta\|^2_{C([0,T];L^2(\Omega))} \le \delta \\ ii) & \|\nabla u - \nabla\Phi_\delta\|^2_{L^2(\Omega\times]0,T[)} \le \delta. \end{cases} \tag{11.48}$$

Then, if one writes

$$u_\varepsilon - u = (u_\varepsilon - \Phi_\delta) + (\Phi_\delta - u),$$

one has

$$\begin{aligned} \|u_\varepsilon - u\|^2_{C([0,T];L^2(\Omega))} &\le 2\left(\|u_\varepsilon - \Phi_\delta\|^2_{C([0,T];L^2(\Omega))} + \|\Phi_\delta - u\|^2_{C([0,T];L^2(\Omega))}\right) \\ &\le 2\|u_\varepsilon - \Phi_\delta\|^2_{C([0,T];L^2(\Omega))} + 2\delta. \end{aligned} \tag{11.49}$$

We will now estimate the term $\|u_\varepsilon - \Phi_\delta\|^2_{C([0,T];L^2(\Omega))}$. To do so, set

$$\begin{cases} \rho_\varepsilon^\delta(t) = \dfrac{1}{2}\|u_\varepsilon(t) - \Phi_\delta(t)\|^2_{L^2(\Omega)} \\ \qquad + \displaystyle\int_0^t \int_\Omega A^\varepsilon(x)(\nabla u_\varepsilon - C^\varepsilon\nabla\Phi_\delta)(x,\tau)\,(\nabla u_\varepsilon - C^\varepsilon\nabla\Phi_\delta)(x,\tau)\,dx\,d\tau. \end{cases} \tag{11.50}$$

Using the ellipticity condition of A^ε, one has

$$\frac{1}{2}\alpha\|u_\varepsilon(t) - \Phi_\delta\|^2_{L^2(\Omega)} + \alpha\int_0^t \|\nabla u_\varepsilon - C^\varepsilon\nabla\Phi_\delta\|^2_{L^2(\Omega)} \le \rho_\varepsilon^\delta(t). \tag{11.51}$$

Then, from Proposition 11.8, we have

$$\limsup_{\varepsilon\to 0} \frac{1}{2}\|u_\varepsilon(t) - \Phi_\delta(t)\|^2_{L^2(\Omega)} \le \limsup_{\varepsilon\to 0} \|\rho_\varepsilon^\delta\|_{C([0,T])} = \|\rho^\delta\|_{C([0,T])}, \tag{11.52}$$

where

$$\begin{cases} \rho^\delta(t) = \dfrac{1}{2}\|u(t) - \Phi_\delta(t)\|^2_{L^2(\Omega)} \\ \qquad + \displaystyle\int_0^t \int_\Omega A^0(x)(\nabla u - \nabla\Phi_\delta)(x,\tau)\,(\nabla u - \nabla\Phi_\delta)(x,\tau)\,dx\,d\tau. \end{cases} \tag{11.53}$$

Using now Proposition 8.3 and (11.48) we obtain

$$\|\rho^\delta\|_{C([0,T])} \le \left(\frac{1}{2} + \frac{\beta^2}{\alpha}\right)\delta. \tag{11.54}$$

Then, from (11.49) and (11.52), we have

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon - u\|^2_{C([0,T];\,L^2(\Omega))} \le \left(1 + \frac{2\beta^2}{\alpha}\right)\delta,$$

which implies (11.42)(i), since δ is arbitrary.

To show (11.42)(ii), let us write $\nabla u_\varepsilon - C^\varepsilon \nabla u$ in the form

$$\nabla u_\varepsilon - C^\varepsilon \nabla u = (\nabla u_\varepsilon - C^\varepsilon \nabla \Phi_\delta) + C^\varepsilon(\nabla \Phi_\delta - \nabla u).$$

From the Hölder inequality, (8.30)(i) and (11.48)(ii), we have

$$\left\{\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_0^T \|\nabla u_\varepsilon(t) - C^\varepsilon \nabla u(t)\|^2_{L^1(\Omega)}\,dt \\
&\quad\le 2\limsup_{\varepsilon\to 0} \int_0^T \|\nabla u_\varepsilon(t) - C^\varepsilon \nabla \Phi_\delta(t)\|^2_{L^1(\Omega)}\,dt \\
&\qquad + 2\limsup_{\varepsilon\to 0} \int_0^T \|C^\varepsilon \nabla \Phi_\delta(t) - C^\varepsilon \nabla u(t)\|^2_{L^1(\Omega)}\,dt \\
&\quad\le \limsup_{\varepsilon\to 0} c_1 \int_0^T \|\nabla u_\varepsilon(t) - C^\varepsilon \nabla \Phi_\delta(t)\|^2_{L^2(\Omega)}\,dt \\
&\qquad + 2\limsup_{\varepsilon\to 0} \int_0^T \|C^\varepsilon\|^2_{L^2(\Omega)} \|\nabla \Phi_\delta(t) - \nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
&\quad\le \limsup_{\varepsilon\to 0} c_1 \int_0^T \|\nabla u_\varepsilon(t) - C^\varepsilon \nabla \Phi_\delta(t)\|^2_{L^2(\Omega)}\,dt + c_2\,\delta.
\end{aligned}\right. \tag{11.55}$$

Let us estimate the integral term in the right-hand side. Using (11.51) written with $t = T$, definitions (11.50), (11.53), and Proposition 11.8, it follows that

$$\limsup_{\varepsilon\to 0} \int_0^T \|\nabla u_\varepsilon - C^\varepsilon \nabla \Phi_\delta\|^2_{L^2(\Omega)} \le \frac{1}{\alpha} \lim_{\varepsilon\to 0} \rho^\delta_\varepsilon(T) = \frac{1}{\alpha}\rho^\delta(T),$$

which, together with (11.54), gives

$$\limsup_{\varepsilon\to 0} \int_0^T \|\nabla u_\varepsilon - C^\varepsilon \nabla \Phi_\delta\|^2_{L^2(\Omega)} \le \frac{1}{\alpha}\left(\frac{1}{2} + \frac{\beta^2}{\alpha}\right)\delta.$$

This, used in (11.55), ends the proof of (11.42)(ii). The proof of Theorem 11.7 is complete. □

12

Homogenization of the wave equation

In this chapter we are concerned with the asymptotic behaviour as $\varepsilon \to 0$ of the solution $u_\varepsilon = u_\varepsilon(x,t)$ of the wave equation introduced in Section 5.2 (Example 5.3), namely,

$$\begin{cases} u_\varepsilon'' - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega \times]0,T[\\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \times]0,T[\\ u_\varepsilon(x,0) = u_\varepsilon^0(x) & \text{in } \Omega \\ u'(x,0) = u_\varepsilon^1(x) & \text{in } \Omega, \end{cases} \tag{12.1}$$

where as in the previous chapter, the operators div and ∇ are taken with respect to the space variable $x \in \Omega$ and $'$ denotes the derivative with respect to the time variable $t \in]0,T[$ with $T > 0$. We suppose we are given the source term f_ε and the initial states u_ε^0 and u_ε^1. The matrix A^ε is Y-periodic and defined by

$$a_{ij}^\varepsilon(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall i,j = 1,\ldots,N \tag{12.2}$$

and

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) = \left(a_{ij}^\varepsilon(x)\right)_{1\le i,j\le N} \quad \text{a.e. on } \mathbb{R}^N, \tag{12.3}$$

where

$$\begin{cases} a_{ij} = a_{ji}, \quad \forall i,j = 1,\ldots,N \\ a_{ij} \text{ is } Y\text{-periodic}, \quad \forall i,j = 1,\ldots,N \\ A = (a_{ij})_{1\le i,j\le N} \in M(\alpha,\beta,Y), \end{cases} \tag{12.4}$$

with $\alpha, \beta \in \mathbb{R}$, such that $0 < \alpha < \beta$ and $M(\alpha,\beta,Y)$ given by Definition 4.11.

Let us point out that in this chapter, contrary to the elliptic and parabolic cases, we assume that the matrix A is symmetric. This assumption is essential in the existence result.

Problem (12.1) is a particular case of a large class of partial differential equations called hyperbolic equations. For general results concerning this kind of equations, we refer the reader to Lions and Magenes (1968a, Chapter 3, 1968b) (see also Wloka, 1987; Lions, 1988). For homogenization results concerning the wave equation we refer to Bensoussan, Lions, and Papanicolaou (1978) for the periodic case, and to Brahim-Otsmane, Francfort, and Murat (1992) for the general non-periodic one.

In Section 12.1 we will show the existence and uniqueness of the solution of (12.1) in a variational framework when f_ε is in $L^2(\Omega\times]0,T[)$, u^0_ε in $H^1_0(\Omega)$ and u^1_ε in $L^2(\Omega)$. For the definition and properties of time-dependent functional spaces used in this chapter, we refer again to Section 3.5.

In Section 12.2 and 12.3 we give the main homogenization results for problem (12.1).

12.1 Existence and uniqueness

Let Ω be a bounded open set in $\mathbb{R}^N$ and consider the following problem:

$$\begin{cases} u'' - \operatorname{div}(B\nabla u) = f & \text{in } \Omega\times]0,T[\\ u = 0 & \text{on } \partial\Omega\times]0,T[\\ u(x,0) = u^0(x) & \text{in } \Omega \\ u'(x,0) = u^1(x) & \text{in } \Omega, \end{cases} \tag{12.5}$$

under the following assumptions:

$$\begin{cases} i) & B \text{ is symmetric and in } M(\alpha,\beta,\Omega) \\ ii) & f \in L^2(\Omega\times]0,T[) \\ iii) & u^0 \in H^1_0(\Omega) \\ iv) & u^1 \in L^2(\Omega). \end{cases} \tag{12.6}$$

Let us introduce the space

$$W_2 = \left\{ v \mid v \in L^2(0,T;\ H^1_0(\Omega)),\ v' \in L^2(\Omega\times]0,T[) \right\},$$

which is clearly a Banach space with respect to the graph norm defined by

$$\|v\|_{W_2} = \|v\|_{L^2(0,T;\ H^1_0(\Omega))} + \|v'\|_{L^2(\Omega\times]0,T[)}.$$

Then, the variational formulation of problem (12.5) is the following one:

$$\begin{cases} \text{Find } u \in W_2 \text{ such that} \\ \langle u''(t), v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \displaystyle\int_\Omega B(x)\nabla u(x,t)\,\nabla v(x)\,dx \\ \qquad = \displaystyle\int_\Omega f(x,t)\,v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H^1_0(\Omega) \\ u(x,0) = u^0(x) \quad \text{in } \Omega \\ u'(x,0) = u^1(x) \quad \text{in } \Omega. \end{cases} \tag{12.7}$$

Remark 12.1. Suppose that u is a solution of (12.7), under assumptions (12.6). Then one has the equality

$$u'' = \operatorname{div}(B\nabla u) + f \quad \text{in } \mathcal{D}'(\Omega\times]0,T[).$$

therefore

$$u'' \in L^2(0,T;\ H^{-1}(\Omega)).$$

Consequently, Theorem 3.58(ii) applied to $u \in \mathcal{W}$ and $u' \in \mathcal{W}_1$ implies

$$u \in C([0,T];\ L^2(\Omega)), \quad u' \in C([0,T];\ H^{-1}(\Omega)), \tag{12.8}$$

so that the initial conditions make sense in $L^2(\Omega)$ and $H^{-1}(\Omega)$, respectively. ◇

We have the following result:

Theorem 12.2. *Suppose that assumptions (12.6) are fulfilled. Then problem (12.7) has a unique solution $u \in \mathcal{W}_2$. Moreover,*

$$u \in L^\infty(0,T;\ H_0^1(\Omega)), \quad u' \in L^\infty(0,T;\ L^2(\Omega)), \quad u'' \in L^2(0,T;\ H^{-1}(\Omega))$$

and there exists a constant c depending on α, β, Ω, and T such that

$$\begin{cases} \|u\|_{L^\infty(0,T;\ H_0^1(\Omega))} & +\|u'\|_{L^\infty(0,T;\ L^2(\Omega))} + \|u''\|_{L^2(0,T;\ H^{-1}(\Omega))} \\ & \leq c\big(\|f\|_{L^2(\Omega\times]0,T[)} + \|u^0\|_{L^2(\Omega)} + \|u^1\|_{H_0^1(\Omega)}\big). \end{cases} \tag{12.9}$$

Before proving this theorem, let us mention that the solution u is even more regular. We recall the following result due to Lions and Magenes (1968a, Chapter 3, Theorem 8.2):

Proposition 12.3. *The solution u given by Theorem 12.2, is such that*

$$u \in C([0,T];\ H_0^1(\Omega)), \quad u' \in C([0,T];\ L^2(\Omega)).$$

Moreover, if the data are more regular, namely

$$\begin{cases} i) & f \in C([0,T];\ H_0^1(\Omega)) \\ ii) & u^0 \in H_0^1(\Omega) \text{ and } \nabla u^0 \in H^1(\Omega), \\ iii) & u^1 \in H_0^1(\Omega) \end{cases}$$

then

$$u' \in C([0,T];\ H_0^1(\Omega)), \quad u'' \in C([0,T];\ L^2(\Omega)).$$

In the proof of Theorem 12.2, we will make use of the following simplified version of the classical Gronwall's lemma:

Lemma 12.4. *Let v a function in $C([0,T])$ and suppose that there exists a positive number γ such that*

$$v(t) \leq \gamma + \int_0^t v(\tau)\, d\tau, \quad \forall\, t \in [0,T]. \tag{12.10}$$

Then

$$v(t) \leq \gamma e^T, \quad \forall\, t \in [0,T].$$

Proof. Inequality (12.10) can be written as follows:

$$\frac{d}{dt}\left(e^{-t}\int_0^t v(\tau)\,d\tau\right) \le e^{-t}\gamma.$$

which by integration leads to

$$\int_0^t v(\tau)\,d\tau \le \gamma(e^t - 1).$$

This, replaced into (12.10) gives the result. □

Proof of Theorem 12.2. As for the heat equation (see section 11.2), we will use the Faedo–Galerkin method.

Step 1. Let (w_ℓ) be the orthonormal basis in $L^2(\Omega)$ given by (iii) from Proposition 8.23 and Remark 8.24 for the choice $B = I$ in problem (8.75).

Denote by V_m be the m dimensional subspace of $H_0^1(\Omega)$, spanned by $w_1, \ldots, w_m$. Introduce also the projection (see (11.9))

$$P_m v = \sum_{i=1}^{m} (v, w_i)_{L^2(\Omega)}\, w_i, \quad \forall v \in L^2(\Omega).$$

We refer to Step 1 of the proof of Theorem 11.2 for the properties of P_m, namely (11.10)–(11.14).

Step 2. Introduce, for any $m \in \mathbb{N}^*$, the finite dimensional approximate problem

$$\begin{cases} \text{Find } u_m = \sum_{j=1}^{m} g_j^m(t) w_j \in V_m \text{ such that} \\ \displaystyle\int_\Omega u_m''(x,t)\, w_k\, dx + \int_\Omega B(x)\nabla u_m(x,t)\,\nabla w_k\, dx \\ \qquad = \displaystyle\int_\Omega f(x,t)\, w_k\, dx. \quad \text{in } \mathcal{D}'(0,T) \quad \forall k = 1,\ldots,m \\ u_m(x,0) = u_m^0(x) \quad \text{in } \Omega \\ u_m'(x,0) = u_m^1(x) \quad \text{in } \Omega, \end{cases} \tag{12.11}$$

where, according to assumptions (12.6)(iii) and (12.6)(iv), we set

$$u_m^0 = P_m u^0, \quad u_m^1 = P_m u^1.$$

From the properties of P_m (see (11.10) and (11.13)), we have

$$\begin{cases} i) & u_m^0 \to u^0 \ \text{ strongly in } H_0^1(\Omega) \\ ii) & u_m^1 \to u^1 \ \text{ strongly in } L^2(\Omega). \end{cases} \tag{12.12}$$

Problem (12.8) is equivalent to the following system of m linear ordinary differential equations of the second order with unknowns $g_1^m, \ldots, g_m^m$:

$$\begin{cases} \dfrac{d^2 g_k^m}{dt^2} + \displaystyle\sum_{j=1}^{m} g_j^m(t) \int_\Omega B \nabla w_j \nabla w_k \, dx = \int_\Omega f(x,t) \, w_k \, dx \\ g_k^m(0) = (u^0, w_k) \\ (g_k^m)'(0) = (u^1, w_k), \end{cases}$$

for any $k = 1, \ldots, m$. Classical results (see, for instance Coddington and Levinson, 1955) give the existence and uniqueness in $C^1([0,T])$ of a solution $\{g_1^m, \ldots, g_m^m\}$ of this system on the interval $[0,T]$. Hence, u_m is determined and u_m and u'_m are in $C([0,T]; V_m)$.

Step 3. We will now prove that u_m satisfies some a priori estimates. To do so, let us multiply the kth equation in (12.11) by $(g_k^m)'$ and sum over k from 1 to m. We obtain

$$\int_\Omega u''_m(x,t) \, u'_m(x,t) \, dx + \int_\Omega B(x) \nabla u_m(x,t) \, \nabla u'_m(x,t) \, dx = \int_\Omega f(x,t) \, u'_m(x,t) \, dx. \tag{12.13}$$

Due to the symmetry of B one has

$$\int_\Omega B(x) \nabla u_m(x,t) \, \nabla u'_m(x,t) \, dx = \frac{1}{2} \frac{d}{dt} \int_\Omega B(x) \nabla u_m(x,t) \, \nabla u_m(x,t) \, dx.$$

Hence, (12.13) can be rewritten as

$$\begin{aligned} \frac{d}{dt}\left(\|u'_m\|^2_{L^2(\Omega)} + \int_\Omega B(x) \nabla u_m(x,t) \, \nabla u_m(x,t) \, dx \right) &\leq 2 \|f\|_{L^2(\Omega)} \|u'_m\|_{L^2(\Omega)} \\ &\leq \|f\|^2_{L^2(\Omega)} + \|u'_m\|^2_{L^2(\Omega)}. \end{aligned}$$

Integrating on $(0,t)$ with $t \leq T$ and using the ellipticity of B, we get

$$\begin{aligned} &\|u'_m(t)\|^2_{L^2(\Omega)} + \alpha \|u_m(x,t)\|^2_{H_0^1(\Omega)} \\ &\leq \; \|u_m^1\|^2_{L^2(\Omega)} + \int_\Omega B \nabla u_m^0 \, \nabla u_m^0 \, dx + \int_0^T \|f\|^2_{L^2(\Omega)} \, dt + \int_0^t \|u'_m(\tau)\|^2_{L^2(\Omega)} \, d\tau \\ &\leq \; \|u_m^1\|^2_{L^2(\Omega)} + \beta \|u_m^0\|^2_{H_0^1(\Omega)} + \|f\|^2_{L^2(\Omega \times (0,T))} + \int_0^t \|u'_m(\tau)\|^2_{L^2(\Omega)} \, d\tau. \end{aligned}$$

Using properties (11.11) and (11.12) of the projection P_m, we finally have

$$\begin{aligned} &\|u'_m(t)\|^2_{L^2(\Omega)} + \alpha \|u_m(x,t)\|^2_{H_0^1(\Omega)} \\ &\leq \; \|u^1\|^2_{L^2(\Omega)} + \beta \|u^0\|^2_{H_0^1(\Omega)} + \|f\|^2_{L^2(\Omega \times (0,T))} \\ &\quad + \int_0^t \left[\|u'_m(\tau)\|^2_{L^2(\Omega)} + \alpha \|u_m(\tau)\|^2_{H_0^1(\Omega)} \right] d\tau. \end{aligned}$$

Applying now Gronwall's lemma (Lemma 12.4), with

$$\gamma = \|u^1\|^2_{L^2(\Omega)} + \beta\|u^0\|^2_{H^1_0(\Omega)} + \|f\|^2_{L^2(\Omega\times(0,T))},$$

we deduce the a priori estimate

$$\begin{aligned}\|u_m\|_{L^\infty(0,T;\,H^1_0(\Omega))} &+ \|u'_m\|_{L^\infty(0,T;\,L^2(\Omega))}\\ &\leq c_1\big(\|f\|_{L^2(\Omega\times]0,T[)} + \|u^0\|_{L^2(\Omega)} + \|u^1\|_{H^1_0(\Omega)}\big),\end{aligned}$$

where c_1 depends only on α, β, Ω and T. It remains to obtain an a priori estimate for u''_m. Observe that the equation in (12.11) implies that

$$(u''_m(t), v)_{L^2(\Omega)} = (-\text{div }(B\nabla)u_m(t) + f, v)_{L^2(\Omega)}, \quad \forall v \in V_m.$$

This signifies that

$$u''_m(t) = -[P_m(\mathcal{F}(u_m) + f)](t),$$

where P_m is defined by (11.9) and $\mathcal{F} = -\text{div }(B\nabla)$. Arguing exactly as in Step 3 of the proof of Theorem 11.2 when showing (11.19), we obtain

$$\|u''_m\|_{L^2(0,T;\,H^{-1}(\Omega))} \leq c_2\big(\|f\|_{L^2(\Omega\times]0,T[)} + \|u^0\|_{L^2(\Omega)} + \|u^1\|_{H^1_0(\Omega)}\big) \leq c_3,$$

where c_2 and c_3 are constants independent of m. Consequently,

$$\begin{aligned}\|u_m\|_{L^\infty(0,T;\,H^1_0(\Omega))} &+ \|u'_m\|_{L^\infty(0,T;\,L^2(\Omega))} + \|u''_m\|_{L^2(0,T;\,H^{-1}(\Omega))}\\ &\leq c\big(\|f\|_{L^2(\Omega\times]0,T[)} + \|u^0\|_{L^2(\Omega)} + \|u^1\|_{H^1_0(\Omega)}\big),\end{aligned} \tag{12.14}$$

where c depends only on α, β, Ω and T.

Step 4. In this step we pass to the limit in the approximate problem. Estimate (12.14) implies, up to a subsequence, the convergences

$$\begin{cases} u_m \rightharpoonup u & \text{weakly* in } L^\infty(0,T;\,H^1_0(\Omega))\\ u'_m \rightharpoonup u' & \text{weakly* in } L^\infty(0,T;\,L^2(\Omega))\\ u''_m \rightharpoonup u'' & \text{weakly in } L^2(0,T;\,H^{-1}(\Omega)),\end{cases} \tag{12.15}$$

where we made use of Theorem 1.26 and Propositions 3.55 and 3.59 (see for details Step 4 of the proof of Theorem 11.2).

Let us now pass to the limit in (12.11) for $m \to \infty$. We again proceed as in the proof of Theorem 11.2. To do so, let $\psi \in \mathcal{D}(0,T)$ and $v \in H^1_0(\Omega)$. Multiplying the equation in (12.11) by $(v, w_k)_{L^2(\Omega)}\psi$, summing over k from 1 to m and integrating in t over $(0,T)$, we get

$$\begin{cases}\displaystyle\int_0^T\!\!\int_\Omega u''_m(x,t)\,\psi(t)\,(P_m v)(x)\,dx\,dt\\ \displaystyle\qquad + \int_0^T\!\!\int_\Omega B(x)\nabla u_m(x,t)\,\psi(t)\,\nabla(P_m v)(x)\,dx\,dt\\ \displaystyle\qquad = \int_0^T\!\!\int_\Omega f(x,t)\,\psi(t)\,(P_m v)(x)\,dx\,dt.\end{cases} \tag{12.16}$$

where we used definition (11.9). By integration by parts with respect to t, one has

$$\begin{cases} -\displaystyle\int_0^T\int_\Omega u'_m(x,t)\,\psi'(t)\,(P_m v)(x)\,dx\,dt \\ \qquad +\displaystyle\int_0^T\int_\Omega B(x)\nabla u_m(x,t)\,\psi(t)\,\nabla(P_m v)(x)\,dx\,dt \\ \qquad =\displaystyle\int_0^T\int_\Omega f(x,t)\,\psi(t)\,(P_m v)(x)\,dx\,dt. \end{cases}$$

Here, all the terms pass to the limit thanks to convergences (12.15) and the strong convergence (11.13). We obtain

$$\begin{cases} -\displaystyle\int_0^T\int_\Omega u'(x,t)\,\psi'(t)\,v(x)\,dx\,dt + \int_0^T\int_\Omega B(x)\nabla u(x,t)\,\psi(t)\,\nabla v(x)\,dx\,dt \\ \qquad\qquad = \displaystyle\int_0^T\int_\Omega f(x,t)\,\psi(t)\,v(x)\,dx\,dt. \end{cases} \tag{12.17}$$

Due to Theorem 3.58 (iii), we have

$$\begin{aligned} &\int_0^T\int_\Omega u'(x,t)\,\psi'(t)\,v(x)\,dx \\ &= -\int_0^T \langle u''(t), \psi(t)v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt + \int_0^T \frac{d}{dt}\int_\Omega u'(x,t)\,\psi(t)\,v(x)\,dx\,dt \\ &= -\int_0^T \langle u''(t), \psi(t)v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt. \end{aligned}$$

since $\psi(0) = \psi(T) = 0$. This, together with (12.17), shows that u satisfies

$$\begin{cases} -\displaystyle\int_0^T\langle u''(t), \psi(t)v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}dt + \int_0^T\int_\Omega B(x)\nabla u(x,t)\,\psi(t)\,\nabla v(x)\,dx\,dt \\ \qquad\qquad = \displaystyle\int_0^T\int_\Omega f(x,t)\,\psi(t)\,v(x)\,dx\,dt. \end{cases} \tag{12.18}$$

It remains to check that the initial conditions $u(x,0) = u^0(x)$ and $u'(x,0) = u^1(x)$ are satisfied. We follow the arguments from Step 4 from the proof of Theorem 11.2. Choose in (12.16) (which is still valid if $\psi \in C^\infty([0,T])$) a function $\psi \in C^\infty([0,T])$ such that $\psi'(0) = 1$ and $\psi'(T) = 0$. Then, from (12.16), we get

$$\begin{aligned} &-\int_0^T\int_\Omega u'_m(x,t)\,\psi'(t)\,(P_m v)(x)\,dx\,dt \\ &+ \int_0^T\int_\Omega B(x)\nabla u_m(x,t)\,\psi(t)\,\nabla(P_m v)(x)\,dx\,dt \\ &= \int_0^T\int_\Omega f(x,t)\,\psi(t)\,(P_m v)(x)\,dx\,dt + \int_\Omega u_m^1(x)\,(P_m v)(x)\,dx. \end{aligned}$$

where we pass to the limit to obtain

$$-\int_0^T\int_\Omega u'(x,t)\psi'(t)\,v(x)\,dx\,dt+\int_0^T\int_\Omega B(x)\nabla u(x,t)\,\psi(t)\,\nabla v(x)\,dx\,dt$$
$$=\int_0^T\int_\Omega f(x,t)\,\psi(t)\,v(x)\,dx\,dt+\int_\Omega u^1(x)\,v(x)\,dx.$$

Again by Theorem 3.58 (iii), as $u\in C([0,T];\,L^2(\Omega))$ (see Remark 12.1), we have

$$\int_0^T\langle u''(t),\psi(t)v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+\int_\Omega u'(x,0)\,v(x)\,dx$$
$$+\int_0^T\int_\Omega B(x)\nabla u(x,t)\,\psi(t)\,\nabla v(x)\,dx\,dt$$
$$=\int_0^T\int_\Omega f(x,t)\,\psi(t)\,v(x)\,dx\,dt+\int_\Omega u^1(x)\,v(x)\,dx.$$

Since (12.18) is still valid for $\psi\in C^\infty([0,T])$, we deduce that

$$\int_\Omega u'(x,0)\,v(x)\,dx=\int_\Omega u^1(x)\,v(x)\,dx,\quad \forall v\in H_0^1(\Omega),$$

which by Theorem 1.44 implies that $u'(x,0)=u^1(x)$.

To obtain the other initial condition, let us choose in (12.16) a function $\psi\in C^\infty([0,T])$ such that $\psi(0)=0$, $\psi'(0)=1$ and $\psi(T)=\psi'(T)=0$. We get, by integrating twice by parts with respect to t,

$$\int_0^T\int_\Omega u_m(x,t)\,\psi''(t)\,(P_mv)(x)\,dx\,dt$$
$$+\int_0^T\int_\Omega B(x)\nabla u_m(x,t)\,\psi(t)\,\nabla(P_mv)(x)\,dx\,dt$$
$$=\int_0^T\int_\Omega f(x,t)\,\psi(t)\,(P_mv)(x)\,dx\,dt-\int_\Omega u_m^0(x)v(x)\,dx,$$

where we pass to the limit and obtain

$$\int_0^T\int_\Omega u(x,t)\,\psi''(t)\,v(x)\,dx\,dt+\int_0^T\int_\Omega B(x)\nabla u(x,t)\,\psi(t)\,\nabla v(x)\,dx\,dt$$
$$=\int_0^T\int_\Omega f(x,t)\,\psi(t)\,v(x)\,dx\,dt-\int_\Omega u^0(x)v(x)\,dx.$$

We integrate by parts with respect to t in the first term, which is allowed by

Remark 12.1. Then, we apply once more Theorem 3.58(iii) and get

$$\int_0^T \langle u''(t), \psi(t)v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt - \int_\Omega u(x,0)\, v(x)\, dx$$
$$+ \int_0^T \int_\Omega B(x)\nabla u(x,t)\, \psi(t)\, \nabla v(x)\, dx\, dt = \int_0^T \int_\Omega f(x,t)\, \psi(t)\, v(x)\, dx\, dt$$
$$- \int_\Omega u^0(x) v(x)\, dx.$$

This, by the same arguments as before, implies $u(x,0) = u^0(x)$.

Step 5. We now prove estimate (11.9). As in the case of the heat equation, we show it for the solution u obtained above by the Faedo–Galerkin method. This is not restrictive, since in Step 6 we will prove the uniqueness of the solution of problem (12.7).

Estimate (12.9) for the solution u defined by (12.15) follows from a priori estimate (12.13). We skip the proof, since it makes use of exactly the same semi-continuity arguments as those from Step 5 in the proof of Theorem 11.2.

Step 6. Let u_1 and u_2 be two solutions corresponding to the same data. Their difference $w = u_1 - u_2$ satisfies (12.5) with $f \equiv 0$, $u^0 \equiv 0$ and $u^1 \equiv 0$, namely

$$\begin{cases} \langle w''(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \displaystyle\int_\Omega B(x)\nabla w(x,t)\, \nabla v(x)\, dx \\ \qquad\qquad = 0 \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega) \\ w(x,0) = 0 \quad \text{in } \Omega \\ w'(x,0) = 0 \quad \text{in } \Omega. \end{cases}$$

If one could use w' as test function in this problem, then one could easily obtain an estimation giving the uniqueness. But this is not allowed, since we only know (Theorem 12.2) that w' is in $L^\infty(0,T; L^2(\Omega))$, so the first term would not make sense. To avoid this difficulty we use an argument from Lions and Magenes (1968a, Chapter 3).

Let $s \in]0,T[$ be fixed and set

$$\psi(x,t) = \begin{cases} -\displaystyle\int_t^s w(x,\tau)\, d\tau & \text{for } t < s \\ 0 & \text{for } t \geq s. \end{cases}$$

Observe that one can take $v = \psi$ in the variational formulation for w. After integration with respect to t over $]0,T[$, we obtain

$$\int_0^T \langle w''(t), \psi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_0^T \int_\Omega B(x)\nabla w(x,t)\, \nabla\psi(x)\, dx = 0.$$

Note also that the function $\psi'(t) \in L^2(\Omega)$, since

$$\psi'(x,t) = \begin{cases} -w(x,t) & \text{for } t < s \\ 0 & \text{for } t \geq s. \end{cases}$$

Then, by using Theorem 3.58, Remark 3.44 and taking into account the initial condition satisfied by w' and the definition of ψ, we have

$$\begin{aligned}
&\int_0^T \langle w''(t), \psi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_0^T \langle \psi'(t), w'\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\
&= \int_0^T \langle w''(t), \psi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_0^T \int_\Omega \psi'(x,t)\, w'(x,t)\, dx\, dt \\
&= \int_0^T \frac{d}{dt} \int_\Omega w'(x,t)\, \psi(x,t)\, dx\, dt \\
&= \int_\Omega w'(x,T)\, \psi(x,T)\, dx - \int_\Omega w'(x,0)\, \psi(x,0)\, dx = 0.
\end{aligned}$$

Consequently, due to the definition of ψ', we have

$$-\int_0^s \int_\Omega w(x,t)\, w'(x,t)\, dx\, dt + \int_0^T \int_\Omega B(x) \nabla\psi'(x,t)\, \nabla\psi(x,t)\, dx = 0,$$

or equivalently, due to the symmetry of B,

$$-\frac{d}{dt} \int_0^s \int_\Omega w^2(x,t)\, dx\, dt + \frac{d}{dt} \int_0^s \int_\Omega B(x) \nabla\psi(x,t)\, \nabla\psi(x,t)\, dx\, dt = 0.$$

Since by definition

$$\nabla\psi(x,s) = 0 \quad \text{a.e. on } \Omega,$$

taking into account the initial condition on w, one gets

$$\|w(s)\|^2_{L^2(\Omega)} + \int_\Omega B(x) \nabla\psi(x,0)\, \nabla\psi(x,0)\, dx = 0.$$

The ellipticity of B implies

$$\|w(s)\|^2_{L^2(\Omega)} + \alpha \|\nabla\psi(x,0)\|^2_{L^2(\Omega)} \leq 0.$$

Hence,

$$w(s) = 0.$$

But s is arbitrary in $]0,T[$, so that $w \equiv 0$. □

Remark 12.5. One can take $f \in L^2(0,T; H^{-1}(\Omega))$ in problem (12.5). Theorem 12.2 can be easily adapted to this case. For simplicity, we took here $f \in L^2(\Omega\times]0,T[)$. ◇

12.2 The homogenization result

Let us now consider problem (12.1) and suppose we are given $f_\varepsilon \in L^2(\Omega\times]0,T[)$, $u_\varepsilon^0 \in H_0^1(\Omega)$ and $u_\varepsilon^0 \in L^2(\Omega)$. The variational formulation of problem (12.1) is

$$\begin{cases} \text{Find } \ u_\varepsilon \in \mathcal{W}_2 \text{ such that} \\ \langle u''_\varepsilon(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \displaystyle\int_\Omega A^\varepsilon(x)\nabla u_\varepsilon(x,t)\,\nabla v(x)\,dx \\ \qquad = \displaystyle\int_\Omega f_\varepsilon(x,t)\,v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega) \\ u_\varepsilon(x,0) = u_\varepsilon^0(x) \quad \text{in } \Omega \\ u'_\varepsilon(x,0) = u_\varepsilon^1(x) \quad \text{in } \ \Omega. \end{cases} \tag{12.19}$$

Theorem 11.2 provides the existence and uniqueness of a solution u_ε such that

$$u_\varepsilon \in L^\infty(0,T;\ H_0^1(\Omega)), \quad u'_\varepsilon \in L^\infty(0,T;\ L^2(\Omega)).$$

We now study the asymptotic behaviour of problem (12.19) as $\varepsilon \to 0$. As for the heat equation, studied in Chapter 11, the oscillations in (12.19) are only due to the variable x, so that in the homogenization process, the variable t will play the role of a parameter. In fact, we have the following result:

Theorem 12.6. *Suppose that $f_\varepsilon \in L^2(\Omega\times]0,T[)$, and $u_\varepsilon^0 \in H_0^1(\Omega)$, $u_\varepsilon^1 \in L^2(\Omega)$. Let u_ε be the solution of (12.19) with A^ε defined by (12.2)–(12.4). Assume that*

$$\begin{cases} i) & u_\varepsilon^0 \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega) \\ ii) & u_\varepsilon^1 \rightharpoonup u^1 \quad \text{weakly in } L^2(\Omega) \\ iii) & f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^2(\Omega\times]0,T[). \end{cases} \tag{12.20}$$

Then, one has the convergences

$$\begin{cases} i) & u_\varepsilon \rightharpoonup u \quad \text{weakly* in } L^\infty(0,T;\ H_0^1(\Omega)) \\ ii) & u'_\varepsilon \rightharpoonup u' \quad \text{weakly* in } L^\infty(0,T;\ L^2(\Omega)) \\ iii) & A^\varepsilon\nabla u_\varepsilon \rightharpoonup A^0\nabla u^0 \quad \text{weakly in } (L^2(\Omega\times]0,T[))^N, \end{cases}$$

where u is the solution of the homogenized problem:

$$\begin{cases} u'' - \operatorname{div}\left(A^0\nabla u\right) = f \quad \text{in } \Omega\times]0,T[\\ u = 0 \quad \text{on } \partial\Omega\times]0,T[\\ u(x,0) = u^0(x) \quad \text{in } \Omega \\ u'(x,0) = u^1(x) \quad \text{in } \Omega, \end{cases} \tag{12.21}$$

and A^0 is the homogenized matrix given in Theorem 6.1 by (6.30).

Proof. As for the heat equation, we apply Tartar's method of oscillating test functions, following along the lines of the proof given in Section 8.1 for the elliptic case.

Since $A^\varepsilon \in M(\alpha, \beta, \Omega)$, from assumption (12.20) and estimate (12.9) we have

$$\begin{cases} \|u_\varepsilon\|_{L^\infty(0,T;\, H_0^1(\Omega))} + \|u'_\varepsilon\|_{L^\infty(0,T;\, L^2(\Omega))} + \|u''_\varepsilon\|_{L^2(0,T;\, H^{-1}(\Omega))} \\ \qquad \leq c\big(\|f_\varepsilon\|_{L^2(\Omega\times]0,T[)} + \|u_\varepsilon^0\|_{L^2(\Omega)} + \|u_\varepsilon^1\|_{H_0^1(\Omega)}\big) \leq c_1, \end{cases} \tag{12.22}$$

where the constant c_1 is independent of ε.

Then, if ξ^ε is defined by

$$\xi^\varepsilon(x,t) = (\xi_1^\varepsilon(x,t), \dots, \xi_N^\varepsilon(x,t)) = A^\varepsilon(t)\nabla u_\varepsilon(x,t), \tag{12.23}$$

from the assumptions on A^ε, one has in particular

$$\|\xi^\varepsilon\|_{(L^2(\Omega\times]0,T[))^N} \leq \beta c_1.$$

These estimations, together with Theorem 3.58, provide the existence of a subsequence, still denoted by ε, such that

$$\begin{cases} i) & u_\varepsilon \rightharpoonup u & \text{weakly* in } L^\infty(0,T;\, H_0^1(\Omega)) \\ ii) & u_\varepsilon \to u & \text{strongly in } L^2(\Omega\times]0,T[) \\ iii) & u'_\varepsilon \rightharpoonup u' & \text{weakly* in } L^\infty(0,T;\, L^2(\Omega)) \\ iv) & \xi^\varepsilon \rightharpoonup \xi^0 & \text{weakly in } (L^2(\Omega\times]0,T[))^N. \end{cases} \tag{12.24}$$

From definition (12.23) and problem (12.19), by using Theorem 3.58 (iii) one has that ξ^ε satisfies

$$\begin{cases} \displaystyle\int_0^T\!\!\int_\Omega \xi^\varepsilon(x,t)\cdot\nabla v(x)\,\varphi(t)\,dx\,dt = \int_0^T\!\!\int_\Omega f_\varepsilon(x,t)\,v(x)\varphi(t)\,dx\,dt \\ \displaystyle\qquad\qquad + \int_0^T\!\!\int_\Omega u'_\varepsilon(x,t)\,v(x)\varphi'(t)\,dx\,dt \end{cases} \tag{12.25}$$

for any $v \in H_0^1(\Omega)$ and $\varphi \in \mathcal{D}(0,T)$, where we can pass to the limit due to convergences (12.20) and (12.24). Using once more Theorem 3.58, we obtain that ξ^0 satisfies

$$\begin{cases} \displaystyle\langle u''(t), v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_\Omega \xi^0(x,t)\cdot\nabla v(x)\,dx \\ \displaystyle\qquad = \int_\Omega f(x,t)\,v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{12.26}$$

Let us prove that

$$\xi^0 = A^0\nabla u^0. \tag{12.27}$$

We will make use again of the oscillating test functions w_λ^ε (see (8.10)), defined by

$$w_\lambda^\varepsilon(x) = \varepsilon w_\lambda\left(\frac{x}{\varepsilon}\right) = \lambda \cdot x - \varepsilon \chi_\lambda\left(\frac{x}{\varepsilon}\right),$$

where w_λ is defined by (6.27).

We recall the following convergences:

$$\begin{cases} i) & w_\lambda^\varepsilon \rightharpoonup \lambda \cdot x \quad \text{weakly in } H^1(\Omega) \\ ii) & w_\lambda^\varepsilon \to \lambda \cdot x \quad \text{strongly in } L^2(\Omega), \end{cases} \tag{12.28}$$

and set $\eta_\lambda^\varepsilon = {}^tA^\varepsilon \nabla w_\lambda^\varepsilon$, which satisfies the convergence (see (8.13))

$$\eta_\lambda^\varepsilon \rightharpoonup \mathcal{M}_Y({}^tA\nabla w_\lambda) = {}^tA^0\lambda \quad \text{weakly in } (L^2(\Omega))^N, \tag{12.29}$$

and the equation (see (8.14))

$$\int_\Omega \eta_\lambda^\varepsilon \cdot \nabla v \, dx = 0, \quad \forall v \in H_0^1(\Omega).$$

Let $\psi \in \mathcal{D}(\Omega)$. Choose here $v = \psi u_\varepsilon \varphi$ and integrate on $]0,T[$. Then

$$\int_0^T\int_\Omega \eta_\lambda^\varepsilon \cdot \nabla u_\varepsilon(x,t)\, \psi(x)\varphi(t)\, dx\, dt + \int_0^T\int_\Omega \eta_\lambda^\varepsilon \cdot \nabla\psi(x)\, u_\varepsilon(x,t)\varphi(t)\, dx\, dt = 0. \tag{12.30}$$

Taking $v = \psi w_\lambda^\varepsilon$ in (12.25) and subtracting from (12.30), we obtain

$$\begin{aligned} &\int_0^T\int_\Omega \xi^\varepsilon(x,t)\cdot\nabla\psi(x)\, w_\lambda^\varepsilon\, \varphi(t)\, dx\, dt - \int_0^T\int_\Omega \eta_\lambda^\varepsilon \cdot\nabla\psi(x)\, u_\varepsilon(x,t)\varphi(t)\, dx\, dt \\ &\quad = \int_0^T\int_\Omega f_\varepsilon(x,t)\, \psi(x)\, w_\lambda^\varepsilon(x)\varphi(t)\, dx\, dt \\ &\quad + \int_0^T\int_\Omega u_\varepsilon'(x,t)\psi(x)\, w_\lambda^\varepsilon(x)\varphi'(t)\, dx\, dt, \end{aligned}$$

where we pass to the limit by using convergences (12.20), (12.24), (12.28) and (12.29). We obtain

$$\begin{aligned} &\int_0^T\int_\Omega \xi^0(x,t)\cdot\nabla\psi(x)\,(\lambda\cdot x)\,\varphi(t)\, dx\, dt - \int_0^T\int_\Omega {}^tA^0\lambda\cdot\nabla\psi(x)\, u\,(x,t)\varphi(t)\, dx\, dt \\ &\quad = \int_0^T\int_\Omega f(x,t)\,\psi(x)\,(\lambda\cdot x)\varphi(t)\, dx\, dt \\ &\quad + \int_0^T\int_\Omega u'(x,t)\psi(x)\,(\lambda\cdot x)\varphi'(t)\, dx\, dt. \end{aligned}$$

From equation (12.26), by the same computation as in Section 8.1, we deduce (12.27).

To show that u satisfies the initial conditions in (12.21), one makes use of the same argument as in Step 4 of the proof of Theorem 12.2.

To do that, one chooses first in (12.30) $\varphi \in C^\infty([0,T])$ such that $\varphi'(0) = 1$ and $\varphi'(T) = 0$ to obtain, when passing to the limit, $u'(x,0) = u^1(x)$, in view of convergences (12.20) and (12.24). Then, choosing in (12.25) a function $\varphi \in C^\infty([0,T])$ such that $\varphi(0) = 0$, $\varphi'(0) = 1$, $\varphi(T) = \varphi'(T) = 0$ and passing again to the limit, one obtains $u(x,0) = u^0(x)$.

Finally, observe that since A^0 is elliptic (see Proposition 6.12), Theorem 12.2 provides the uniqueness of the solution of problem (12.21). Consequently, the whole sequences in (12.24) converge. This concludes the proof. □

12.3 Convergence of the energy

In this section we prove, under suitable assumptions, the convergence of the energy associated to problem (12.1) to the energy of the homogenized problem. As already seen in the elliptic and parabolic cases, this property is essential in the proof of the corrector result which will be given in Section 12.4.

Let us define the energies associated to problems (12.1) and (12.21) respectively, by

$$\begin{cases} E^\varepsilon(u_\varepsilon)(t) = \dfrac{1}{2}\|u'_\varepsilon(t)\|^2_{L^2(\Omega)} + \dfrac{1}{2}\displaystyle\int_\Omega A^\varepsilon(x)\nabla u_\varepsilon(x,t)\,\nabla u_\varepsilon(x,t)\,dx\,dt \\ E(u)(t) = \dfrac{1}{2}\|u'(t)\|^2_{L^2(\Omega)} + \dfrac{1}{2}\displaystyle\int_\Omega A^0(x)\nabla u(x,t)\,\nabla u(x,t)\,dx\,dt. \end{cases} \tag{12.31}$$

We will need the following result, which, when $f = 0$, is known as 'the conservation of the energy':

Proposition 12.7. *Suppose that assumptions (12.6) are fulfilled. Then the solution u of problem (12.7) satisfies the following identity:*

$$\begin{cases} \dfrac{1}{2}\|u'(t)\|^2_{L^2(\Omega)} + \dfrac{1}{2}\displaystyle\int_\Omega B(x)\nabla u(x,t)\,\nabla u(x,t)\,dx\,dt \\ \quad = \dfrac{1}{2}\|u^1\|^2_{L^2(\Omega)} + \dfrac{1}{2}\displaystyle\int_\Omega B\nabla u^0\,\nabla u^0\,dx + \int_0^T\int_\Omega f(x,\tau)\,u'(x,\tau)\,dx\,d\tau. \end{cases} \tag{12.32}$$

Proof. If one could use u' as test function in (12.7) then (12.32) would be immediate, but this is not allowed, since we only know (Theorem 12.2) that u' is in $L^\infty(0,T;\ L^2(\Omega))$. To avoid this difficulty, we use a density argument (see for instance, Lions, 1988, Chapter 1, Lemma 3.5). Let $\{u_h^0\}_{h\in\mathbb{N}}$, $\{u_h^1)\}_{h\in\mathbb{N}}$ and $\{f_h\}_{h\in\mathbb{N}}$ be three sequences in $\mathcal{D}(\Omega)$ such that

$$\begin{cases} i) & u_h^0 \to u^0 \quad \text{strongly in } H_0^1(\Omega) \\ ii) & u_h^1 \to u^1 \quad \text{strongly in } L^2(\Omega\times]0,T[) \\ iii) & f_h \to f \quad \text{strongly in } L^2(\Omega\times]0,T[). \end{cases} \tag{12.33}$$

as $h \to \infty$. Consider for any h, the solution u_h of the problem

$$\begin{cases} u_h'' - \operatorname{div}(B\nabla u_h) = f_h & \text{in } \Omega\times]0,T[\\ u_h = 0 \quad \text{on } \partial\Omega\times]0,T[\\ u_h(x,0) = u_h^0(x) & \text{in } \Omega \\ u_h'(x,0) = u_h^1(x) & \text{in } \Omega. \end{cases} \tag{12.34}$$

Due to the regularity of data in (12.34), from the regularity result in Proposition 12.3, one can choose u_h' as test function in the variational formulation of this problem (see (12.7). Due to the symmetry of B, one has, after integration over $]0,t[$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big[\int_0^t \|u_h'\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega B(x)\nabla u_h(x,\tau)\,\nabla u_h(x,\tau)\,dx\,d\tau\Big] \\ = \int_0^T \int_\Omega f_h(x,\tau)\,u_h'(x,\tau)\,dx\,d\tau, \end{aligned}$$

which implies that

$$\begin{cases} \frac{1}{2}\|u_h'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega B(x)\nabla u_h(x,t)\,\nabla u_h(x,t)\,dx\,dt \\ \quad = \frac{1}{2}\|u_h^1\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega B\nabla u_h^0\,\nabla u_h^0\,dx + \int_0^T\int_\Omega f_h(x,\tau)\,u_h'(x,\tau)\,dx\,d\tau. \end{cases} \tag{12.35}$$

Observe now that from estimate (12.9) applied to problems (12.7) and (12.34) one has, by linearity,

$$\begin{aligned} &\|u - u_h\|_{L^\infty(0,T;\,H_0^1(\Omega))} + \|u' - u_h'\|_{L^\infty(0,T;\,L^2(\Omega))} \\ &\leq c\big(\|f - f_h\|_{L^2(\Omega\times]0,T[)} + \|u^0 - u_h^0\|_{L^2(\Omega)} + \|u^1 - u_h^1\|_{H_0^1(\Omega)}\big), \end{aligned}$$

where c depends only on α, β, Ω, and T. Consequently,

$$\begin{cases} i) & u_h \to u \quad \text{strongly in } L^\infty(0,T;\,H_0^1(\Omega)) \\ ii) & u_h' \to u' \quad \text{strongly in } L^\infty(0,T;\,L^2(\Omega)), \end{cases}$$

as $h \to \infty$. This, together with (12.33), allows us to pass to the limit in (12.35) to get the claimed result. □

From this proposition and definition (12.31) one has immediately the following result:

Corollary 12.8. *Let $E^\varepsilon(u_\varepsilon)$ and $E(u)$ be given by (12.31). Then*

$$\begin{cases} i) \quad E^\varepsilon(u_\varepsilon)(t) = \frac{1}{2}\|u_\varepsilon^1\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega A^\varepsilon \nabla u_\varepsilon^0 \, \nabla u_\varepsilon^0 \, dx \\ \qquad\qquad + \int_0^T \int_\Omega f_\varepsilon(x,\tau) u'_\varepsilon(x,\tau)\, dx\, d\tau \\ ii) \quad E(u)(t) = \frac{1}{2}\|u^1\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega A^0 \nabla u^0 \, \nabla u^0 \, dx \\ \qquad\qquad + \int_0^T \int_\Omega f(x,\tau)\, u'(x,\tau)\, dx\, d\tau. \end{cases} \tag{12.36}$$

In Chapter 11. we proved the convergence of the energy associated with the heat equation to that of the corresponding homogenized problem. To do so, we needed to suppose a strong convergence of the initial condition.

For the case of the wave equation. the situation is more complicated. One has to suppose the strong convergence of f_ε and u_ε^1 and, moreover, to make a special assumption on u_ε^0. The peculiarity of this assumption is that it does not concern u_ε^0 but $\text{div}(A^\varepsilon \nabla u_\varepsilon^0)$. namely one suppose that there exists an element $U^0 \in H^{-1}(\Omega)$ such that

$$-\text{div}\left(A^\varepsilon \nabla u_\varepsilon^0\right) \to U^0 \quad \text{strongly in } H^{-1}(\Omega). \tag{12.37}$$

Observe that any element $U^0 \in H^{-1}(\Omega)$ can be written in the form $U^0 = -\text{div}(A^0 \nabla u^0)$ for some $u^0 \in H_0^1(\Omega)$. For that, one has just to solve the problem

$$\begin{cases} -\text{div}\,(A^0 \nabla u^0) = U^0 & \text{in } \Omega \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.38}$$

which has a unique solution $u^0 \in H_0^1(\Omega)$, due to the ellipticity of the matrix A^0 (Proposition 6.12) and Theorem 4.16. Consequently, convergence (12.37) is equivalent to

$$\text{div}(A^\varepsilon \nabla u_\varepsilon^0) \to \text{div}(A^0 \nabla u^0) \quad \text{strongly in } H^{-1}(\Omega).$$

This implies in particular the following convergences:

$$\begin{cases} i) & u_\varepsilon^0 \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega) \\ ii) & A^\varepsilon \nabla u_\varepsilon^0 \rightharpoonup A^0 \nabla u^0 & \text{weakly in } (L^2(\Omega))^N. \end{cases} \tag{12.39}$$

which are an immediate consequence of Theorem 8.16 applied to the problem

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon^0) = F^\varepsilon & \text{in } \Omega \\ u_\varepsilon^0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.40}$$

where $F^\varepsilon = -\text{div}(A^\varepsilon \nabla u_\varepsilon^0)$. Observe that (12.37) is a stronger assumption than (12.39). Indeed. convergence (12.39)(ii) only implies the weak convergence of $\text{div}(A^\varepsilon \nabla u_\varepsilon^0)$ to $\text{div}(A^0 \nabla u^0)$ in $H^{-1}(\Omega)$.

Let us prove the following convergence result:

Proposition 12.9. *Let $f_\varepsilon \in L^2(\Omega\times]0,T[)$, $u^0_\varepsilon \in H^1_0(\Omega)$, $u^1_\varepsilon \in L^2(\Omega)$ and let u_ε be the solution of (12.1) with A^ε defined by (12.2)–(12.4). Suppose that*

$$\begin{cases} \text{There exists } u^0 \in H^1_0(\Omega) \text{ such that} \\ -\text{div}\left(A^\varepsilon \nabla u^0_\varepsilon\right) \rightarrow -\text{div}(A^0\nabla u^0) \quad \text{strongly in } H^{-1}(\Omega) \end{cases} \tag{12.41}$$

and

$$\begin{cases} i) \quad u^1_\varepsilon \rightarrow u^1 \quad \text{strongly in } L^2(\Omega) \\ ii) \quad f_\varepsilon \rightarrow f \quad \text{strongly in } L^2(\Omega\times]0,T[). \end{cases} \tag{12.42}$$

Then

$$E^\varepsilon(u_\varepsilon) \rightarrow E(u) \quad \text{in } C([0,T]), \tag{12.43}$$

where u is the solution of the homogenized problem

$$\begin{cases} u'' - \text{div}\left(A^0\nabla u\right) = f & \text{in } \Omega\times]0,T[\\ u = 0 \quad \text{on } \partial\Omega\times]0,T[\\ u(x,0) = u^0(x) & \text{in } \Omega \\ u'(x,0) = u^1(x) & \text{in } \Omega. \end{cases}$$

and A^0 is the homogenized matrix given in Theorem 6.1 by (6.30).

Proof. Since hypothesis (12.41) implies (12.20)(i) (see (12.39)), all the assumptions of Theorem 12.6 are satisfied, hence u_ε converges to u. Recall that in particular, one has the estimate

$$\|u_\varepsilon\|_{L^\infty(0,T;\, H^1_0(\Omega))} + \|u'_\varepsilon\|_{L^\infty(0,T;\, L^2(\Omega))} \leq c_1 \tag{12.44}$$

where the constant c_1 is independent of ε.

The proof of (12.43) follows the same outlines as that of Proposition 11.5. We prove first that

$$\begin{cases} i) \quad |E^\varepsilon(u_\varepsilon)(t)| \leq c, \quad \forall t \in [0,T] \\ ii) \quad |E^\varepsilon(u_\varepsilon)(t+h) - E^\varepsilon(u_\varepsilon)(t)| \leq \theta(h), \\ \qquad \text{uniformly with respect to } \varepsilon, \forall t \in [0,T-h], \forall h > 0, \end{cases} \tag{12.45}$$

where θ tends to zero as $h \rightarrow 0$.

Statement (i) is straightforward by assumptions (12.4), (12.42) and the former estimate (12.44).

On the other hand, from Corollary 12.8, assumption (12.42) and estimate (12.44), one has

$$\begin{aligned} |E^\varepsilon(u_\varepsilon)(t+h) - E^\varepsilon(u'_\varepsilon)(t)| &= \left| \int_t^{t+h} \int_\Omega f_\varepsilon(x,\tau) u_\varepsilon(x,\tau)\, dx\, d\tau \right| \\ &\leq h^{\frac{1}{2}} \|u'_\varepsilon\|_{L^\infty(0,T;\, L^2(\Omega))} \|f_\varepsilon\|_{L^2(\Omega\times]0,T[)} \leq c_2 h^{\frac{1}{2}}, \end{aligned}$$

where c_2 is independent of ε.

Properties (12.45) mean that $E^\varepsilon(u_\varepsilon)$ belongs to a compact set in $C([0,T])$, according to the Ascoli–Arzelà theorem. Hence, there exists a subsequence (still denoted by ε) and $\zeta \in C([0,T])$ such that

$$E^\varepsilon(u_\varepsilon) \to \zeta \quad \text{in } C([0,T]).$$

Turning back to problem (12.40), in view of (12.41) and (12.39) one can apply Theorem 8.16 written for $f^\varepsilon = -\text{div}\left(A^\varepsilon \nabla u_\varepsilon^0\right)$ to have the convergence of energy,

$$\int_\Omega A^\varepsilon \nabla u_\varepsilon^0\, \nabla u_\varepsilon^0\, dx \to \int_\Omega A^0 \nabla u^0\, \nabla u^0\, dx.$$

Then, due to the convergences from Theorem 12.6 and assumptions (12.41), we can pass to the pointwise limit in (12.36)(i), to get

$$\lim_{\varepsilon\to 0} E^\varepsilon(u_\varepsilon)(t) = E(u)(t), \qquad \forall t \in [0,T].$$

Hence, $\zeta = E(u)$ and this ends the proof. □

12.4 A corrector result

We end this chapter by a corrector result. We will use, for its proof, arguments from Brahim-Otsmane, Francfort, and Murat (1992). As already mentioned, this result is based on the convergence of the energy. The proof is done in the same spirit as that for the heat equation (see Section 11.4). The corrector matrix $C^\varepsilon = (C_{ij}^\varepsilon)_{1\le i,j\le N}$ is still that introduced for the elliptic case, and is defined by

$$\begin{cases} C_{ij}^\varepsilon(x) = & C_{ij}\left(\dfrac{x}{\varepsilon}\right) \quad \text{a.e. on } \Omega \\ C_{ij}(y) = & \delta_{ij} - \dfrac{\partial \widehat{\chi}_j}{\partial y_i}(y) = \dfrac{\partial \widehat{w}_j}{\partial y_i}(y) \quad \text{a.e. on } Y, \end{cases} \tag{12.46}$$

where $\widehat{\chi}_j$ and $\widehat{w}_j$ are defined by (6.14), (6.15) and (6.16).

Theorem 12.10. *Let u_ε be the solution of (12.1) with A^ε defined by (12.2)-(12.4). Suppose that the data satisfy (12.40) and (12.41). Then*

$$\begin{cases} i) & u_\varepsilon' \to u' \quad \text{strongly in } C([0,T];\, L^2(\Omega)) \\ ii) & \nabla u_\varepsilon - C^\varepsilon \nabla u \to 0 \quad \text{strongly in } (C([0,T];\, L^1(\Omega)))^N. \end{cases} \tag{12.47}$$

Moreover, if $C \in (L^r(Y))^{N\times N}$ for some r such that $2 \le r \le \infty$, and $\nabla u \in (L^s(\Omega))^N$ for some s such that $2 \le s < \infty$, then

$$\nabla u_\varepsilon - C^\varepsilon \nabla u \to 0 \quad \text{strongly in } (C([0,T];\, L^t(\Omega)))^N,$$

where

$$t = \min\left\{2, \frac{rs}{r+s}\right\}.$$

The proof of this result is based on the following proposition, which is analogous to Proposition 11.8.

Proposition 12.11. *Suppose that the assumptions of Theorem 12.10 are fulfilled. Set for any* $\Phi \in C^\infty([0,T]; \mathcal{D}(\Omega))$

$$\begin{aligned}\rho_\varepsilon(t) &= \frac{1}{2}\|u'_\varepsilon(t) - \Phi'(t)\|^2_{L^2(\Omega)} \\ &+ \int_0^T\int_\Omega A^\varepsilon(x)(\nabla u_\varepsilon - C^\varepsilon\nabla\Phi)(x,\tau)\,(\nabla u_\varepsilon - C^\varepsilon\nabla\Phi)(x,\tau)\,dx\,d\tau.\end{aligned}$$

Then

$$\rho_\varepsilon \rightarrow \rho \quad \text{strongly in } C([0,T]).$$

where

$$\rho(t) = \frac{1}{2}\|u'(t)-\Phi'(t)\|^2_{L^2(\Omega)}+\int_0^T\int_\Omega A^0(x)(\nabla u-\nabla\Phi)(x,\tau)\,(\nabla u-\nabla\Phi)(x,\tau)\,dx\,d\tau.$$

Proof. We follow the lines of the proof of Proposition 11.8. Write first ρ_ε in the form

$$\rho_\varepsilon = \rho^1_\varepsilon + \rho^2_\varepsilon + \rho^3_\varepsilon, \tag{12.48}$$

where

$$\begin{cases}\rho^1_\varepsilon(t) &= \frac{1}{2}\|u'_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega A^\varepsilon\nabla u_\varepsilon\nabla u_\varepsilon\,dx\,d\tau \\ \rho^2_\varepsilon(t) &= \frac{1}{2}\|\Phi'(t)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega A^\varepsilon\,(C^\varepsilon\nabla\Phi)\,(C^\varepsilon\nabla\Phi)\,dx\,d\tau \\ \rho^3_\varepsilon(t) &= \int_\Omega u'_\varepsilon\,\Phi'\,dx + \int_0^T\int_\Omega A^\varepsilon\,(C^\varepsilon\nabla\Phi)\,\nabla u_\varepsilon\,dx\,d\tau \\ &\quad + \int_0^T\int_\Omega A^\varepsilon\nabla u_\varepsilon\,(C^\varepsilon\nabla\Phi)\,dx\,d\tau.\end{cases} \tag{12.49}$$

We will see that all these terms converge in $C([0,T])$.

First term. Notice that ρ^1_ε is the energy $E^\varepsilon(u_\varepsilon)$ and hence by Proposition 12.9 it follows that

$$\rho^1_\varepsilon \rightarrow E(u) = \frac{1}{2}\|u'(t)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega A^0(x)\nabla u(x,\tau)\,\nabla u(x,\tau)\,dx\,d\tau, \text{in } C([0,T]). \tag{12.50}$$

Second term. For the second term ρ^2_ε, we argue exactly as we did when proving (11.46). Thus we obtain

$$\rho^2_\varepsilon \rightarrow \frac{1}{2}\|\Phi'(t)\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega A^0(x)\nabla\Phi(x,\tau)\,\nabla\Phi(x,\tau)\,dx\,d\tau, \quad \text{in } C([0,T]). \tag{12.51}$$

Third term. The pointwise convergence

$$\rho_\varepsilon^3 \to \int_\Omega u'\,\Phi'\,dx + \int_0^T \int_\Omega A^0 \nabla\Phi \nabla u\,dx\,d\tau + \int_0^T \int_\Omega A^0 \nabla u \nabla \Phi\,dx\,d\tau \quad \text{for any } t \in [0,T],$$

follows by using convergences (12.24) and the same computations as in (8.40) and (8.41). It is easy to prove that ρ_ε^3 is bounded in $H^1(0,T)$. A priori estimates (12.22) for u_ε, convergence (8.30)(i) and the assumptions on A imply immediately the boundedness of $\|\rho_\varepsilon^3\|_{L^\infty(0,T)}$. On the other hand, from Theorem 3.58 (iii), one has

$$\begin{aligned}(\rho_\varepsilon^3)'(t) &= \langle u_\varepsilon''(\cdot,t), \Phi'(\cdot,t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega u_\varepsilon'(x,t)\,\Phi''(x,t)\,dx \\ &\quad + \int_\Omega A^\varepsilon\,(C^\varepsilon \nabla\Phi)\,\nabla u_\varepsilon\,dx + \int_\Omega A^\varepsilon \nabla u_\varepsilon\,(C^\varepsilon\nabla\Phi)\,dx.\end{aligned}$$

From this expression, the boundedness of $\|(\rho_\varepsilon^3)'\|_{L^2(0,T)}$ is obvious by the same arguments as above. Hence, one has the convergence

$$\rho_\varepsilon^3 \to \int_\Omega u\,\Phi\,dx + \int_0^T \int_\Omega A^0 \nabla\Phi \nabla u\,dx\,d\tau + \int_0^T \int_\Omega A^0 \nabla u \nabla\Phi\,dx\,d\tau \quad \text{in } C([0,T]), \tag{12.52}$$

due to the compact injection of $H^1(0,T)$ in $C([0,T])$ (see Theorem 3.27(iii)). Recalling (12.48), from definition (12.49) and convergences (12.50), (12.51) and (12.52), the claimed result follows easily. □

Proof of Theorem 12.10. The proof is analogous to that of Theorem 11.7.

Let us just point out the main difference. One has to study the convergence of the term $\|u_\varepsilon' - u'\|^2_{C([0,T];\,L^2(\Omega))}$ instead of $\|u_\varepsilon - u\|^2_{C([0,T];\,L^2(\Omega))}$.

To do so, for $\delta > 0$, by using Proposition 3.60, one introduces $\Phi_\delta \in C^\infty([a,b];\,\mathcal{D}(\Omega))$, such that

$$\begin{cases} i) & \|u' - \Phi_\delta'\|^2_{C([a,b];\,L^2(\Omega))} \le \delta \\ ii) & \|\nabla u - \nabla\Phi_\delta\|^2_{L^2(\Omega\times]0,T[)} \le \delta. \end{cases}$$

Then, to prove the result, we write

$$u_\varepsilon - u = (u_\varepsilon - \Phi_\delta) + (\Phi_\delta - u),$$

and argue exactly as in the proof of Theorem 11.7. □

13

General approaches to the non-periodic case

In this chapter we present some results concerning the convergence of the solutions of partial differential equations with non-periodic coefficients. For simplicity, we will only present results concerning the elliptic problems. All the results can be extended to the heat and wave equations, as well as to the linearized elasticity system.

As in the previous chapters, Ω denotes a bounded open set in $\mathbb{R}^N$ and $\varepsilon > 0$ is a parameter taking its values in a sequence which tends to zero.

Let $\alpha, \beta \in \mathbb{R}$, be such that $0 < \alpha < \beta$. Recall (see Definition 4.11) that $M(\alpha, \beta, \Omega)$ denotes the set of $N \times N$ matrices $A = (a_{ij})_{1 \leq i,j \leq N} \in (L^\infty(\Omega))^{N \times N}$ such that

$$\begin{cases} i) & (A(x)\gamma, \gamma) \geq \alpha|\gamma|^2 \\ ii) & |A(x)\gamma| \leq \beta|\gamma| \,, \end{cases} \tag{13.1}$$

for any $\gamma \in \mathbb{R}^N$ and a.e. on Ω.

In this chapter we consider the general elliptic problem

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ \quad u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{13.2}$$

where f is given in $H^{-1}(\Omega)$ and $\{A^\varepsilon\}$ is a sequence of matrices in $M(\alpha, \beta, \Omega)$.

In Section 13.1 we recall the notions of G-convergence and H-convergence introduced respectively by Spagnolo (1967) and Tartar (1977a). These definitions deal with the convergence of the solutions of problem (13.2). In Section 13.2 we present the compensated compactness due to Murat and Tartar (see Murat, 1978b and Tartar, 1979) and a corrector result.

Finally, in Section 13.3 we give some optimal bounds for the eigenvalues of a homogenized matrix in the general case. We refer for that to Tartar (1985), Lurie and Cherkaev (1984, 1986).

Throughout this book, we have restricted ourselves to the linear case. For the study of the asymptotic behaviour of a large class of nonlinear problems one has a general mathematical theory introduced by E. De Giorgi (see De Giorgi, 1975, De Giorgi and Spagnolo, 1973, De Giorgi and Franzoni, 1975) and called Γ-convergence. It deals with the convergence of the minima of functionals. There is now a wide range of results in this field. We refer in particular to Dal Maso (1993) for a general exposition. We refer also the reader to references in their work.

13.1 G-convergence and H-convergence

In Section 5.1 we have shown some properties of problem (13.2). First, it has a unique solution satisfying the estimate

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \le \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}.$$

Furthermore, the vector $A^\varepsilon \nabla u^\varepsilon$ satisfies the estimate

$$\|A^\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega)} \le \frac{\beta}{\alpha}\|f\|_{H^{-1}(\Omega)},$$

so that there exists a subsequence such that

$$\begin{cases} i) & u^{\varepsilon'} \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega) \\ ii) & A^{\varepsilon'}\nabla u^{\varepsilon'} \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^N. \end{cases} \tag{13.3}$$

Moreover, one has

$$-\text{div}\ \xi^0 = f \quad \text{in } \Omega.$$

The question is still whether one can find a relation between u^0 and ξ^0 and a limit equation satisfied by u^0.

From the end of the sixties these questions have been widely investigated.

The first significant work on this subject is due to S. Spagnolo who, in a paper of 1967 (see Spagnolo, 1967). introduced the notion of G-convergence, which deals with the convergence of the solutions of elliptic problems of the type (13.2) as well as of the corresponding heat equation. In this framework, the matrices A^ε are supposed to be symmetric.

Definition 13.1. Let $\{A^\varepsilon\}$ be a sequence of symmetric matrices in $M(\alpha, \beta, \Omega)$. We say that it *G-converges* to the symmetric matrix $A^0 \in M(\alpha,\beta,\Omega)$ iff for every function f of $H^{-1}(\Omega)$, the solution u^ε of

$$\begin{cases} -\text{div}\ (A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

is such that

$$u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1,$$

where u^0 is the unique solution of the problem

$$\begin{cases} -\text{div}\ (A^0 \nabla u^0) = f & \text{in } \Omega \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

The G-convergence has the following main properties:

Theorem 13.2. *One has*

i) (uniqueness). The G-limit of a G-converging sequence $\{A^\varepsilon\} \in M(\alpha,\beta,\Omega)$ *is unique.*

ii) (locality). Let $\{A^\varepsilon\}$ *and* $\{B^\varepsilon\}$ *be two sequences of symmetric matrices in* $M(\alpha,\beta,\Omega)$ *which G-converge respectively to* A^0 *and* B^0. *If for some* $\omega \subset \Omega$ *one has*

$$A^\varepsilon = B^\varepsilon \quad \text{in } \omega, \quad \text{for every } \varepsilon,$$

then

$$A^0 = B^0 \quad \text{in } \omega.$$

iii) (compactness). Let $\{A^\varepsilon\}$ *be a sequence of symmetric matrices in* $M(\alpha,\beta,\Omega)$. *Then there exists a subsequence* $\{A^{\varepsilon'}\}$ *and a matrix* $A^0 \in M(\alpha,\beta,\Omega)$ *such that* $\{A^{\varepsilon'}\}$ *G-converges to* A^0.

iv) A sequence $\{A^\varepsilon\}$ *of symmetric matrices in* $M(\alpha,\beta,\Omega)$ *G-converges iff all its G-converging subsequences have the same limit.*

This kind of convergence has been extended to sequences of matrices in $M(\alpha,\beta,\Omega)$ which are not necessarily symmetric. This leads to the notion of H-convergence, introduced by Tartar (1977a) and developed by F. Murat and L. Tartar (see Murat, 1978a, Murat and Tartar, 1997a).

Definition 13.3. A sequence $\{A^\varepsilon\}$ in $M(\alpha,\beta,\Omega)$ H-converges to $A^0 \in M(\alpha',\beta',\Omega)$ iff for every function f of $H^{-1}(\Omega)$, the solution u_ε of

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{13.4}$$

is such that

$$\begin{cases} i) & u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega) \\ ii) & A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0 \quad \text{weakly in } (L^2(\Omega))^N, \end{cases}$$

where u^0 is the unique solution of the problem

$$\begin{cases} -\text{div}\,(A^0 \nabla u^0) = f & \text{in } \Omega \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{13.5}$$

Let us point out the main difference between these two notions of convergence. G-convergence deals with symmetric matrices and supposes the convergence of the solutions u^ε only. H-convergence is defined for general sequences (not necessarily symmetric) and suppose not only the convergence of the solutions u^ε but also that of $A^\varepsilon \nabla u^\varepsilon$. The main feature of H-convergence is that the additional condition on the convergence of $A^\varepsilon \nabla u^\varepsilon$ is essential in order to keep the main properties stated in Theorem 13.2. Indeed one has

Theorem 13.4. *One has the following properties:*

i) *(uniqueness). The H-limit of a H-converging sequence* $\{A^\varepsilon\} \in M(\alpha, \beta, \Omega)$ *is unique.*

ii) *(locality). Let* $\{A^\varepsilon\}$ *and* $\{B^\varepsilon\}$ *be two sequences in* $M(\alpha, \beta, \Omega)$ *which H-converge respectively to* A^0 *and* B^0. *If for some* $\omega \subset \Omega$ *one has*

$$A^\varepsilon = B^\varepsilon \quad \text{in } \omega, \quad \text{for every } \varepsilon,$$

then

$$A^0 = B^0 \quad \text{in } \omega.$$

iii) *(compactness). Let* $\{A^\varepsilon\}$ *be a sequence in* $M(\alpha, \beta, \Omega)$. *Then there exists a subsequence* $\{A^{\varepsilon'}\}$ *and a matrix* $A^0 \in M(\alpha, \beta^2/\alpha, \Omega)$ *such that* $\{A^{\varepsilon'}\}$ *H-converges to* A^0.

iv) *A sequence* $\{A^\varepsilon\}$ *of symmetric matrices in* $M(\alpha, \beta, \Omega)$ *H-converges iff all its H-converging subsequences have the same limit.*

Remark 13.5. Observe that the H-limit A^0 provided by compactness in Theorem 13.4 is in a larger class then $M(\alpha, \beta, \Omega)$, namely in $M(\alpha, \beta^2/\alpha, \Omega)$. ◇

A natural question is what is the relation between the two convergences for a sequence of symmetric matrices? The answer is given by

Proposition 13.6. *For a a sequence* $\{A^\varepsilon\}$ *of symmetric matrices in* $M(\alpha, \beta, \Omega)$ *G-convergence is equivalent to H-convergence.*

The proof of this result makes use of the following comparison theorem, which generalizes Theorem 8.12 to the non-periodic case. We refer for it to De Giorgi and Spagnolo (1973) and to Tartar (1977a).

Theorem 13.7. *Let* $\{A^\varepsilon\}$ *and* $\{B^\varepsilon\}$ *be two sequences in* $M(\alpha, \beta, \Omega)$ *which H-converge respectively to* A^0 *and* B^0. *Suppose that for any* ε, *the matrix* B^ε *is symmetric and that*

$$B^\varepsilon \leq A^\varepsilon,$$

in the sense of Definition 8.11. Then

$$B^0 \leq A^0.$$

Proof of Proposition 13.6. Observe first that for any sequence $\{A^\varepsilon\}$ in $M(\alpha, \beta, \Omega)$, if I denotes the $N \times N$ identity matrix, one has

$$A^\varepsilon \leq \beta I, \quad \forall \varepsilon,$$

in the sense of Definition 8.11. Then, if $\{A^\varepsilon\}$ is a sequence of symmetric matrices which H-converges to some $A^0 \in M(\alpha, \beta^2/\alpha, \Omega)$, from Theorem 13.7 one has

$$A^0 \leq \beta I.$$

This implies that A^0 is in $M(\alpha, \beta, \Omega)$ and then that the sequence $\{A^\varepsilon\}$ G-converges to A^0 .

Conversely, suppose that $\{A^\varepsilon\}$ is a sequence of symmetric matrices in $M(\alpha, \beta, \Omega)$ which G-converges to some A^0. From Theorem 13.4 there exists a subsequence $\{A^{\varepsilon'}\}$ and a matrix $B^0 \in M(\alpha, \beta^2/\alpha, \Omega)$ such that $\{A^{\varepsilon'}\}$ H-converges to B^0. Due to the implication already proved, one has that $\{A^{\varepsilon'}\}$ G-converges to B^0. But since obviously, the subsequence $\{A^{\varepsilon'}\}$ also G-converges to A^0, the uniqueness of the G-limit provided by Theorem 13.2 shows that $A^0 = B^0$. Then, all the H-convergent subsequences of $\{A^\varepsilon\}$ converge to A^0. This, thanks to Theorem 13.4, proves that the whole sequence $\{A^\varepsilon\}$ H-converges to A^0. □

Let us now give one important consequence of Proposition 13.6.

Corollary 13.8. *Let $\{A^\varepsilon\}$ be a sequence of symmetric matrices in $M(\alpha, \beta, \Omega)$ which G-converges to A^0. Then*

$$A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0 \quad \text{weakly in } (L^2(\Omega))^N.$$

Remark 13.9. Let $\{A^\varepsilon\}$ be the sequence of periodic matrices defined by (6.2)–(6.4). Then. Theorem 6.1, together with Proposition 6.12, states precisely the H-convergence of this sequence to the constant matrix A^0 defined by (6.30). ◊

The proof of the main theorems 13.2 and 13.4 are quite difficult and delicate. They can be found in Spagnolo (1967), Murat and Tartar (1997a) and also in Oleinik, Shamaev, and Yosifian (1992) and in Jikov, Kozlov, and Oleinik (1994).

13.2 Compensated compactness and correctors

One of the main tools for proving Theorem 13.4 above is the compensated compactness due to F. Murat and L. Tartar (see for instance. Murat, 1978a and Tartar, 1979). We recall this result as well as some related properties in this section.

As we have seen throughout this book, the product of two weakly convergent sequences does not converge, in general, to the product of the limits, and this is the principal difficulty when trying to characterize ξ^0, given in (13.3) in terms of u^0. The compensated compactness shows that under some additional assumptions, the product of two weak convergent sequences in $L^2(\Omega)$ converges in the sense of distributions to the product of the limits.

This result is interesting in itself and is widely used in many applications.

Theorem 13.10. *Let $\{U^\varepsilon\}$ and $\{V^\varepsilon\}$ two sequences in $(L^2(\Omega))^N$ such that*

$$\begin{cases} U^\varepsilon \rightharpoonup U^0 & \text{weakly in } (L^2(\Omega))^N \\ V^\varepsilon \rightharpoonup V^0 & \text{weakly in } (L^2(\Omega))^N. \end{cases}$$

Suppose that $\{\text{div } U^\varepsilon\}$ *is compact in* $H^{-1}(\Omega)$ *and* $\{\text{curl } V^\varepsilon\}$ *is bounded in* $(L^2(\Omega))^{N\times N}$, *where the matrix* $\text{curl } V^\varepsilon = \big((\text{curl } V^\varepsilon)_{ij}\big)_{1\le i,j\le N}$ *is defined by*

$$(\text{curl } V^\varepsilon)_{ij} = \frac{\partial V_i^\varepsilon}{\partial x_j} - \frac{\partial V_j^\varepsilon}{\partial x_i}, \quad \text{for } i,j = 1,\dots,N.$$

Then

$$U^\varepsilon V^\varepsilon \longrightarrow U^0 V^0 \quad \text{in } \mathcal{D}'(\Omega).$$

In the framework of H-convergence, the interest of Theorem 13.10 is that it can be applied to the case

$$U^\varepsilon = A^\varepsilon \nabla u^\varepsilon, \quad V^\varepsilon = \nabla v^\varepsilon,$$

where u^ε solves a problem of the form (13.1). Indeed,

Corollary 13.11. *Suppose that* $\{A^\varepsilon\}$ *H-converges to* A^0 *and let* u^ε *be the solution of problem (13.4). Suppose further that* $\{v^\varepsilon\}$ *is a sequence in* $H^1(\Omega)$ *such that*

$$v^\varepsilon \rightharpoonup v^0 \quad \text{weakly in } H^1(\Omega).$$

Then one has

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \, \nabla v^\varepsilon \, \varphi \, dx \longrightarrow \int_\Omega A^0 \nabla u^0 \, \nabla v^0 \, \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

where u^0 *is the solution of (13.5).*

Proof. By assumption, one has

$$\begin{cases} A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0 & \text{weakly in } (L^2(\Omega))^N \\ \nabla v^\varepsilon \rightharpoonup \nabla v^0 & \text{weakly in } (L^2(\Omega))^N. \end{cases}$$

Consequently, Theorem 13.10 applies, since $\text{div } A^\varepsilon \nabla u^\varepsilon = -f$ is fixed in $H^{-1}(\Omega)$ and obviously, $\text{curl } V^\varepsilon = 0$. □

This implies in particular that if $\{A^\varepsilon\}$ H-converges to A^0, then one has convergence of the energy.

Proposition 13.12. *Suppose that* $\{A^\varepsilon\}$ *H-converges to* A^0 *and let* u^ε *be the solution of problem (13.4). Then,*

$$A^\varepsilon \nabla u^\varepsilon \, \nabla u^\varepsilon \longrightarrow A^0 \nabla u^0 \, \nabla u^0 \quad \text{in } \mathcal{D}'(\Omega),$$

where u^0 *is the solution of (13.5).*

Proof. The proof is straightforward from Corollary 13.11 by choosing $v^\varepsilon = u^\varepsilon$. □

At this point, let us also mention that most of the results proved in Chapter 8 for the periodic case can be extended to the case of an H-convergent sequence. In particular one can construct a sequence of local corrector matrices, in order to improve locally the weak convergence of ∇u^ε to ∇u^0 supposed in Definition 13.3.

To do so, we introduce first, for any open subset ω of Ω such that $\overline{\omega} \subset \Omega$, a family of auxiliary functions w^ε_λ as follows.

Let ω_1 be an open subset of Ω such that $\overline{\omega} \subset \omega_1 \subset \overline{\omega}_1 \subset \Omega$ and $\varphi \in \mathcal{D}(\omega_1)$ such that $\varphi = 1$ on ω. Further, let $\{A^\varepsilon\}$ be a sequence H-converging to A^0.

Consider, for any $\lambda \in \mathbb{R}^N$, the solution w^ε_λ of the problem

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla w^\varepsilon_\lambda) = -\operatorname{div}\big(A^0 \nabla[(\lambda \cdot x)\varphi(x)]\big) & \text{in } \omega_1 \\ w^\varepsilon_\lambda = 0 \quad \text{on } \partial\omega_1. \end{cases} \tag{13.6}$$

The existence and uniqueness of the solution $w^\varepsilon_\lambda \in H^1_0(\omega_1)$ is given by Theorem 4.16.

Observe that, due to the particular form of φ, one has

$$-\operatorname{div}(A^\varepsilon \nabla w^\varepsilon_\lambda) = -\operatorname{div}(A^0 \lambda) \quad \text{in } \omega. \tag{13.7}$$

Notice also, that according to Definition 13.4, w^ε_λ converges weakly in $H^1_0(\omega_1)$ to the solution w^0_λ of

$$\begin{cases} -\operatorname{div}(A^0 \nabla w^0_\lambda) = -\operatorname{div}\big[A^0 \nabla\big((\lambda \cdot x)\varphi(x)\big)\big] & \text{in } \omega_1 \\ w^0_\lambda = 0 \quad \text{on } \partial\omega_1. \end{cases}$$

Then, by uniqueness, one must have

$$w^0_\lambda(x) = (\lambda \cdot x)\varphi(x) \quad \text{in } \omega_1,$$

so that, since $\varphi = 1$ on ω, one has the convergence

$$w^\varepsilon_\lambda \rightharpoonup \lambda \cdot x \quad \text{weakly in } H^1(\omega). \tag{13.8}$$

In the sequel we set

$$w^\varepsilon_i = w^\varepsilon_{e_i}, \tag{13.9}$$

where $(e_i)_{i=1}^N$ is the canonical basis of $\mathbb{R}^N$.

Remark 13.13. It is interesting to remark that for $\lambda = e_i$ the relations (13.7) and (13.8) are the essential properties of the functions $\widehat{w}^\varepsilon_i$ introduced in Section 6.1 in the periodic case (see (8.32) and (8.34)). ◊

We are now in position to define the corrector matrix in the context of H-convergence.

Definition 13.14. Suppose that $\{A^\varepsilon\}$ H-converges to A^0 and let u^ε be the solution of problem (13.4). The *corrector matrix* $C^\varepsilon = (C^\varepsilon_{ij})_{1 \le i,j \le N} \in (L^2(\omega))^{N \times N}$ is defined by

$$C^\varepsilon_{ij} = \frac{\partial w^\varepsilon_j}{\partial y_i}.$$

It is easily seen from (13.8) and (13.9) that

$$C^\epsilon \rightharpoonup I \quad \text{weakly in } (L^2(\omega))^{N\times N}, \tag{13.10}$$

where I is the unit $N \times N$ matrix. Then, H-convergences implies in particular that

$$\nabla u^\epsilon - C^\epsilon \nabla u^0 \rightharpoonup 0 \quad \text{weakly in } (L^1(\omega))^N. \tag{13.11}$$

Remark 13.15. Observe that the corrector matrix C^ϵ depends on the choice of ω_1 and φ. But, since (13.7) and (13.8) are independent on ω_1 and φ, for any choice of ω_1 and φ one has convergences (13.10) and (13.11). ◇

Remark 13.16. Observe that Definition 13.14 generalizes the definition of the correctors given for the periodic case in Section 8.2. Convergences (13.10) and (13.11) correspond to (8.30)(i) and (8.36). ◇

The main corrector result is

Theorem 13.17. *Suppose that $\{A^\varepsilon\}$ H-converges to A^0 and let u^ϵ be the solution of problem (13.4). Let $\{C^\epsilon\}$ be any sequence of corrector matrices given by Definition 13.14. Then,*

$$\nabla u^\epsilon - C^\epsilon \nabla u^0 \to 0 \quad \text{strongly in } (L^1(\omega))^N. \tag{13.12}$$

Moreover, if C^ϵ is bounded in $(L^r(\omega))^{N\times N}$ for some r such that $2 \le r \le \infty$, and $\nabla u^0 \in (L^s(\omega))^N$ for some s such that $2 \le s < \infty$, then

$$\nabla u^\epsilon - C^\epsilon \nabla u^0 \to 0 \quad \text{strongly in } (L^t(\omega))^N, \tag{13.13}$$

where

$$t = \min\left\{2, \frac{rs}{r+s}\right\}.$$

13.3 Optimal bounds

The compactness result in Theorem 13.4 states that any sequence A^ϵ in $M(\alpha, \beta, \Omega)$ has a subsequence that H-converges to some matrix A^0 in $M(\alpha', \beta', \Omega)$. Suppose now that we are in the isotropic case, that is A^ε is of the form $A^\varepsilon = \gamma_\epsilon I$, where I is the identity matrix and

$$\alpha \le \gamma_\varepsilon(x) \le \beta, \quad \text{a.e. on } \Omega,\ \forall \varepsilon.$$

Then, problem (13.4) reads

$$\begin{cases} -\mathrm{div}(\gamma_\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.14}$$

In this case, since A^ϵ is symmetric, A^0 is also the G-limit of A^ϵ, so that A^0 is symmetric too and belongs to $M(\alpha, \beta, \Omega)$. Consequently, $A^0(x)$ admits N eigenvalues $\lambda_1(x), \dots, \lambda_N(x)$ defined a.e. on Ω, which are strictly positive.

In Section 5.1 we presented the model of a periodic physical case, where the conductivity of the periodic composite material was given by

$$\gamma_\varepsilon(x) = \gamma_1 \chi_1\left(\frac{x}{\varepsilon}\right) + \gamma_2 \chi_2\left(\frac{x}{\varepsilon}\right).$$

This case corresponds then to a periodic mixture of two (homogeneous) materials. In particular, Theorem 6.1 applies to this situation and provides explicit formulas for the homogenized matrix A^0. It will turn out that in general, A^0 is not isotropic.

In the non-periodic case, one cannot explicitly describe A^0.

A natural question is if one can at least characterize the H-limit A^0 when A^ε is of the form $A^\varepsilon = \gamma_\varepsilon I$ and

$$\gamma_\varepsilon(x) = \begin{cases} \gamma_1 & \text{if } x \in \Omega_1^\varepsilon \\ \gamma_2 & \text{if } x \in \Omega_2^\varepsilon, \end{cases}$$

i.e.

$$\gamma_\varepsilon(x) = \gamma_1 \chi_{\Omega_1^\varepsilon}(x) + \gamma_2 \chi_{\Omega_2^\varepsilon}(x), \tag{13.15}$$

with

$$\Omega_1^\varepsilon \cup \Omega_2^\varepsilon = \Omega, \quad \Omega_1^\varepsilon \cap \Omega_2^\varepsilon = \emptyset.$$

This corresponds to a (non-periodic) mixture of two materials. Here $\chi_{\Omega_i^\varepsilon}$, for $i = 1, 2$, is the characteristic function of the set Ω_i^ε. Obviously $\chi_{\Omega_1^\varepsilon} = 1 - \chi_{\Omega_2^\varepsilon}$.

Suppose that the proportion of the two materials is kept constant, i.e.

$$\frac{|\Omega_1^\varepsilon|}{|\Omega|} = \theta \in]0, 1[, \quad \frac{|\Omega_2^\varepsilon|}{|\Omega|} = 1 - \theta \tag{13.16}$$

and that the homogenized material is isotropic, i.e. A^0 is of the form λI. In this case there are some well-known bounds on the eigenvalues of the matrix A^0. These bounds are known in physics, mechanics or chemistry under various names, such as Hashin-Shtrikman (see Hashin and Shtrikman, 1962), Clausius–Mossoti, Lorentz–Lorenz or Maxwell Garnett bounds.

Suppose now that there exists a function $\theta(x)$ such that

$$\chi_{\Omega_1^\varepsilon} \rightharpoonup \theta \quad \text{weakly* in } L^\infty(\Omega). \tag{13.17}$$

Then from (13.15) one has obviously

$$\gamma_\varepsilon \rightharpoonup \theta\gamma_1 + (1-\theta)\gamma_2 \quad \text{weakly* in } L^\infty(\Omega). \tag{13.18}$$

In this case and with A^0 not necessarily isotropic, one still has bounds on the eigenvalues of the matrix A^0. This is a general mathematical result due to Tartar (1985) for the N-dimensional case (see also Murat and Tartar, 1997b). They were also obtained for the two-dimensional case independently by Lurie and Cherkaev (1984). We refer also to Lurie and Cherkaev (1986, 1997).

Theorem 13.18. *Let $A^\varepsilon \in M(\alpha, \beta, \Omega)$ such that $A^\varepsilon = \gamma_\varepsilon I$ where γ_ε is given by (13.15) with $\gamma_1 \leq \gamma_2$. Suppose that $A^{\varepsilon'} \in M(\alpha, \beta, \Omega)$ is a subsequence which H-converges to A^0. Then the eigenvalues $\lambda_1(x), \ldots, \lambda_N(x)$ of A^0 satisfy a.e. on Ω, the following inequalities:*

$$\begin{cases} \gamma_g(x) \leq \lambda_i(x) \leq \gamma_a(x), \quad \forall i = 1, \ldots, N, \\ \displaystyle\sum_{i=1}^{N} \frac{1}{\lambda_i(x) - \gamma_1} \leq \frac{1}{\gamma_g(x) - \gamma_1} + \frac{N-1}{\gamma_a(x) - \gamma_1} \\ \displaystyle\sum_{i=1}^{N} \frac{1}{\gamma_2 - \lambda_i(x)} \leq \frac{1}{\gamma_2 - \gamma_g(x)} + \frac{N-1}{\gamma_2 - \gamma_a(x)}, \end{cases} \tag{13.19}$$

where

$$\begin{cases} \gamma_a(x) = \theta(x)\gamma_1 + (1 - \theta(x))\gamma_2, \\ \gamma_g(x) = \left(\dfrac{\theta(x)}{\gamma_1} + \dfrac{1 - \theta(x)}{\gamma_2} \right)^{-1}. \end{cases} \tag{13.20}$$

Conversely, if the eigenvalues of a symmetric matrix A^0 satisfy (13.19) for some function θ such that $0 \leq \theta(x) \leq 1$ a.e. on Ω, then there exists a sequence of matrices A^ε of the form $\gamma_\varepsilon I$ satisfying (13.15) and (13.17), which H-converges to A^0.

In the two-dimensional case and under assumption (13.16), we can give a geometric interpretation of the bounds in (13.19) by using Fig. 13.1.

The point $C_1 = C_1(\theta)$ for a fixed $\theta \in [0, 1]$ is defined by

$$C_1(\theta) = (\gamma_g, \gamma_a) = \left(\frac{\gamma_1\gamma_2}{(\gamma_2 - \gamma_1)\theta + \gamma_1}, \gamma_2 - (\gamma_2 - \gamma_1)\theta \right).$$

and a simple computation made by using (13.20) shows that, when θ varies between 0 and 1, C_1 describes the hyperbola

$$(h_1) : y_1 = \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 - y_2}.$$

Observe that

$$C_1(0) = (\gamma_2, \gamma_2), \quad C_1(1) = (\gamma_1, \gamma_1).$$

The point $C_2 = C_2(\theta)$ is the reflection of C_1 with respect to the line $y_1 = y_2$, that is to say $C_2(\theta) = (\gamma_a, \gamma_g)$. It describes the hyperbola

$$(h_2) : y_2 = \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 - y_1}.$$

The points $B_1 = B_1(\theta)$ and $B_2 = B_2(\theta)$ are given by

$$B_1(\theta) = (\gamma_a, \gamma_a), \quad B_2(\theta) = (\gamma_g, \gamma_g).$$

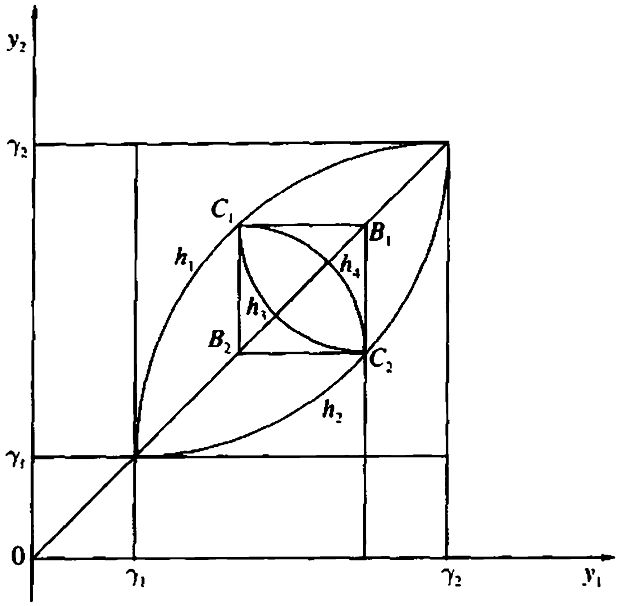


Fig. 13.1

The first line in (13.19) means that the eigenvalues are contained in the square $B_1 C_2 B_2 C_1$.

The other two inequalities in (13.19) say that actually the eigenvalues are contained in the dashed area between the two hyperbola h_3 and h_4 whose equations are

$$(h_3)\ :\ \frac{1}{y_1-\gamma_1}+\frac{1}{y_2-\gamma_1}=\frac{1}{\gamma_g-\gamma_1}+\frac{1}{\gamma_a-\gamma_1},$$
$$(h_4)\ :\ \frac{1}{\gamma_2-y_1}+\frac{1}{\gamma_2-y_2}=\frac{1}{\gamma_2-\gamma_g}+\frac{1}{\gamma_2-\gamma_a}.$$

Remark 13.19. Let $N = 2$ and suppose that the matrix A^0 is isotropic. Then, one shows that inequalities (13.19) reduces to

$$\lambda_- \leq \lambda \leq \lambda_+,$$

where

$$\lambda_- = \frac{\theta\gamma_1 + (1-\theta)\gamma_2 + \gamma_2}{(1-\theta)\gamma_1 + \theta\gamma_2 + \gamma_1}\gamma_1,$$
$$\lambda_+ = \frac{\theta\gamma_1 + (1-\theta)\gamma_2 + \gamma_1}{(1-\theta)\gamma_1 + \theta\gamma_2 + \gamma_2}\gamma_2.$$

These are actually the well-known Hashin–Shtrikman (or Clausius–Mossoti or Lorentz–Lorenz or Maxwell–Garnett) inequalities mentioned above. ◇

Bibliography

R. A. Adams (1975) *Sobolev Spaces*, Academic Press. New York.

S. Agmon, A. Douglis and L. Nirenberg (1959) Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary value conditions, I. *Comm. Pure Appl. Math.* **12**, 623–727.

G. P. Akilov and L. V. Kantorovich (1981) *Analyse Fonctionnelle*, Mir, Moscow.

G. Allaire (1992) Homogenization and two-scale convergence. *SIAM J. Math. Anal.* **23**, 1482–1518.

G. Allaire (1994) Two-scale convergence: a new method in periodic homogenization. *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vol. XII*, ed. H. Brezis and J.-L. Lions. Pitman Research Notes in Mathematics, Longman, Harlow, pp. 1–14.

J. P. Aubin (1963) Un théorème de compacité. *C. R. Acad. Sci.* **256**, 5042–5044.

I. Babuška (1976) Homogenization and its applications, in *Numerical Solutions of Partial Differential Equations* III, ed. B. Hubbard, Academic Press, pp. 89–116.

N. S. Bakhvalov (1974) Averaged characteristic of bodies with periodic structure, *Dokl. Akad. Nauk. SSSR* **218**, 1046–1048. English translation: *Phys. Dokl.* **19**, 1974–1975.

N. Bakhvalov and G. Panasenko (1989) *Homogenization: Averaging Processes in Periodic Media*, Kluwer, Dortmund, Boston, London.

A. Bensoussan, J.-L. Lions and G. Papanicolaou (1978) *Asymptotic Analysis for Periodic Structures*, North Holland, Amsterdam.

L. Boccardo and P. Marcellini (1976) Sulla convergenza delle soluzioni di disequazioni variazionali, *Annali di Mat. Pura e Applicata* **90**, 137–159.

L. Boccardo and F. Murat (1982) Homogénéisation de problèmes quasi-linéaires, in *Atti del Convegno su Studio dei Problemi dell'Analisi Funzionale, Bressanone 7–9 settembre 1981*, Pitagora, Bologna pp. 13–53.

S. Brahim-Otsmane, G. A. Francfort and F. Murat (1992) *Correctors for the homogenization of the wave and heat equations*, *J. Math. Pures et Appl.* **71**, 197–231.

H. Brezis (1987) *Analyse fonctionnelle. théorie et applications*. Masson, Paris.

G. Buttazzo (1989) *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations*, Pitman Research Notes in Mathematics 207, London.

T. Cazenave and A. Haraux (1998) *An Introduction to Semilinear Evolution Equations*, Oxford Lecture in Mathematics and its Applications, no. 13, Clarendon Press, Oxford.

P. G. Ciarlet (1982) *Introduction à l'analyse numérique matricielle et à l'optimisation*, Masson, Paris.

D. Cioranescu and J. Saint Jean Paulin (1999) *Homogenization of Reticulated Structures*, Applied Mathematical Sciences, 139, Springer-Verlag, New York.

D. Cioranescu and F. Murat (1982) Un terme étrange venu d'ailleurs, in *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vol. II and III*, ed. by H. Brezis and J.-L. Lions, Research Notes in Mathematics, 60 and 70, Pitman, London, pp. 93–138, 154–178.

E. Coddington and N. Levinson (1955) *Theory of Ordinary Differential Equations*, McGraw Hill, New York.

F. Colombini and S. Spagnolo (1977) Sur la convergence des solutions d'équations paraboliques, *J. Math. Pures Appl.* **56**, 263–306.

R. Courant and D. Hilbert (1962) *Methods of Mathematical Physics*, Interscience Publishers, New York.

B. Dacorogna (1989) *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin.

G. Dal Maso (1993) *An introduction to Γ-convergence*, Birkhäuser, Boston.

E. De Giorgi (1975) Sulla convergenza di alcune successioni di integrali del tipo dell'area, *Rend. Mat.* **8**, 277–294.

E. De Giorgi and T. Franzoni (1975) Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat.* **58**, 842–850.

E. De Giorgi and S. Spagnolo (1973) Sulla convergenza degli integrali dell'energia, *Boll. Un. Mat. It.* **8**, 391–411.

J. Diestel and J. J. Uhl (1977) *Vector Measures*, Mathematical Surveys, 15, American Mathematical Society, Providence.

N. Dinculeanu (1967) *Vector Measures*, Pergamon Press, New York.

P. Donato (1983a) Una stima per la differenza di H-limiti e qualche applicazione a problemi di omogeneizzazione, *Rend. di Matematica* **4**, 623–640.

P. Donato (1983b) Alcune osservazioni sulla convergenza debole di funzioni non uniformemente oscillanti, *Ricerche di Matematica*, XXXII **2**, 203–219.

P. Donato (1985) Weak convergence of non-uniformly oscillating functions, in *Nonlinear variational problems*, ed. A. Marino, L. Modica, S. Spagnolo and M. Degiovanni, Research Notes in Mathematics 127, Pitman, London, pp. 102–104.

N. Dunford and J. T. Schwartz (1958) *Linear Operators*, Interscience Publishers, New York.

G. Duvaut (1978) Analyse fonctionnelle et mécanique des milieux continus. Applications à l'étude des matériaux composites élastiques à structure périodique, homogénéisation, in *Theoretical and Applied Mechanics*. ed. W. T. Koiter, North Holland, Amsterdam.

G. Duvaut and J. L. Lions (1972) *Les inéquations en mécanique et en physique*, Dunod, Paris.

R. Edwards (1965) *Functional Analysis*, Holt. Rinehart & Winston, New York.

H. Ene and G. Paşa (1987) *Metoda omogenizării. Aplicaţii la teoria materialelor compozite*, Ed. Academiei. Bucarest.

D. Gilbarg and N. Trudinger (1977) *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag. Berlin.

V. Girault and P. A. Raviart (1981) *Finite Element Methods for Navier Stokes Equations*, Lecture Notes in Mathematics 749. Springer-Verlag, Berlin.

Z. Hashin and S. Shtrikman (1962) A variational approach to the theory of effective magnetic permeability of multiphase materials. *J. Appl. Phys.* **33**, 3125–3131.

E. Hopf (1957) On nonlinear partial differential equations in *Lectures Series of the Symposium on Partial Differential Equations. Berkeley, 1955*, Ed. Univ. Kansas, pp. 1 29.

V .V. Jikov, S. M. Kozlov and O. A. Oleinik (1994) *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin. Heidelberg.

S. Kesavan (1979) Homogenization of elliptic eigenvalue problems, *Appl. Math. and Optim.* **5**. Part I. 153 167. Part II. 197 216.

D. Kinderlehrer and G. Stampacchia (1980) *An Introduction to Variational Inequalities and their Applications*. Academic Press, New York.

V. A. Kondratiev and O. A. Oleinik (1989a) On Korn's inequalities. *C. R. Acad. Sci.* **308**, 483 487.

V. A. Kondratiev and O. A. Oleinik (1989b) On the dependence of the constants in the Korn's inequality on the parameters characterizing the geometry of the domain, *Usp. Mat. Nauk* **44** (6), 153–160. English translation: *Russ. Math. Surv.* **43** (1989), 187 195.

V. A. Kondratiev and O. A. Oleinik (1990) Hardy's and Korn's inequalities, *Rend. Mat. Appl., Ser.* VII **10**(3), 641–666.

O. A. Ladyzhenskaya and N. Uraltseva (1968) *Linear and Quasilinear Elliptic Equations*, Academic Press, New York.

S. Lang (1993) *Introduction to Linear Algebra*, Springer-Verlag, New York, Berlin.

F. Lené (1984) Contribution à l'étude des matériaux composites et de leur endommagement. Thèse d'Etat, Université Paris VI.

J. L. Lions (1969) *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris.

J.-L. Lions (1978) Remarques sur les aspects numériques de la méthode de l'homogénéisation, in *Etudes numériques des grands systèmes, Novosibirsk, 1976*, Dunod, Paris, pp. 15–38.

J.-L. Lions (1981) *Some Methods in the Mathematical Analysis of Systems and Their Control*, Beijing, Gordon and Breach, New York.

J. L. Lions (1988) *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Masson, Paris.

J.-L. Lions and E. Magenes (1968a) *Problèmes aux limites non homogènes et applications. Vol. 1*. Dunod, Paris.

J.-L. Lions and E. Magenes (1968b) *Problèmes aux limites non homogènes et applications, Vol. 2*. Dunod, Paris.

E. H. Love (1944) *A Treatise on the Mathematical Theory of Elasticity*. Dover, New York.

K. A. Lurie and A. V. Cherkaev (1984) Exact estimates of conductivity of composites, *Proc. Royal Soc. Edinburgh* **A99**. 71–84.

K. A. Lurie, and A. V. Cherkaev (1986/1997) Effective characteristics of composite materials and optimal design of structural elements. *Uspekhi Mekaniki*, Advances in Mechanics, **92**. English translation, in *Topics in the Mathematical Modeling of Composite Materials*. ed. R. V. Kohn and A. V. Cherkaev, Birkhauser, Boston. pp. 175 258.

V. G. Mazya (1985) *Sobolev Spaces*. Springer-Verlag. Berlin.

N. G. Meyers (1963) An L^p-estimate for the gradient of solutions of second order elliptic divergent equations. *Ann. Sc. Norm. Sup. Pisa* **17**. 189 206.

F. Murat (1978a) H-Convergence. Séminaire d'analyse fonctionnelle et numérique de l'Université d'Alger, duplicated. pp. 34.

F. Murat (1978b) Compacité par compensation. *Ann. Sc. Norm. Sup. Pisa* **5**, 489 507.

F. Murat and L. Tartar (1997a) H-Convergence. in *Topics in the Mathematical Modelling of Composite Materials.* ed. A. Cherkaev and R. Kohn, Birkhäuser. Boston, pp. 21-43.

F. Murat and L. Tartar (1997b) *Calcul des variations et homogénéisation* in Les métodes de l'homogénéisation: Théorie et applications en physique, Eyrolles, 1985. pp. 319-369, English translation: Calculus of Variations and homogenization in *Topics in the Mathematical Modelling of Composite Materials,* ed. A. Cherkaev and R. Kohn. Birkhäuser, Boston. pp. 139-173.

J. Nečas (1967) *Les méthodes directes en théorie des équations elliptiques,* Masson, Paris.

G. Nguetseng (1989) A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20**. 608-629.

O. A. Oleinik. A. S. Shamaev and G. A. Yosifian (1992) *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam.

A. Pazy (1974) *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Lecture Notes. Unversity of Maryland.

W. Rudin (1966) *Real and Complex Analysis.* McGraw Hill. New York.

W. Rudin (1973) *Functional Analysis.* McGraw Hill. New York.

E. Sanchez-Palencia (1970a) Solutions périodiques par rapport aux variables d'espace et applications, *C. R. Acad. Sci.* **271**. 1129-1132.

E. Sanchez-Palencia (1970b) Equations aux dérivées partielles dans un type de milieux hétérogènes. *C. R. Acad. Sci.* **272**, 1410-1411.

E. Sanchez-Palencia (1980) *Non homogeneous Media and Vibration Theory*, Lecture Notes in Physics. 127. Springer-Verlag. Berlin.

J. Sanchez-Hubert and E. Sanchez-Palencia (1992) *Introduction aux méthodes asymptotiques et àl'homogénéisation.* Masson. Paris.

L. Schwartz (1951) *Théorie des distributions.* Hermann. Paris.

J. Simon (1987) Compact sets in the space $L^p(0,T;B)$. *Annali di Mat. Pura ed Appl.* **146**, IV. 65 96.

S. Spagnolo (1967) Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore. *Ann. Sc. Norm. Sup. Pisa* **21**. 657 699.

S. Spagnolo (1968) Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche, *Ann. Sc. Norm. Sup. Pisa* **22**. 571 597.

L. Tartar (1977a) *Cours Peccot au Collège de France* (unpublished). partially written in Murat (1978a).

L. Tartar (1977b) Estimations des coefficients homogénéisés, in *Lectures Notes in Mathematics 704*, Springer-Verlag, Berlin, pp. 364–373. English translation, Estimations of homogenized coefficients, in *Topics in the Mathematical Modeling of Composite Materials*, ed. A. Cherkaev and R. Kohn, Birkhäuser, Boston, pp. 9–20.

L. Tartar (1978) Quelques remarques sur l'homogénéisation, in *Functional Analysis and Numerical Analysis, Proc. Japan-France Seminar 1976*, ed. H. Fujita, Japanese Society for the Promotion of Science, 468–482.

L. Tartar (1979) Compensated compactness and applications to partial differential equations, in *Nonlinear Analysis and Mechanics, Heriott Watt Symposium*, ed. R. J. Knops, vol. IV, Research Notes in Mathematics, 39 Pitman, pp. 136–212.

L. Tartar (1985) Estimations fines de coefficients homogénéisés, in *Ennio De Giorgi Colloquium*, ed. P. Kree, Research Notes in Mathematics 125, Pitman, London, pp. 168–187.

R. Temam (1979) *Navier-Stokes Equations*, North Holland, Amsterdam.

G. M. Troianiello (1987) *Elliptic differential equations and obstacle problems*, Plenum Press, New York.

C. Truesdell and R. Toupin (1960) *The Classical Field Theory*, Handbuch des Physics, Bank III, 1, Springer, Berlin.

J. Wloka (1987) *Partial differential equations*, Cambridge University Press, Cambridge.

K. Yosida (1964) *Functional Analysis*. Springer-Verlag, Berlin.

E. Zeidler (1980) *Nonlinear Functional Analysis and its Applications*. Part I and II, Springer-Verlag, Berlin.

Index

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